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On the Local Uniqueness of the Fixed Point of the Probabilistic q-Contraction in Fuzzy Metric Spaces

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Abstract. In this paper we prove the local uniqueness of the fixed point of the probabilistic q-contraction in fuzzy metric space.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

The first fixed point theorem in probabilistic metric spaces was proved by Sehgal and Bharucha-Reid [21] for mappings $f : S \to S$, on Menger space $(S, \mathcal{F}, T_{\mathbf{M}})$, where $T_{\mathbf{M}} = \min$. The real operation of triangular norms was introduced in the theory of probabilistic metric spaces by K. Menger [15], see [7–9, 11, 20, 22]. It turns out that t-norms are crucial operations in several fields, e.g., in statistics by copulas ([13, 14]), fuzzy sets, fuzzy logics (see [11]) and their applications, but also, among other fields, in the theory of generalized measures [11, 17, 23] and in nonlinear differential and difference equations [17]. Further investigations of the fixed point theory in a more general Menger space $(S, \mathcal{F}, T)$ was connected with investigations of the structure of the t-norm T, see [1, 4, 7]. Further development of the fixed point theory was obtained in a more general space - fuzzy metric spaces, see [2, 3, 6, 7, 16, 24].

We present in this paper a result on the local uniqueness of fixed point in fuzzy metric space. In Section 2 we give some results related t-norms. In Section 3 we give the definition of fuzzy metric space and Section 4 is devoted to the main result of the paper, the local uniqueness of the fixed point of the probabilistic q-contraction in fuzzy metric space.

2. Triangular Norms

A triangular norm (t-norm for short) is a binary operation on the unit interval $[0, 1]$, i.e., a function $T : [0, 1]^2 \to [0, 1]$ which is commutative, associative, monotone and $T(x, 1) = x$ for every $x \in [0, 1]$. A method of construction a new t-norm from a system of given t-norms is given in the following theorem, see [7, 11].

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Theorem 1 *Let $(T_k)_{k\in K}$ be a family of t-norms and let $(]\alpha_k, \beta_k[)_{k\in K}$ be a family of pairwise disjoint open subintervals of the unit interval $[0,1]$ (i.e., K is an at most countable index set). Consider the linear transformations $\varphi_k : [\alpha_k, \beta_k] \to [0,1], k \in K$, given by*

$$\varphi_k(u) = \frac{u - \alpha_k}{\beta_k - \alpha_k}.$$

Then the function $T : [0,1]^2 \to [0,1]$ defined by

$$T(x,y) = \begin{cases} \varphi_k^{-1}(T_k(\varphi_k(x), \varphi_k(y))) & \text{if } (x,y) \in \,]\alpha_k, \beta_k[^2 \,, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

is a triangular norm, which is called the ordinal sum of $(T_k)_{k\in K}$.

An arbitrary t-norm T can be extended (by associativity) in a unique way to an n-ary operation taking for $(x_1, \dots, x_n) \in [0,1]^n$, $n \in \mathbb{N}$, the values $T(x_1, \dots, x_n)$ which is defined by

$$\mathop{\top}_{i=1}^{0} x_i = 1, \quad \mathop{\top}_{i=1}^{n} x_i = T\Big(\mathop{\top}_{i=1}^{n-1} x_i, x_n\Big) = T(x_1, \dots, x_n).$$

Specially, we have $T_{\mathbf{L}}(x_1, \dots, x_n) = \max\left(\sum_{i=1}^{n} x_i - (n-1), 0\right)$ and $T_{\mathbf{M}}(x_1, \dots, x_n) = \min(x_1, \dots, x_n)$.

We can extend T to a countable infinitary operation taking for any sequence $(x_n)_{n\in\mathbb{N}}$ from $[0,1]$ the values

$$\mathop{\top}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathop{\top}_{i=1}^{n} x_i. \tag{1}$$

The limit on the right side of (1) exists since the sequence $\left(\mathop{\top}_{i=1}^{n} x_i\right)_{n\in\mathbb{N}}$ is non-increasing and bounded from below.

In the fixed point theory it is of interest to investigate the classes of t-norms T and sequences $(x_n)_{n\in\mathbb{N}}$ from the interval $[0,1]$ such that $\lim_{n\to\infty} x_n = 1$, and

$$\lim_{n\to\infty} \mathop{\top}_{i=n}^{\infty} x_i = \lim_{n\to\infty} \mathop{\top}_{i=1}^{\infty} x_{n+i} = 1.$$

In the classical case $T = T_{\mathbf{P}}$ we have $\left(T_{\mathbf{P}}\right)_{i=1}^{n} = \prod_{i=1}^{n} x_i$ and for every sequence $(x_n)_{n\in\mathbb{N}}$ from the interval $[0,1]$ with $\sum_{i=1}^{\infty}(1 - x_n) < \infty$ it follows that

$$\lim_{n\to\infty} \left(T_{\mathbf{P}}\right)_{i=n}^{\infty} = \lim_{n\to\infty} \prod_{i=n}^{\infty} x_i = 1.$$

The equivalence

$$\sum_{i=1}^{\infty}(1 - x_i) < \infty \quad \Leftrightarrow \quad \lim_{n\to\infty} \mathop{\top}_{i=n}^{\infty} x_i = 1$$

holds also for $T \geqslant T_{\mathbf{L}}$.

In the paper [4] the condition

$$\lim_{n\to\infty} \mathop{\top}_{i=n}^{\infty} x_i = 1$$

is investigated for some classes of t-norms T and sequences $(x_i)_{i\in\mathbb{N}}$ from $[0,1]$.

3. Fuzzy Metric Spaces

By [12] we have the following definition.

Definition 2 *A fuzzy metric space in the sense of Kramosil and Michálek is a triple* (X, M, T), *where* X *is a nonempty set,* T *is a t-norm and* M *is a fuzzy set on* $X^2 \times [0, \infty[$ *satisfying the following conditions for all* $x, y, z \in X$ *and* $s, t > 0$

(FM-1) $M(x, y, 0) = 0$;

(FM-2) $M(x, y, t) = 1$, *for all* $t > 0$ *if and only if* $x = y$;

(FM-3) $M(x, y, t) = M(y, x, t)$;

(FM-4) $M(x, z, t + s) \geqslant T(M(x, y, t), M(y, z, s))$;

(FM-5) $M(x, y, \cdot) : \mathbb{R}^+ \to [0, 1]$ *is left continuous.*

We additionally suppose that $M(x, y, t) > 0$ for $t > 0$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in X is a Cauchy sequence if for every $\varepsilon > 0$ and $\lambda \in]0, 1[$ there exists $n_0(\varepsilon, \lambda) \in \mathbb{N}$ such that $M(x_n, x_m, \varepsilon) > 1 - \lambda$, for every $n, m \geqslant n_0(\varepsilon, \lambda)$. A fuzzy metric space is complete if every Cauchy sequence converges.

4. A Fixed Point Theorem in Fuzzy Metric Spaces

It is well known that the uniqueness of a fixed point of probabilistic q-contraction does not follow immediately, as in the case of a Menger space, since $\lim_{t\to\infty} M(x, y, t) = 1$ does not hold generally. One of the solution of this problem is to assume that on (X, M, T) the following condition holds

$$M(x, y, t) \equiv C, \text{ for every } t > 0 \text{ implies } C = 1. \tag{2}$$

In this paper we shall prove that a kind of the local uniqueness can be obtained without condition (2).

Let $\mathrm{Fix}(f)$ denote the set of fixed points of a function $f : X \to X$.

Definition 3 *Let* (X, M, T) *be a fuzzy metric space. A mapping* $f : X \to X$ *is a probabilistic* q*-contraction* ($q \in]0, 1[$) *if*

$$M(fp_1, fp_2, x) \geqslant M\Big(p_1, p_2, \frac{x}{q}\Big)$$

for every $p_1, p_2 \in X$ *and every* $x \in \mathbb{R}^+$.

Theorem 4 *Let* (X, M, T) *be a complete fuzzy metric space,* T *acontinuous t-norm at the point* $(1, 1)$, $f : X \to X$ *a probabilistic* q*-contraction and there exists* $x_0 \in X$ *such that*

$$\lim_{n\to\infty} \mathop{\boldsymbol{T}}_{i=n}^{\infty} M\Big(x_0, fx_0, \frac{1}{q^i}\Big) = 1. \tag{3}$$

If $x = \lim_{n\to\infty} f^n x_0$ *and*

$$A = \{y \mid y \in X, \lim_{t\to\infty} M(x_0, y, t) = 1\},$$

then $A \cap \mathrm{Fix}(f) = \{x\}$.

Proof. Condition (3) implies the existence of $\lim_{n\to\infty} f^n x_0$, as in the case of Menger spaces, and the continuity of f implies that $x \in \mathrm{Fix}(f)$, see [7].

Firstly, we shall prove that $x \in A$, i.e., that

$$\lim_{t\to\infty} M(x_0, x, t) = 1. \tag{4}$$

In order to prove (4) we shall prove that for every $\lambda \in]0,1[$ there exists $t' > 0$ such that $M(x_0, x, t') > 1 - \lambda$. Let $n, m \in \mathbb{N}$. Then

$$\begin{aligned} M\Big(x_0, fx_0, \frac{1}{q^n}\Big) &= T\Big(1, M\Big(x_0, fx_0, \frac{1}{q^n}\Big)\Big) \\ &= \underbrace{T\Big(T\Big(\cdots\Big(T}_{(\mathrm{m})-\mathrm{times}}\Big(T\Big(1, M\Big(x_0, fx_0, \frac{1}{q^n}\Big)\Big)\ldots\Big) \\ &\geqslant \mathop{\mathsf{T}}_{i=n}^{\infty} M\Big(x_0, fx_0, \frac{1}{q^i}\Big). \end{aligned}$$

Therefore by (3) we obtain

$$\lim_{n\to\infty} M(x_0, fx_0, \frac{1}{q^n}) = 1.$$

Since $M(x_0, fx_0, \cdot)$ is nondecreasing we obtain that

$$\lim_{t\to\infty} M(x_0, fx_0, t) = 1. \tag{5}$$

Since for every $m \in \mathbb{N}$ and $t > 0$ we have

$$\begin{aligned} M\Big(f^m x_0, f^{m+1} x_0, t\Big) &\geqslant M\Big(f^{m-1} x_0, f^m x_0, \frac{t}{q}\Big) \\ &\geqslant \cdots \\ &\geqslant M\Big(x_0, fx_0, \frac{t}{q^m}\Big), \end{aligned}$$

(5) implies that for every fixed $m \in \mathbb{N}$ we obtain

$$\lim_{t\to\infty} M(f^m x_0, f^{m+1} x_0, t) = 1. \tag{6}$$

Let n be an arbitrary but fixed natural number. Then for every $t > 0$ we have

$$\begin{aligned} M(x_0, f^n x_0, t) &\geqslant T\Big(M\Big(x_0, f^{n-1} x_0, \frac{t}{2}\Big), M\Big(f^{n-1} x_0, f^n x_0, \frac{t}{2}\Big)\Big) \\ &\geqslant \cdots \\ &\geqslant \underbrace{T\Big(T\Big(\cdots\Big(T}_{(\mathrm{n}-1)-\mathrm{times}}\Big(M(x_0, fx_0, \frac{t}{2^{n-1}}\Big), M(fx_0, f^2 x_0, \frac{t}{2^{n-1}}\Big)\Big), \ldots, M\Big(f^{n-1} x_0, f^n x_0, \frac{t}{2}\Big)\Big). \end{aligned}$$

Since the t-norm T is continuous at the point $(1,1)$ then (6) implies that

$$\lim_{t\to\infty} M(x_0, f^n x_0, t) = 1 \tag{7}$$

for a fixed $n \in \mathbb{N}$. Let $\lambda \in]0,1[$, $t > 0$, and $\delta(\lambda) \in]0,1[$ such that

$$T(1-\delta, 1-\delta) > 1-\lambda.$$

Since $\lim_{n\to\infty} f^n x_0 = x$ there exists $n_0(t,\delta) \in \mathbb{N}$ such that

$$M\Big(x, f^{n_0}x_0, \frac{t}{2}\Big) > 1-\delta.$$

By (7) we obtain that there exists $t(\delta) > 0$ such that

$$M\Big(x_0, f^{n_0}x_0, \frac{t(\delta)}{2}\Big) > 1-\delta.$$

Let $t' = \max\{t, t(\delta)\}$. Then we obtain

$$\begin{aligned} M(x,x_0,t') &\geqslant T\Big(M\Big(x, f^{n_0}x_0, \frac{t}{2}\Big), M\Big(f^{n_0}x_0, x_0, \frac{t(\delta)}{2}\Big)\Big) \\ &> T(1-\delta, 1-\delta) \\ &> 1-\lambda. \end{aligned}$$

Therefore $x \in A \cap \mathrm{Fix}(f)$.

If $y \in A \cap \mathrm{Fix} f$ then $y = fy$ and $\lim_{t\to\infty} M(x_0, y, t) = 1$. Then

$$\begin{aligned} M(x,y,t) &= M(fx, fy, t) \\ &\geqslant M\Big(x, y, \frac{t}{q}\Big) \\ &\dots \\ &\geqslant M\Big(x, y, \frac{t}{q^n}\Big) \\ &\geqslant T\Big(M\Big(x, y, \frac{t}{2q^n}\Big), M\Big(x, y, \frac{t}{2q^n}\Big)\Big). \end{aligned}$$

Therefore

$$M(x,y,t) \geqslant T\Big(\lim_{n\to\infty} M\Big(x, y, \frac{t}{2q^n}\Big), \lim_{n\to\infty} M\Big(x, y, \frac{t}{2q^n}\Big)\Big) = T(1,1) = 1.$$

Hence $x = y$ and so $A \cap \mathrm{Fix}(f) = \{x\}$. □

Remark 5 *For a class of φ-probabilistic contraction and t-norm of H-type Fang [2] proved a similar result (Theorem 4.1) about the local uniqueness of the fixed point in fuzzy metric spaces.*

References

[1] J. Aczél, Lectures on Functional Equations and their Applications, Academic Press, New York, 1969.
[2] J. X. Fang., On φ-contractions in probabilistic and fuzzy metric spaces, Fuzzy Sets and Syst. 267 (2015) 86–99.
[3] A. George, P. Veeramani, On some results in fuzzy metric spaces Fuzzy Sets Syst., 64 (1994), 395-399.
[4] O. Hadžić, E. Pap, M. Budinčević, On some classes of t-norms important in the fixed point theory, Kybernetika 38,3 (2002) 363–382
[5] O. Hadžić, E. Pap, On some classes of t-norms important in the fixed point theory, Bull. Acad. Serbe Sci. Art. Sci. Math. 121, 25 (2000) 15–28.
[6] O. Hadžić, E. Pap, A fixed point theorem for multivalued mappings in probabilistic metric spaces and an application in fuzzy metric spaces, Fuzzy Sets and Syst.. 127, 3 (2002) 333–344.
[7] O. Hadžić, E. Pap, Fixed Point Theory in Probabilistic Metric Spaces, Kluwer Academic Publishers, Dordrecht, 2001.
[8] O. Hadžić, E. Pap, Probabilistic multi-valued contractions and decomposable measures, Internat. J. Uncertainty, Fuzziness, Knowledge-Based Systems 10, Supplement (2002) 59–74.

[9] O. Hadžić, E. Pap, V. Radu, Some generalized contraction mapping principles in probabilistic metric spaces, Acta Math. Hungarica 101 (1-2) (2003) 111–128.
[10] O. Kaleva, S. Seikalla, On fuzzy metric spaces, Fuzzy Sets and Syst. 12 (1984) 215–229.
[11] E. P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Trends in Logic 8, Dordrecht, 2000.
[12] I. Kramosil, J. Michálek, Fuzzy metrics and statistical metric spaces Kybernetika, 11 (1975) 336–344
[13] E. P. Klement, R. Mesiar, E. Pap, Uniform approximation of associative copulas by strict and non-strict copulas, Illinois J. Math. 45, No. 4 (2001), 1393–1400.
[14] E. P. Klement, R. Mesiar, E. Pap, Archimax copulas and invariance under transformations, C. R. Math. Acad. Sci. Paris-Mathematics 340 (2005) 755–758.
[15] K. Menger (1942). Statistical metric, *Proc. Nat. Acad. USA* 28 (142) 535–537.
[16] D. Miheț, A class of contractions in fuzzy metric spaces Fuzzy Sets Syst., 161 (2010) 1131–1137.
[17] E. Pap, Null-Additive Set Functions, Kluwer Academic Publishers, Dordrecht, 1995.
[18] E. Pap, O. Hadžić, R. Mesiar, A fixed point theorem in probabilistic metric spaces and applications in fuzzy set theory, J. Math. Anal. Appl. 202 (1996) 433–449.
[19] V. Radu, Lectures on probabilistic analysis, Surveys, Lectures Notes and Monographs Series on Probability, Statistics & Applied Mathematics, No 2, Universitatea de Vest din Timişoara, 1994.
[20] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, Elsevier North - Holland, New York, 1983.
[21] V. M. Sehgal, A. T. Bharucha-Reid, Fixed points of contraction mappings on probabilistic metric spaces, Math. Syst. Theory 6 (1972) 97–102.
[22] R. M. Tardiff , Contraction maps on probabilistic metric spaces, J. Math. Anal. Appl. 165 (1992) 517–523.
[23] S. Weber, $\perp$-decomposable measures and integrals for Archimedean t-conorm $\perp$, J. Math. Anal. Appl. 101 (1984) 114–138.
[24] J.Z. Xiao, X.H. Zhu, X. Jin, Fixed point theorems for nonlinear contractions in Kaleva-Seikkala's type fuzzy metric spaces, Fuzzy Sets Syst. 200 (2012) 65–83.