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Some Notes on Weak Subdifferential

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Abstract. Some necessary conditions for having nonempty weak subdifferential of a function are presented and the positively homogeneous of the weak subdifferential operator is proved. Necessary and sufficient conditions for achieving a global minimum of a weak subdifferentiable function is stated, as well as a link between subdifferential and the Fréchet differential with a weak subdifferential. A result about the equality of the fuzzy sum rule inclusion is also investigated. Finally, some examples are included.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

The notion of weak subdifferential which is a generalization of the classic subdifferential, is introduced by Azimov and Gasimov [1]. It uses explicitly defined supporting conic surfaces instead of supporting hyperplanes. Recall that a convex set has a supporting hyperplane at each boundary point. This leads to one of the central notions in convex analysis, that of a subgradient of a possible nonsmooth even extended real valued function [4]. The main reason of difficulties arising when passing from the convex analysis to the nonconvex one is that the nonconvex cases may arise in many different forms and each case may require a special approach. The main ingredient is the method of supporting the given nonconvex set. Subgradients plays an important role in deriving of optimality conditions and duality theorems. The first canonical generalized gradient was introduced by Clarke [4]. He applied the generalized gradient systematically to nonsmooth problems in a variety of problems. Since a nonconvex set has no supporting hyperline at each boundary point, the notion of subgradient have been generalized by most researchers on optimality conditions for nonconvex problems [3, 4]. By using the notion of subgradients, a collection of zero duality gap conditions for a wide class of nonconvex optimization problems was derived [1]. In this study we give some important properties of the weak subdifferentials. By using the definition and properties of the weak subdifferential which are described in [1, 2, 10, 11], we present some facts concerning weak subdifferential in the nonsmooth and nonconvex analysis. It is also obtained Necessary and sufficient optimality conditions by using the weak subdifferential.

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This paper is organized as follows. The definition and some preliminaries of the weak subdifferential are given in Section 2. In Section 3, some theorems connecting operations on the weak subdifferential in the non-smooth and non-convex analysis are provided. Also, a necessary condition in which a function attains its global minimum by applying weak subdifferential is stated.

2. Preliminaries

Throughout this paper let X be a real normed space and let X^* be the topological dual of X. By $\|\cdot\|$ we denote the norm of X and by $\langle x^*, x\rangle$ the value of the linear functional $x^* \in X^*$ at the point $x \in X$.

Definition 2.1 ([10, 11]). *Let $f : X \to \mathbb{R}$ be a function and $\bar{x} \in X$ be a given point. The set*

$$\partial f(\bar{x}) = \left\{x^* \in X^* : (\forall x \in X)\ \ f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle\right\}$$

is called the subdifferential of f at $\bar{x} \in X$.

Definition 2.2 ([10, 11]). *Let $f : X \to \mathbb{R}$ be a function and $\bar{x} \in X$ be a given point. A pair $(x^*, c) \in X^* \times \mathbb{R}^+$ where $\mathbb{R}^+$, the set of nonnegative real numbers, is called the weak subgradient of f at $\bar{x} \in X$ if the following inequality holds:*

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

The set

$$\partial^w f(\bar{x}) = \left\{(x^*, c) \in X^* \times \mathbb{R}^+ : (\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|\ \right\}$$

of all weak subgradients of f at $\bar{x} \in X$ is called the weak subdifferential of f at $\bar{x} \in X$. If $\partial^w f(\bar{x}) \neq \emptyset$, then f is called weakly subdifferentiable at $\bar{x}$.

Remark 2.3. *It is obvious from the definition of weak subgradient that if $\partial^w f(\bar{x})$ is nonempty then it contains uncountable members. Because if $(x^*, \bar{c}) \in \partial^w f(\bar{x})$, then we have*

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - \bar{c}\|x - \bar{x}\|.$$

Hence

$$(\forall x \in X, \forall c \geq \bar{c}) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|,$$

which the last inequality means that $(x^*, c) \in \partial^w f(\bar{x})$. This completes proof of the assertion.

Remark 2.4. *It is clear that when f is subdifferentiable at $\bar{x}$, then f is also weakly subdifferentiable at $\bar{x}$; that is, if $x^* \in \partial f(\bar{x})$, then by the definition of weak subgradient we get $(x^*, c) \in \partial^w f(\bar{x})$ for every $c \geq 0$. But the following example shows that the converse may fail.*

Example 2.5. *Let $X = \mathbb{R}$ and $f(x) = -|x|$. Then it follows from the definition of weak subdifferential that*

$$(a, c) \in \partial^w f(0) \Longleftrightarrow (a, c) \in \mathbb{R} \times \mathbb{R}^+ \quad \text{and} \quad (\forall x \in X) \quad -|x| \geq ax - c|x|.$$

Hence the weak subdifferential can be explicitly written as

$$\partial^w f(0) = \left\{(a, c) \in \mathbb{R} \times \mathbb{R}^+;\ |a| \leq c - 1\right\}.$$

On the other hand, it follows from the definition of the subdifferential that $\partial f(0) = \emptyset$.

Remark 2.6. *It follows from Definition 2.2 that the pair $(x^*, c) \in X^* \times \mathbb{R}^+$ is a weak subdifferential of f at $\bar{x} \in X$ if and only if there exists a continuous (super linear) concave function*

$$g(x) = \langle x^*, x - \bar{x}\rangle + f(\bar{x}) - c\|x - \bar{x}\|,$$

such that

$$(\forall x \in X) \quad g(x) \leq f(x) \quad \text{and} \quad g(\bar{x}) = f(\bar{x}).$$

The class of weakly subdifferentiable functions are wider than the class of subdifferentiable functions. The weak subdifferential is a strong tool for studying nonconvex optimization problems, for instance, see [1, 12]. It is worth noting that the calculation of weak subdifferential by using its definition is not easy in general. The calculation of weak subdifferential for some functions is given in [14]. M.Kucuk, et al. presented the very useful method for calculation of weak subdifferential of functions that represented as the infimum of support functions, the functions that represented as difference of two sublinear functions , and convex functions.

Definition 2.7 ([13]). *A function $f : X \to \mathbb{R}$ is called locally Lipschitz at $\bar{x} \in X$ if there exist a nonnegative number L (Lipschitz constant) and a neighborhood $N(\bar{x})$ of $\bar{x}$ such that*

$$(\forall x \in N(\bar{x})) \qquad |f(x) - f(\bar{x})| \leq L\|x - \bar{x}\|.$$

If the above inequality holds for all $x \in X$, then f is called Lipschitz with the Lipschitz constant L.

Theorem 2.8 ([10]). *Let the weak subdifferential of $f : X \to \mathbb{R}$ at $\bar{x}$ be nonempty. Then the set $\partial^w f(\bar{x})$ is closed and convex.*

3. Main Result

In this section we follow the main results given in [10]. In the sequel we need the following definition .

Definition 3.1 ([13]). *A function $f : X \to (-\infty, +\infty]$ is lower semicontinuous at $\bar{x} \in X$ if*

$$x_n \to \bar{x} \;\to\; \liminf f(x_n) \geq f(\bar{x}).$$

It is worth noting that Definition 3.1 was called sequentially lower semicontinuity by some authors while they defined the lower semicontinuity of f at the point $\bar{x} \in X$ as

$$\liminf_{x \to \bar{x}} f(x) \geq f(\bar{x}).$$

It is clear that the lower semicontinuity at a point implies the sequentially lower semicontinuity at the point.

The next result provides a necessary condition for weak subdifferentiability of a function at a point.

Proposition 3.2. *Let f be weak subdifferentiable at $\bar{x} \in X$. Then f is lower semicontinuous at $\bar{x} \in X$.*

Proof. The weak subdifferentiability of f at $\bar{x}$ implies that $\partial^w f(\bar{x}) \neq \emptyset$. Hence there exists the pair $(x^*, c) \in X^* \times \mathbb{R}^+$ such that

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

The result follows by taking the limit inferior of the both sides of the last inequality when $x \to \bar{x}$. □

The following example shows that the converse of Proposition 3.2 may fail.

Example 3.3. *Let $X = \mathbb{R}$ and $f(x) = -x^2$. It is easy to see that $\partial^w f(0) = \emptyset$ while f is a continuous function.*

The next definition is important in this paper.

Definition 3.4 ([7]). *Let $f : X \to \mathbb{R}$ be a function. If there is a continuous linear map $f'(\bar{x}) : X \to \mathbb{R}$ with the property*

$$\lim_{\|h\| \to 0} \frac{|f(\bar{x} + h) - f(\bar{x}) - \langle f'(\bar{x}), h\rangle|}{\|h\|} = 0,$$

then $f'(\bar{x}) : X \to \mathbb{R}$ is called the Fréchet derivative of f at $\bar{x} \in X$ and f is called the Fréchet differentiable at $\bar{x}$.

The next conclusion provides a link between Fréchet differentiability and weak subdifferentiability of a function.

Proposition 3.5. *Assume $f : X \to \mathbb{R}$ is subdifferentiable and Fréchet differentiable at $\bar{x}$. Then*

$$\{(f'(\bar{x}), c); c \geq 0\} \subset \partial^w f(\bar{x}).$$

Proof. Since f is subdifferentiable at $\bar{x} \in X$, then there exists $x^* \in \partial f(\overline{x}) \subset X^*$ such that

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle.$$

By taking

$$x = \bar{x} + te \text{ s.t. } t \geq 0, \; e \in X, \; \|e\| = 1,$$

we get

$$f(\bar{x} + te) - f(\bar{x}) \geq \langle x^*, te \rangle.$$

Therefore,

$$\frac{f(\bar{x} + te) - f(\bar{x})}{t} \geq \frac{\langle x^*, te \rangle}{t}.$$

Now it is obvious from Fréchet differentiability of f at $\bar{x}$, by letting $t \to 0^+$, that

$$\langle x^* - f'(\bar{x}), e \rangle \leq 0.$$

Hence $x^* = f'(\bar{x})$ and $f'(\bar{x}) \in \partial f(\bar{x})$. Then $f'(\bar{x}) \in \partial f(\overline{x})$ and so it follows from Remark 2.4 that

$$\{(f'(\bar{x}), c); c \geq 0\} \subset \partial^w f(\bar{x}).$$

This completes the proof. □

The following example shows that the conclusion in Proposition 3.5 may be strict.

Example 3.6. *Let $X = \mathbb{R}, f \equiv 0$ and $\bar{x} = 0$. Then by the definition of weak subdifferential and Fréchet differentiability of f at $\bar{x}$ we have, respectively,*

$$\partial^w f(0) = \{(a, c) \in \mathbb{R} \times \mathbb{R}^+ ; |a| \leq c\}$$

and

$$A = \{(f'(0), c); c \geq 0\} = \{(0, c); c \geq 0\}.$$

It is clear that $A \subsetneq \partial^w f(0)$.

The following example shows that the subdifferentiability of f at $\bar{x}$ in Proposition 3.5 is essential.

Example 3.7. *Let $X = \mathbb{R}$ and $f(x) = -x^2$. Then it is easy to verify that*

$$\partial f(0) = \emptyset, \quad \partial^w f(\bar{x}) = \emptyset \quad \text{and} \quad f'(0) = 0.$$

Remark 3.8. *It is well known that if f is convex and Fréchet differentiable at $\bar{x}$ then $\partial f(\bar{x}) = \left\{f'(\bar{x})\right\}$. Hence by Proposition 3.5 f is weak subdifferentiable at $\bar{x}$.*

The next result gives a characterization of having global minimum for a weakly subdifferentiable function.

Proposition 3.9. *Suppose $f : X \to (-\infty, +\infty]$ is weakly subdifferentiable at $\bar{x} \in X$. Then f has a global minimum at $\bar{x}$ if and only if $(0, c) \in \partial^w f(\bar{x})$ for all $c \geq 0$.*

Proof. The proof directly follows from the definition of weak subdifferentiability of f at $\bar{x} \in X$. □

The next conclusion asserts that the operator weak subdifferential (∂^w) is positively homogeneous.

Proposition 3.10. *Let $f : X \to \mathbb{R}$ be weakly subdifferentiable at $\bar{x} \in X$. Then*

$$(\forall \alpha > 0) \qquad \partial^w(\alpha f)(\bar{x}) = \alpha \partial^w f(\bar{x}).$$

Proof. If $(x^*, c) \in \alpha \partial^w f(\bar{x})$, then

$$\frac{1}{\alpha}(x^*, c) \in \partial^w f(\bar{x}).$$

Hence

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \left\langle \frac{x^*}{\alpha}, x - \bar{x} \right\rangle - \frac{c}{\alpha}\|x - \bar{x}\|.$$

Thus,

$$(\forall x \in X) \qquad \alpha f(x) - \alpha f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - \bar{c}\|x - \bar{x}\|.$$

This means that $\alpha \partial^w f(\bar{x}) \subset \partial^w \alpha f(\bar{x})$. Now we prove that the converse of the inclusion. Since αf with the first part of proof is weakly subdifferentiable at $\bar{x}$, then there exists a pair $(x^*, c) \in \partial^w \alpha f(\bar{x})$ such that

$$(\forall x \in X) \qquad \alpha f(x) - \alpha f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - \bar{c}\|x - \bar{x}\|.$$

Hence

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \left\langle \frac{x^*}{\alpha}, x - \bar{x} \right\rangle - \frac{c}{\alpha}\|x - \bar{x}\|.$$

This implies that

$$\left(\frac{x^*}{\alpha}, \frac{c}{\alpha}\right) \in \partial^w f(\bar{x}).$$

Consequently, $(x^*, c) \in \alpha \partial^w f(\bar{x})$ and therefore $\partial^w \alpha f(\bar{x}) \subset \alpha \partial^w f(\bar{x})$. This completes the proof. □

Remark 3.11. *Note that $\partial^w(f(\alpha \bar{x})) = \partial^w \alpha f(\bar{x})$ may drop. Consider $X = \mathbb{R}$, $\bar{x} = 1$, $\alpha = \sqrt{2}$, and define*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}^c, \\ 0, & x \in \mathbb{Q}. \end{cases}$$

Then we have

$$\partial^w f(1) = \left\{(a, c) \in \mathbb{R} \times \mathbb{R}^+;\ |a| \leq c\right\}, \quad \partial^w f(\sqrt{2}) = \emptyset.$$

Now we are interested to find a sufficient condition that the following equality holds.

Proposition 3.12. *If f is a positively homogeneous function and weak subdifferentiable at $\bar{x}$ and $\alpha \bar{x}$, where α is a positive real number, then*

$$\partial^w(f(\alpha \bar{x})) = \partial^w f(\bar{x}).$$

Proof. It follows from the hypothesis that

$$\begin{aligned}(x^*, c) \in \partial^w f(\alpha\bar{x}) &\iff f(\alpha x) - f(\alpha\bar{x}) \geq \langle x^*, \alpha x - \alpha\bar{x}\rangle - c\|\alpha x - \alpha\bar{x}\| \\ &\iff \alpha(f(x) - f(\bar{x})) \geq \alpha(\langle x^*, x - \bar{x}\rangle - \bar{c}\|x - \bar{x}\|) \\ &\iff (x^*, c) \in \partial^w f(\bar{x}).\end{aligned}$$

This completes the proof. □

In the next we recall the fuzzy sum rule and we investigate sufficient condition which the equality holds.

Proposition 3.13 ([10]). *If $f_1 : X \to \mathbb{R}$ and $f_2 : X \to \mathbb{R}$ are weak subdifferential at $\bar{x}$, then $f_1 + f_2$ is weak subdifferential at $\bar{x}$ and*

$$\partial^w f_1(\bar{x}) + \partial^w f_2(\bar{x}) \subseteq \partial^w (f_1 + f_2)(\bar{x}).$$

Remark 3.14. *The simple example $X = \mathbb{R}$, $f_1(x) = \sin x$, $f_2(x) = -\sin x$, $\bar{x} = 0$, shows that the inclusion of Proposition 3.13 may be strict.*

The following proposition provides sufficient conditions in which the equality of Proposition 3.13 holds.

Proposition 3.15. *Assume that $f_1 : X \to \mathbb{R}$ is weak subdifferentiable at $\bar{x}$, $f_2 : X \to \mathbb{R}$ is subdifferentiable and Fréchet differentiable at $\bar{x}$ and $-f_2$ is subdifferentiable at $\bar{x}$. Then*

$$\partial^w f_1(\bar{x}) + \partial^w f_2(\bar{x}) = \partial^w (f_1 + f_2)(\bar{x}).$$

Proof. If $(x^*, c) \in \partial^w (f_1 + f_2)(\bar{x})$, then

$$(\forall x \in X) \qquad (f_1 + f_2)(x) - (f_1 + f_2)(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

Since $f_2 : X \to \mathbb{R}$ is Fréchet differentiable at $\bar{x}$ and $-f_2$ is subdifferentiable at $\bar{x}$, we get, see Proposition 3.5,

$$(\forall x \in X) \qquad -f_2(x) + f_2(\bar{x}) \geq \langle -f_2'(\bar{x}), x - \bar{x}\rangle.$$

It follows from the first inequality that

$$(\forall x \in X) \qquad (f_1(x) - f_1(\bar{x})) + (f_2(x) - f_2(\bar{x})) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

Hence

$$(\forall x \in X) \qquad (f_1(x) - f_1(\bar{x})) \geq -(f_2(x) - f_2(\bar{x})) + \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

Now the hypothesis implies

$$f_1(x) - f_1(\bar{x}) \geq \langle(-f_2'(\bar{x}), x - \bar{x})\rangle + \langle x^*, x - \bar{x}\rangle \; - c\|x - \bar{x}\|.$$

Therefore

$$(f_1(x) - f_1(\bar{x})) \geq \langle x^* - f_2'(\bar{x}), x - \bar{x})\rangle - c\|x - \bar{x}\|.$$

Then

$$(x^* - f_2'(\bar{x}), c) \in \partial^w f_1(\bar{x}) \;,\; (f_2'(\bar{x}), 0) \in \partial^w f_2(\bar{x}).$$

This means that

$$\partial^w (f_1 + f_2)(\bar{x}) \subseteq \partial^w f_1(\bar{x}) + \partial^w f_2(\bar{x}).$$

The reverse side of the inclusion follows from Proposition 3.13 and so the proof is completed. □

Corollary 3.16. *If for all but at most one of the weak subdifferentiable functions f_i at $\bar{x}$, $f_i, -f_i$ are Fréchet differentiable and subdifferentiable at $\bar{x}$, then*

$$\sum_{i=1}^{n} \partial^w f_i(\bar{x}) = \partial^w \left(\sum_{i=1}^{n} f_i \right)(\bar{x}).$$

Remark 3.17. *It is easy to check that if $f : X \to \mathbb{R}$ is Fréchet differentiable at $\bar{x}$, then $f, -f$ are subdifferentiable at $\bar{x}$ if and only if*

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) = \langle f'(\bar{x}), x - \bar{x} \rangle.$$

Proposition 3.18. *Let $f_1 : X \to \mathbb{R}$ be a function, $-f_1$ be Fréchet differentiable and subdifferentiable at $\bar{x}$ and $f_2 : X \to \mathbb{R}$ be a function. If $f_1 + f_2$ attains a global minimum at $\bar{x}$, then $(-f_1'(\bar{x}), 0) \in \partial^w f_2(\bar{x})$.*

Proof. Since $f_1 + f_2$ attains a global minimum at $\bar{x}$ then

$$(\forall x \in X) \qquad (f_1 + f_2)(x) \geq (f_1 + f_2)(\bar{x})$$

and so we can rewrite the inequality as

$$(\forall x \in X) \qquad f_2(x) - f_2(\bar{x}) \geq f_1(\bar{x}) - f_1(x).$$

Hence the subdifferentability and Fréchet differentiabability of $-f_1$, similar to the proof of Proposition 3.15, imply that

$$(\forall x \in X) \qquad f_2(x) - f_2(\bar{x}) \geq \langle -f_1'(\bar{x}), x - \bar{x} \rangle.$$

This means that

$$(-f_1'(\bar{x}), 0) \in \partial^w f_2(\bar{x})$$

and so the proof is completed. □

Proposition 3.19. *Let $f : X \to \mathbb{R}$ be weak subdifferentiable at $\bar{x}$ and $g - f$ attain a global minimum at $\bar{x}$. Then*

$$\partial^w f(\bar{x}) \subset \partial^w g(\bar{x}).$$

Proof. The weak subdifferentiability of f at $\bar{x}$ implies that $\partial^w f(\bar{x}) \neq \emptyset$. Hence there exists $(x^*, c) \in X^* \times \mathbb{R}^+$ such that

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\|.$$

Since $g - f$ attains a global minimum at $\bar{x}$ then

$$(\forall x \in X) \qquad (g - f)(x) \geq (g - f)(\bar{x}).$$

Therefore,

$$(\forall x \in X) \qquad g(x) - g(\bar{x}) \geq f(x) - f(\bar{x}).$$

Consequently, the above inequalities imply that

$$(\forall x \in X) \qquad g(x) - g(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\|.$$

This means that $(x^*, c) \in \partial^w g(\bar{x})$, which is the desired result and the proof is completed. □

Corollary 3.20. *If f attains a global minimum at $\bar{x}$ then $\partial^w f(\bar{x})$ contains the weak subdifferentiable of the zero function at $\bar{x}$, that is*

$$\partial^w 0(\bar{x}) \subset \partial^w f(\bar{x}).$$

Example 3.21. *The example, $f(x) = -|x|$ for all $x \in \mathbb{R}$ and $\overline{x} = 0$ shows that the condition $\overline{x}$ is a global minimum of f in the previous corollary is essential.*

Proposition 3.22. *If $g - f$ is a constant function on X, then*

$$(\forall \bar{x} \in X) \qquad \partial^w f(\bar{x}) = \partial^w g(\bar{x}).$$

Proof. From

$$(\forall x \in X) \qquad (f-g)(x) \geq (f-g)(\bar{x})$$

by Proposition 3.19 we have

$$\partial^w g(\bar{x}) \subset \partial^w f(\bar{x}).$$

Similarly it follows from

$$(\forall x \in X) \qquad (g-f)(x) \geq (g-f)(\bar{x})$$

that

$$\partial^w f(\bar{x}) \subset \partial^w g(\bar{x}).$$

This completes the proof. □

Let Y be a real normed space and Y^* denote the topological dual space of Y. For any $y^* \in Y^*$, we consider the scalar function $\langle y^*, h\rangle$ is defined by the equality

$$(\forall u \in X) \qquad \langle y^*, h\rangle(u) = \langle y^*, h(u)\rangle,$$

where $h : X \to Y$ is a function and X is a real normed space.

Let $g : Y \to \mathbb{R}$ be a function and $\bar{y} = h(\bar{x})$. In the next result we will concentrate on the composition $f(u) = g(h(u))$, $u \in X$, and the projection operator $\pi : X^* \times \mathbb{R} \to X^*$, such that $\pi(x^*, t) = x^*$ for all $(x^*, t) \in X^* \times \mathbb{R}$.

Proposition 3.23. *Assume that g is weak subdifferentiable at $\bar{y}$ and $\langle y^*, h\rangle$ is weak subdifferentiable at $\bar{x}$ for some $y^* \in \pi(\partial^w g(\bar{y}))$. If h is locally Lipschitz at $\bar{x}$ with the constant Lipschitz L, then f is weak subdifferentiable at $\bar{x}$ and*

$$\pi(\partial^w \langle y^*, h\rangle(\bar{x})) \subset \pi(\partial^w f(\bar{x})).$$

Proof. If $w \in \pi(\partial^w \langle y^*, h\rangle(\bar{x}))$ then there exists a nonnegative number c such that

$$(\forall x \in X) \quad \langle y^*, h\rangle(x) - \langle y^*, h\rangle(\bar{x}) \geq \langle w, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

Since $y^* \in \pi(\partial^w g(\bar{y}))$ then there exists $\bar{c} \geq 0$ such that

$$(\forall y \in Y) \quad g(y) - g(\bar{y}) \geq \langle y^*, y - \bar{y}\rangle - \bar{c}\|y - \bar{y}\|,$$

and so

$$(\forall x \in X) \quad g(h(x)) - g(h(\bar{x})) \geq \langle y^*, h(x) - h(\bar{x})\rangle - \bar{c}\|h(x) - h(\bar{x})\|.$$

This means that

$$\begin{aligned} f(x) - f(\bar{x}) &\geq \langle y^*, h(x) - h(\bar{x})\rangle - \bar{c}\|h(x) - h(\bar{x})\| \\ &\geq \langle w, x - \bar{x}\rangle - c\|x - \bar{x}\| - \bar{c}L\|x - \bar{x}\| \\ &= \langle w, x - \bar{x}\rangle - (c + \bar{c}L)\|x - \bar{x}\|, \end{aligned}$$

then $(w, c + \bar{c}L) \in \partial^w f(\bar{x})$. Consequently, $w \in \pi(\partial^w f(\bar{x}))$. This completes the proof. □

It is worth noting that the conclusion of Proposition 3.23 can be rewritten in the following form:

$$\bigcup\Big\{\pi(\partial^w\langle y^*,h\rangle(\bar{x})) : y^*\in\partial^* g(\bar{y})\Big\}\subset\pi(\partial^w f(\bar{x})).$$

Proposition 3.24. *If f and $-g$ is weak subdifferentiable, respectively, at $\bar{x}$ and $\bar{y}$. If h is Lipschitz function with the constant Lipschitz L, then for any $y^*\in\pi(\partial^w(-g(\bar{y})))$ the function $\langle y^*,h\rangle$ is weak subdifferential at $\bar{x}$ and*

$$\pi(\partial^w f(\bar{x}))\subset\pi(\partial^w\langle -y^*,h\rangle(\bar{x})).$$

Proof. If $x^*\in\pi(\partial^w f(\bar{x}))$, then there exists a nonnegative number c such that

$$(\forall x\in X)\qquad f(x)-f(\bar{x})\geq\langle x^*,x-\bar{x}\rangle-c\|x-\bar{x}\|.$$

Also if $(y^*,\bar{c})\in\partial^w(-g)(\bar{y})$, then we have

$$(\forall y\in Y)\qquad -g(y)+g(\bar{y})\geq\langle y^*,y-\bar{y}\rangle-\bar{c}\|y-\bar{y}\|.$$

Consequently,

$$(\forall y\in Y)\qquad -\langle y^*,h\rangle(x)+\langle y^*,h\rangle(\bar{x})\ \geq\ g(y)-g(\bar{y})-\bar{c}\|y-\bar{y}\|.$$

Therefore,

$$\begin{aligned}-\langle y^*,h\rangle(x)+\langle y^*,h\rangle(\bar{x}) &\geq f(x)-f(\bar{x})-\bar{c}\|h(x)-h(\bar{x})\|\\ &\geq \langle x^*,x-\bar{x}\rangle-c\|x-\bar{x}\|-\bar{c}\|h(x)-h(\bar{x})\|.\end{aligned}$$

Thus,

$$\begin{aligned}-\langle y^*,h\rangle(x)+\langle y^*,h\rangle(\bar{x}) &\geq \langle x^*,x-\bar{x}\rangle-c\|x-\bar{x}\|-\bar{c}L\|x-\bar{x}\|\\ &= \langle x^*,x-\bar{x}\rangle-(c+\bar{c}L)\|x-\bar{x}\|.\end{aligned}$$

This means that $(x^*,c+\bar{c}L)\in\partial^w(\langle -y^*,h\rangle(\bar{x}))$. Hence

$$x^*\in\pi(\partial^w(\langle -y^*,h\rangle(\bar{x})).$$

This completes the proof. □

By combining Propositions 3.23 and 3.24 we obtain the following result.

Corollary 3.25. *Let f be weak subdifferentiable at $\bar{x}$ and g be Fréchet differentiable at $\bar{y}$, and g , $-g$ is subdifferentiable at $\bar{y}$. If h is locally Lipschitz function with the constant Lipschitz L at $\bar{x}$, then*

$$\pi(\partial f(\bar{x}))=\pi(\partial^w\langle g'(\bar{y}),h\rangle(\bar{x})).$$

In the following we present some examples.

Example 3.26. *The example*

$$f(x)=\begin{cases}1, & x\in\mathbb{Q}^c,\\ 0, & x\in\mathbb{Q},\end{cases}\qquad g(x)=\begin{cases}0, & x\in\mathbb{Q}^c,\\ 1, & x\in\mathbb{Q},\end{cases}$$

shows that the weak subdifferentability of $f\circ g$ at $\bar{x}$ may not imply the weak subdifferentability of f and g at $\bar{x}$.

The next example shows that the composition of two weak subdifferentiable functions is not necessarily weak subdifferentiable.

Example 3.27. *Take $f(x) = x^2$ and $g(x) = -x$. Then f, g are weak subdifferentiable at $\bar{x} = 0$, but $(g \circ f)(x) = -x^2$ is not weak subdifferentiable at $\bar{x} = 0$.*

The next example shows that the product of two weak subdifferentiable functions is not necessarily weak subdifferentiable.

Example 3.28. *Let $f(x) = x$, $g(x) = -x$. Then f, g are weak subdifferentiable at $\bar{x} = 0$ while $(fg)(x) = -x^2$ is not weak subdifferentiable at $\bar{x} = 0$.*

The next example shows that the weak subdifferentability of fg at $\bar{x}$ may not imply the weak subdifferentability of f and g at $\bar{x}$.

Example 3.29. *Consider*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}^c, \\ 0, & x \in \mathbb{Q}, \end{cases} \qquad and \qquad g(x) = \begin{cases} 0, & x \in \mathbb{Q}^c, \\ 1, & x \in \mathbb{Q}. \end{cases}$$

Then g is not weak subdifferentiable at $x = 0$ while $(fg)(x) = 0$ is weak subdifferentiable at each point of the real number.

Proposition 3.30. *If all f_i, $i \in I$ (I is a finite nonempty set) and $f(u) = \sup_{i \in I} f_i(u)$, $u \in X$, are finite at $\bar{x}$, then the closure of the convex hull of the set $\bigcup_{i \in I_0(\bar{x})} \partial^w f_i(\bar{x})$ is a subset of $\partial^w f(\bar{x})$, i.e.,*

$$\mathrm{cl}\left(\mathrm{co}\left(\bigcup_{i \in I_0(\bar{x})} \partial^w f_i(\bar{x})\right)\right) \subset \partial^w f(\bar{x}),$$

where $I_0(\bar{x}) = \left\{i \in I \,:\, f_i(\bar{x}) = f(\bar{x})\right\}$.

Proof. Suppose that

$$\sum_{i \in I_0(\bar{x})} \alpha_i (x_i^*, c_i) \in \mathrm{co} \bigcup_{i \in I_0(\bar{x})} \partial^w f_i(\bar{x}),$$

such that $\sum_{i \in I_0(\bar{x})} \alpha_i = 1$, $\alpha_i \geq 0$, $(x_i^*, c_i) \in \partial^w f_i(\bar{x})$. Then we have

$$\left(\forall x \in X,\ \forall i \in I_0(\bar{x})\right) \qquad f_i(x) - f_i(\bar{x}) \geq \langle x_i{}^*, x - \bar{x} \rangle - c_i \|x - \bar{x}\|.$$

Therefore,

$$(\forall x \in X) \qquad \sum_{i \in I_0(\bar{x})} \alpha_i f_i(x) - \sum_{i \in I_0(\bar{x})} \alpha_i f_i(\bar{x}) \geq \sum_{i \in I_0(\bar{x})} \alpha_i \langle x_i{}^*, x - \bar{x} \rangle - \sum_{i \in I_0(\bar{x})} \alpha_i c_i \|x - \bar{x}\|.$$

Since $f(x) = \sup_{i \in I} f_i(x)$, $x \in X$, we have $I_0(\bar{x}) = \left\{i \in I \,:\, f_i(\bar{x}) = f(\bar{x})\right\}$, so that

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \left\langle \sum_{i \in I_0(\bar{x})} \alpha_i x_i{}^*, x - \bar{x} \right\rangle - \sum_{i \in I_0(\bar{x})} \alpha_i c_i \|x - \bar{x}\|$$

and

$$\sum_{i \in I_0(\bar{x})} \alpha_i (x_i{}^*, c_i) \in \partial^w f(\bar{x}).$$

Consequently,

$$\mathrm{co}\left(\bigcup_{i\in I_0(x)} \partial^w f_i(\bar{x})\right) \subset \partial^w f(\bar{x}).$$

Now the clossedness of the set $\partial^w f(\bar{x})$ completes the proof. □

The next proposition states necessary conditions that with them a weakly subdifferentiable function obtains a global maximum.

Proposition 3.31. *Let f at $\bar{x}$ attain a global maximum. If $(x^*, c) \in \partial^w f(\bar{x})$, then $\|x^*\| \le c$.*

Proof. Since f has a global maximum at $\bar{x}$, then we have

$$(\forall x \in X) \qquad f(x) \le f(\bar{x}).$$

It follows from $(x^*, c) \in \partial^w f(\bar{x})$, that

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \ge \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

Hence

$$(\forall x \in X) \qquad 0 \ge \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|,$$

Consequently

$$(\forall x \in X) \qquad \langle x^*, x - \bar{x}\rangle \le c\|x - \bar{x}\|,$$

and so

$$\|x^*\| \le c.$$

This completes the proof. □

Recall that in [11, 12, 14, 15], the well-known theorem about the representation of the directional derivative of the convex functions as a point wise maximum of subgradients of that function is generalized to a nonconvex case by using the notion of a subgradient. They worked on special class of invex functions that includes the class of convex functions. It should be noted that the results given in [12] is a generalization of the results presented in [11] for a special class of invex functions .The optimality condition formulated in [12], guarantees the existence of the weak subgradient, that is the pair consisting of some linear functional and some real number such that the graph of the homogeneous function defined by this paper, is a conical surface which separates the optimal point from the given (non convex) set. In the sequel we establish a new version of the main result of [12], for the Fréchet differentiable functions in the setting of infinite dimensional normed spaces.

Proposition 3.32. *If f is subdifferentiable and Fréchet differentiable at $\bar{x}$, then f has a global minimum at $\bar{x}$ if and only if*

$$(\forall x \in X) \qquad \langle f'(\bar{x}), x - \bar{x}\rangle = 0.$$

Proof. Suppose that f has a global minimum at $\bar{x}$, then we have

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \ge 0.$$

From the Fréchet differentiability of f at $\bar{x}$, we get

$$\lim_{\|h\|\to 0} \frac{|f(\bar{x} + h) - f(\bar{x}) - \langle f'(\bar{x}), h\rangle|}{\|h\|} = 0.$$

If we take

$$h = \lambda(x - \bar{x}),$$

then we obtain

$$0 = \lim_{\lambda \to 0^+} \frac{f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x}) - \langle f'(\bar{x}), \lambda(x - \bar{x}) \rangle}{\|\lambda(x - \bar{x})\|}$$

and so, since $\bar{x}$ is a global minimum of f, we have

$$0 \geq \lim_{\lambda \to 0^+} \frac{-\langle f'(\bar{x}), \lambda(x - \bar{x}) \rangle}{\|\lambda(x - \bar{x})\|}.$$

Consequently, by the linearity of $f'(\bar{x})$, we can deduce that

$$(\forall x \in X) \qquad \langle f'(\bar{x}), x - \bar{x} \rangle = 0.$$

Conversely, by using our assumptions, we have

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle f'(\bar{x}), x - \bar{x} \rangle = 0.$$

Then

$$(\forall x \in X) \qquad f(x) \geq f(\bar{x})$$

and this shows that $\bar{x}$ is a global minimum of f. Hence the proof is completed. □

Proposition 3.33. *If f is subdifferentiable and Fréchet differentiable at $\bar{x}$, then f is weakly subdifferentiable at $\bar{x}$ if and only if $f'(\bar{x})$ is weakly subdifferentiable at 0, the zero element of X, and*

$$\partial^w(f(\bar{x})) = \partial^w(f'(\bar{x}))(0).$$

Proof. From the Fréchet differentiability f at $\bar{x}$, we have

$$\lim_{\|h\| \to 0} \frac{|f(\bar{x} + h) - f(\bar{x}) - \langle f'(\bar{x}), h \rangle|}{\|h\|} = 0.$$

By taking

$$h = \lambda(x - \bar{x})$$

and by using the weak subdifferentiability of f at $\bar{x}$, there exist $(x^*, c) \in \partial^w f(\bar{x})$, such that

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \ \geq \ \langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\|.$$

Hence

$$0 = \lim_{\lambda \to 0^+} \frac{f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x}) - \langle f'(\bar{x}), \lambda(x - \bar{x}) \rangle}{\|\lambda(x - \bar{x})\|}$$

and from the weak subdifferentiability of f at $\bar{x}$ we get

$$(\forall x \in X) \qquad 0 \geq \lim_{\lambda \to 0^+} \frac{\langle x^*, \lambda(x - \bar{x}) \rangle - c\|\lambda(x - \bar{x})\| - \langle f'(\bar{x}), \lambda(x - \bar{x}) \rangle}{\|\lambda(x - \bar{x})\|}$$

and equally

$$(\forall x \in X) \qquad \frac{\langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\| - \langle f'(\bar{x}), x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0.$$

Therefore,

$$(\forall x \in X) \qquad \langle x^*, x-\bar{x}\rangle - c\|x-\bar{x}\| - \langle f'(\bar{x}), x-\bar{x}\rangle \leq 0$$

and so by taking $z = x - \bar{x}$, we obtain

$$(\forall z \in X) \qquad \langle f'(\bar{x}), z\rangle \geq \langle x^*, z\rangle - c\|z\|.$$

Now, it follows from $f'(\bar{x})(0) = 0$, that $(x^*, c) \in \partial^w(f'(\bar{x}))(0)$. Conversely, if $(x^*, c) \in \partial^w(f'(\bar{x}))(0)$, then we can write

$$(\forall x \in X) \qquad \langle f'(\bar{x}), x\rangle \geq \langle x^*, x\rangle - c\|x\|.$$

Hence

$$(\forall x \in X) \qquad \langle f'(\bar{x}), x-\bar{x}\rangle \geq \langle x^*, x-\bar{x}\rangle - c\|x-\bar{x}\|,$$

and by applying the subdifferentiability and Fréchet differentiability f at $\bar{x}$, we get

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle f'(\bar{x}), x-\bar{x}\rangle.$$

Then

$$(\forall x \in X) \qquad f(x) - f(\bar{x}) \geq \langle x^*, x-\bar{x}\rangle - c\|x-\bar{x}\|.$$

This means that $(x^*, c) \in \partial^w(f(\bar{x}))$ and proof is completed. □

Proposition 3.34. *If f is subdifferentiable and Fréchet differentiable at $\bar{x}$, then*

$$\langle f'(\bar{x}), x-\bar{x}\rangle = \sup\Big\{\langle x^*, x-\bar{x}\rangle - c\|x-\bar{x}\| \,:\, (x^*, c) \in \partial^w f(\bar{x})\Big\}.$$

Proof. From the hypothesis and by using a similar proof as in Proposition 3.33, we deduce that

$$\langle f'(\bar{x}), x-\bar{x}\rangle \geq \sup\Big\{\langle x^*, x-\bar{x}\rangle - c\|x-\bar{x}\| \,:\, (x^*, c) \in \partial^w f(\bar{x})\Big\}.$$

Since $(f'(\bar{x}), 0) \in \partial^w f(\bar{x})$, then

$$\langle f'(\bar{x}), x-\bar{x}\rangle \in \Big\{\langle x^*, x-\bar{x}\rangle - c\|x-\bar{x}\| \,:\, (x^*, c) \in \partial^w f(\bar{x})\Big\},$$

and the desired equality is obtained. □

Corollary 3.35. *We note that under above assumptions, if f attains a global minimum at $\bar{x}$, then*

$$\sup\Big\{\{\langle x^*, x-\bar{x}\rangle - c\|x-\bar{x}\| \,:\, (x^*, c) \in \partial^w f(\bar{x})\Big\} = 0.$$

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