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On Some Ćirić Type Results in Partial b-Metric Spaces

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Abstract. In this paper, some fixed point results of Ćirić type in partial b-metric spaces are complemented and generalized. An example is given to support the obtained results.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Intoductions and Preliminars

Partial metric spaces [29] and b-metric spaces [8] are two well known generalizations of usual metric spaces. Also, the Banach contraction principle is a fundamental result in the fixed point theory, which has been used and extended in many different directions.

The following two definitions are consistent with Matthews [29] and [8].

Definition 1.1. *A partial metric on a nonempty set X is a function $p : X \times X \to [0, \infty)$ such that for all $u, v, w \in X$, if*

p1) $u = v$ if and only if $p(u,u) = p(u,v) = p(v,v)$,
p2) $p(u,u) \le p(u,v)$,
p3) $p(u,v) = p(v,u)$,
p4) $p(u,w) \le p(u,v) + p(v,w) - p(v,v)$.
A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.
For a partial metric p on X, the function $d^p : X \times X \to [0, \infty)$ given by

$$d^p(u,v) = 2p(u,v) - p(u,u) - p(v,v) \tag{1}$$

is a (usual) metric on X. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\left\{B_p(u,\varepsilon) : u \in X, \varepsilon > 0\right\}$, where $B_p(u,\varepsilon) = \{v \in X : p(u,v) < p(u,u) + \varepsilon\}$ for all $u \in X$ and $\varepsilon > 0$.

For more details on partial metric spaces see [7], [29], [31], [32], [34], [39], [47] and [48].

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Definition 1.2. *[8] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $b : X \times X \to [0, \infty)$ is called a b-metric on X if, for all $u, v, w \in X$, the following conditions hold:*
b1) $b(u, v) = 0$ if and only if $u = v$,
b2) $b(u, v) = b(v, u)$,
b3) $b(u, w) \leq s[b(u, v) + b(v, w)]$.
In this case, the pair (X, b) is called a b-metric space.

As a generalization and unification of partial metric and b-metric spaces, Shukla [47] introduced the concept of partial b-metric space as follows:

Definition 1.3. *A partial b-metric on a nonempty set X is a mapping $p_b : X \times X \to [0, \infty)$ such that for all $u, v, w \in X$, if*
pb1) $u = v$ if and only if $p_b(u, u) = p_b(u, v) = p_b(v, v)$,
pb2) $p_b(u, u) \leq p_b(u, v)$,
pb3) $p_b(u, v) = p_b(v, u)$,
pb4) $p_b(u, w) \leq s[p_b(u, v) + p_b(v, w)] - p_b(v, v)$.
A partial b-metric space is a pair (X, p_b) such that X is a nonempty set and p_b is a partial b-metric on X. The number $s \geq 1$ is called the coefficient of (X, p_b).

In a partial b-metric space (X, p_b), if $u, v \in X$ and $p_b(u, v) = 0$, then $u = v$, but the converse may not be true. It is clear that every partial metric space is a partial b-metric space with the coefficient $s \geq 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of these facts does not necessarily hold. For such examples see [32], [47] and [48].

In [32] authors say that (X, p_b) is a partial b-metric space if pb4) is substituted for the following: for all $u, v, w \in X$,

$$p_b(u, w) \leq s[p_b(u, v) + p_b(v, w) - p_b(v, v)] + \frac{1 - s}{2}(p_b(u, u) + p_b(v, v)). \tag{2}$$

Further, for all definitions of notions such as p_b-convergence, p_b-completeness, p_b-Cauchy sequence in the setting of partial b-metric spaces, the reader refers to [32], [47] and [48].

Definition 1.4. *A triple $(X, \preceq, p_b)$ is called an ordered partial b-metric space if $(X, \preceq)$ is a partially ordered set and p_b is a partial b-metric on X.*

Let $(X, \preceq)$ be a partially ordered set and let f, g be two self-maps on X. We shall use the following terminology [46]:
1) the elements $u, v \in X$ are called comparable if $u \preceq v$ or $v \preceq u$ holds;
2) a subset K of X is said to be well ordered if every two elements of K are comparable;
3) f is called nondecreasing w.r.t. $\preceq$ if $u \preceq v$ implies $fu \preceq fv$;
4) the pair (f, g) is said to be weakly increasing if $fu \preceq gfu$ and $gu \preceq fgu$ for all $u \in X$;
5) f is said to be g-weakly isotone increasing if for all $u \in X$ satisfies $fx \preceq gfu \preceq fgfu$.

Otherwise, fixed point results in partially ordered metric spaces were firstly presented by Ran and Reurings [44] and then by Nieto and López [35], [36]. Subsequently, many authors obtained several interesting results in ordered metric spaces, ordered b-metric spaces and ordered partial metric spaces (see [4], [5], [7], [22], [34], [37], [45], [46]). Otherwise, for more details from the general theory of fixed point consider [4]-[48].

Altering distance functions were introduced by Khan et al. in [28] as follows.

Definition 1.5. *A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following items are satisfied:*
***a)** ψ is continuous and nondecreasing;*
***b)** $\psi(t) = 0$ if and only if $t = 0$.*

So far, many authors have studied fixed point theorems which are based on altering distance functions (see, e.g., [7], [12], [16], [22], [38]).

In [32] authors introduced the following denotations and notions and proved the corresponding fixed point theorems.

Let $(X, \preceq, p_b, s > 1)$ be an ordered partial b-metric space, and let $f, g : X \to X$ be mappings. Set

$$M_s^{f,g}(u,v) = \max\left\{p_b(u,v), p_b(u,fu), p_b(v,gv), \frac{p_b(u,gv) + p_b(fu,v)}{2s}\right\} \tag{3}$$

and

$$M_s^{f}(u,v) = \max\left\{p_b(u,v), p_b(u,fu), p_b(v,fv), \frac{p_b(u,fv) + p_b(fu,v)}{2s}\right\}. \tag{4}$$

Definition 1.6. *Let $(X, p_b, s > 1)$ be a partial b-metric space, and let ψ and φ be altering distance functions. The pair (f, g) of self-mappings $f, g : X \to X$ is called a generalized $(\psi, \varphi)_{s,2}$-contraction pair if*

$$\psi\left(s^2 p_b(fu, gv)\right) \le \psi\left(M_s^{f,g}(u,v)\right) - \varphi\left(M_s^{f,g}(u,v)\right) \tag{5}$$

for all $u, v \in X$.

Theorem 1.7. *Let $(X, \preceq, p_b, s > 1)$ be an ordered p_b-complete partial b-metric space, and let $f, g : X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that (f, g) is a generalized $(\psi, \varphi)_{s,2}$-contraction pair for some altering distance functions ψ and φ. If f and g are continuous (resp. $(X, \preceq, p_b, s > 1)$ is a regular), then f and g have a common fixed point.*

Definition 1.8. *Let $(X, \preceq, p_b, s > 1)$ be an ordered partial b-metric space, and let ψ and φ be altering distance functions. A mapping $f : X \to X$ is called a generalized $(\psi, \varphi)_{s,1}$-weakly contractive mapping if*

$$\psi\left(s p_b(fu, fv)\right) \le \psi\left(M_s^{f}(u,v)\right) - \varphi\left(M_s^{f}(u,v)\right) \tag{6}$$

for all comparable $u, v \in X$.

Theorem 1.9. *Let $(X, \preceq, p_b, s > 1)$ be a p_b-complete ordered partial b-metric space. Let $f : X \to X$ be a nondecreasing, with respect to $\preceq$, continuous (resp. $(X, \preceq, p_b, s > 1)$ is a regular) mapping. Suppose that f is a generalized $(\psi, \varphi)_{s,1}$-weakly contractive mapping. If there exists $u_0 \in X$ such that $u_0 \preceq fu_0$, then f has a fixed point.*

It shows, specifically, the following crucial lemma is often used in proving all main results in [32].

Lemma 1.10. *Let $(X, p_b, s > 1)$ be a partial b-metric space and suppose that $\{a_n\}$ and $\{b_n\}$ are convergent to a and b, respectively. Then we have*

$$\begin{aligned}
&\frac{1}{s^2} p_b(a,b) - \frac{1}{s} p_b(a,a) - p_b(b,b) \\
&\le \liminf_{n\to\infty} p_b(a_n, b_n) \le \limsup_{n\to\infty} p_b(a_n, b_n) \\
&\le s p_b(a,a) + s^2 p_b(b,b) + s^2 p_b(a,b).
\end{aligned} \tag{7}$$

In particular, if $p_b(a,b) = 0$, then we have $\lim_{n\to\infty} p_b(a_n, b_n) = 0$.

Moreover, for each $c \in X$, we have

$$\frac{1}{s} p_b(a,c) - p_b(a,a) \le \liminf_{n\to\infty} p_b(a_n, c) \le \limsup_{n\to\infty} p_b(a_n, c) \le s p_b(a_n, c) + s p_b(a,a). \tag{8}$$

In particular, if $p_b(a,a) = 0$, then we have

$$\frac{1}{s} p_b(a,c) \le \liminf_{n\to\infty} p_b(a_n, c) \le \limsup_{n\to\infty} p_b(a_n, c) \le s p_b(a,c). \tag{9}$$

In the next section the following new result in the framework of b-metric spaces which also holds as well as for partial b-metric spaces, will be used.

Lemma 1.11. *([30] Lemma 2.2.) Every sequence $\{a_n\}$ of elements from a b-metric space $(X, d, s \geq 1)$, having the property that there exists $\gamma \in [0, 1)$ such that $d(a_{n+1}, a_n) \leq \gamma\, d(a_n, a_{n-1})$ for every $n \in \mathbb{N}$, is a b-Cauchy.*

2. Main Results

In what follows, two new concepts which generalize Definition 1.6 and Definition 1.8 greatly will be introduced. Further, very simple proofs of some common fixed point theorems in the new framework for not only without considering the assumptions of Theorem 1.7 and Theorem 1.9, but also without utilizing Lemma 1.10 in the proofs will be presented.

Definition 2.1. *Let $(X, \preceq, p_b, s > 1)$ be an ordered partial b-metric space, and let ψ and φ be altering distance functions. The pair (f, g) of self-mappings $f, g : X \to X$ is called a generalized $(\psi, \varphi)_{s,\varepsilon}$-contraction pair if*

$$\psi\left(s^{\varepsilon} p_b\left(fu, gv\right)\right) \leq \psi\left(M_s^{f,g}\left(u, v\right)\right) - \varphi\left(M_s^{f,g}\left(u, v\right)\right) \tag{10}$$

for all comparable $u, v \in X$, where $\varepsilon > 1$ is a real constant.

Remark 2.2. *Definition 1.6 is the special case of Definition 2.1. Indeed, take $\varepsilon = 2$ in Definition 2.1, then generalized $(\psi, \varphi)_{s,\varepsilon}$-contraction pair is reduced to generalized $(\psi,\ \varphi)_{s,2}$-contraction pair. Accordingly, Definition 2.1 is more useful and meaningful in applications.*

Theorem 2.3. *Let $(X, \preceq, p_b, s > 1)$ be a p_b-complete ordered partial b-metric space, and let $f, g : X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that (f, g) is a generalized $(\psi, \varphi)_{s,\varepsilon}$-contraction pair for some altering distance functions ψ and φ and $\varepsilon > 1$. If f and g are continuous (resp. $(X, \preceq, p_b, s > 1)$ is a regular), then f and g have a common fixed point.*

Proof. It is clear that (10) implies

$$s^{\varepsilon} p_b\left(fu, gv\right) \leq M_s^{f,g}\left(u, v\right), \tag{11}$$

for all comparable $u, v \in X$.

Now, it follows immediately from (11) that $z \in X$ is a fixed point of f if and only if z is a fixed point of g. Take $u_0 \in X$ and construct a sequence $\{u_n\}$ in X such that $u_{2n+1} = fu_{2n}$ and $u_{2n+2} = gu_{2n+1}$ for all nonnegative integers n. Since f and g are weakly increasing with respect to $\preceq$, we have that

$$fu_0 = u_1 \preceq u_2 \preceq u_3 \preceq \cdots \preceq u_n \preceq u_{n+1} \preceq \cdots. \tag{12}$$

If $u_{2n} = u_{2n+1}$ for some n or $u_{2n+1} = u_{2n+2}$ for some n, then obviously f and g have at least one common fixed point. Therefore, we may, assume without loss of generality, that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Now, we complete the proof for three cases:

Case I: We shall prove that

$$p_b\left(x_n, x_{n+1}\right) \leq k p_b\left(x_{n-1}, x_n\right) \tag{13}$$

for all $n \in \mathbb{N}$, where $k \in [0, \frac{1}{s}) \subseteq [0, 1)$.

Indeed, by (12), u_{2n} and u_{2n+1} are comparable, then from (11) it establishes that

$$\begin{aligned}
&s^{\varepsilon} p_b\left(u_{2n+1}, u_{2n+2}\right) = s^{\varepsilon} p_b\left(fu_{2n}, gu_{2n+1}\right) \leq M_s^{f,g}\left(u_{2n}, u_{2n+1}\right) \\
&= \max\left\{p_b\left(u_{2n}, u_{2n+1}\right), p_b\left(u_{2n+1}, u_{2n+2}\right), \frac{p_b\left(u_{2n}, u_{2n+2}\right) + p_b\left(u_{2n+1}, u_{2n+1}\right)}{2s}\right\} \\
&\leq \max\left\{p_b\left(u_{2n}, u_{2n+1}\right), p_b\left(u_{2n+1}, u_{2n+2}\right), \frac{p_b\left(u_{2n}, u_{2n+1}\right) + p_b\left(u_{2n+1}, u_{2n+2}\right)}{2}\right\} \\
&\leq \max\left\{p_b\left(u_{2n}, u_{2n+1}\right), p_b\left(u_{2n+1}, u_{2n+2}\right)\right\}.
\end{aligned} \tag{14}$$

If $p_b(u_{2n}, u_{2n+1}) \le p_b(u_{2n+1}, u_{2n+2})$, then (14) becomes

$$s^{\varepsilon} p_b(u_{2n+1}, u_{2n+2}) \le p_b(u_{2n+1}, u_{2n+2}),$$

which leads to a contradiction (because $s^{\varepsilon} > 1$). Accordingly, we deduce that

$$s^{\varepsilon} p_b(u_{2n+1}, u_{2n+2}) \le p_b(u_{2n}, u_{2n+1}). \tag{15}$$

Again by (12), x_{2n} and x_{2n-1} are comparable, then from (11) it establishes that

$$\begin{aligned} & s^{\varepsilon} p_b(u_{2n}, u_{2n+1}) = s^{\varepsilon} p_b(gu_{2n-1}, fu_{2n}) = s^{\varepsilon} p_b(fu_{2n}, gu_{2n-1}) \le M_s^{f,g}(u_{2n}, u_{2n-1}) \\ &= \max\left\{p_b(u_{2n}, u_{2n-1}), p_b(u_{2n}, u_{2n+1}), \frac{p_b(u_{2n}, u_{2n}) + p_b(u_{2n+1}, u_{2n-1})}{2s}\right\} \\ &\le \max\left\{p_b(u_{2n}, u_{2n-1}), p_b(u_{2n}, u_{2n+1}), \frac{p_b(u_{2n+1}, u_{2n}) + p_b(u_{2n}, u_{2n-1})}{2}\right\} \\ &\le \max\left\{p_b(u_{2n}, u_{2n-1}), p_b(u_{2n}, u_{2n+1})\right\}. \end{aligned} \tag{16}$$

If $p_b(u_{2n-1}, u_{2n}) \le p_b(u_{2n}, u_{2n+1})$, then

$$s^{\varepsilon} p_b(u_{2n}, u_{2n+1}) \le p_b(u_{2n}, u_{2n+1}).$$

This is a contradiction (because $s^{\varepsilon} > 1$). Consequently, we demonstrate that

$$s^{\varepsilon} p_b(u_{2n}, u_{2n+1}) \le p_b(u_{2n-1}, u_{2n}). \tag{17}$$

Hence by (16) and (17), we get (13), where $k = \frac{1}{s^{\varepsilon}} \in [0, \frac{1}{s}) \subseteq [0, 1)$.

Case II: We shall prove that $\{x_n\}$ is a p_b-Cauchy sequence. In order to end this, for $m, n \in \mathbb{N}$ and $m < n$, applying the triangle-type inequality pb4), we arrive at

$$\begin{aligned} & p_b(u_m, u_n) \\ &\le s\left[p_b(u_m, u_{m+1}) + p_b(u_{m+1}, u_n)\right] \\ &\le s p_b(u_m, u_{m+1}) + s^2\left[p_b(u_{m+1}, u_{m+2}) + p_b(u_{m+2}, u_n)\right] \\ &\le \cdots \le s p_b(u_m, u_{m+1}) + s^2 p_b(u_{m+1}, u_{m+2}) + \cdots \\ &\quad + s^{n-m-1}\left[p_b(u_{n-2}, u_{n-1}) + p_b(u_{n-1}, u_n)\right] \\ &\le s p_b(u_m, u_{m+1}) + s^2 p_b(u_{m+1}, u_{m+2}) \\ &\quad + \cdots + s^{n-m-1} p_b(u_{n-2}, u_{n-1}) + s^{n-m} p_b(u_{n-1}, u_n). \end{aligned}$$

Note that (13) and $sk < 1$, it is easy to see that

$$\begin{aligned} p_b(u_m, u_n) &\le \left(sk^m + s^2 k^{m+1} + \ldots + s^{n-m} k^{n-1}\right) p_b(u_0, u_1) \\ &= s\lambda^m \left[1 + (sk) + \ldots + (sk)^{n-m-1}\right] p_b(u_0, u_1) \\ &\le \frac{sk^m}{1 - sk} p_b(u_0, u_1) \to 0, \text{as } m \to \infty. \end{aligned} \tag{18}$$

It follows that $\{u_n\}$ is a p_b-Cauchy sequence. Since (X, p_b) is p_b-complete, then from [15](Lemma 1), it implies that $\{u_n\}$ converges to some $z \in X$. Again by [15], Lemma 1 it may be verified that

$$\lim_{n\to\infty} p_b(u_n, z) = \lim_{n,m\to\infty} p_b(u_n, u_m) = 0 = p_b(z, z). \tag{19}$$

Case III. (Existence of a common fixed point).

(i) Let f and g be continuous. Then, by using pb4), we acquire that

$$\frac{1}{s}p_b(z, fz) \leq p_b(z, fu_{2n}) + p_b(fu_{2n}, fz) \tag{20}$$

and

$$\frac{1}{s}p_b(z, gz) \leq p_b(z, gu_{2n+1}) + p_b(gu_{2n+1}, gz). \tag{21}$$

Letting $n \to \infty$ in (20) and (21), and using the continuity of f and g together with (19), we claim that

$$\frac{1}{s}p_b(z, fz) \leq p_b(fz, fz), \qquad \frac{1}{s}p_b(z, gz) \leq p_b(gz, gz). \tag{22}$$

Now, we derive from (22) and pb2) that

$$\frac{1}{s}\max\{p_b(z, fz), p_b(z, gz)\} \leq p_b(fz, gz). \tag{23}$$

In view of $p_b(z, z) \leq p_b(z, fz)$ or $p_b(z, z) \leq p_b(z, gz)$, then by (11), it establishes that

$$\begin{aligned} s^{\varepsilon}p_b(fz, gz) &\leq \max\left\{p_b(z, z), p_b(z, fz), p_b(z, gz), \frac{p_b(z, gz) + p_b(fz, z)}{2s}\right\} \\ &\leq \max\left\{0, p_b(z, fz), p_b(z, gz), \max\left\{\frac{p_b(z, fz)}{s}, \frac{p_b(z, gz)}{s}\right\}\right\} \\ &= \max\{p_b(z, fz), p_b(z, gz)\}. \end{aligned} \tag{24}$$

Further, combining (23) and (24), we speculate that

$$\frac{1}{s}\max\{p_b(z, fz), p_b(z, gz)\} \leq \frac{1}{s^{\varepsilon}}\max\{p_b(z, fz), p_b(z, gz)\}. \tag{25}$$

If $p_b(z, fz) > 0$ or $p_b(z, gz) > 0$, then from (25) it leads to a contradiction. Hence, we have proved that f and g have at least one common fixed point.

(ii) Let $(X, \preceq, p_b)$ be a regular ordered partial b-metric space. Using the given assumption on $(X, \preceq, p_b)$, we have that $u_n \preceq z$ for all $n \in \mathbb{N}$. Finally, we show that $fz = gz = z$. Actually, by (11), it ensures us that

$$\begin{aligned} & s^{\varepsilon}p_b(u_{2n+1}, gz) \\ &\leq \max\left\{p_b(u_{2n}, z), p_b(u_{2n}, u_{2n+1}), p_b(z, gz), \frac{p_b(u_{2n}, gz) + p_b(u_{2n+1}, z)}{2s}\right\} \\ &\leq \max\left\{p_b(u_{2n}, z), p_b(u_{2n}, u_{2n+1}), p_b(z, gz), \frac{p_b(u_{2n}, u_{2n+1}) + p_b(u_{2n+1}, gz) + p_b(u_{2n+1}, z)}{2}\right\} \end{aligned} \tag{26}$$

Noting that $p_b(u_{2n+1}, gz) \to p_b(z, gz), p_b(u_{2n}, z) \to 0 = p_b(z, z)$ and $p_b(u_{2n}, u_{2n+1}) \to 0$ as $n \to \infty$ and taking the limit from both sides of (26), we claim that

$$s^{\varepsilon}p_b(z, gz) \leq \max\left\{0, 0, p_b(z, gz), \frac{0 + p_b(z, gz)}{2}\right\} = p_b(z, gz). \tag{27}$$

This is a contradiction if $p_b(z, gz) > 0$. That is to say, $z = fz$. Similarly, we can show $z = gz$. Therefore, z is a common fixed point of f and g. □

Remark 2.4. *Since any generalized $(\psi, \varphi)_{s,2}$-contraction pair must be a generalized $(\psi, \varphi)_{s,\varepsilon}$-contraction pair, thus Theorem 2.3 greatly improves and expands Theorems 3, 4 as well as Corollaries 3 and 4 of [32].*

Remark 2.5. *The proof of Theorem 2.3 does not rely on Lemma 1.6 as compared to the proofs of the main results of [15]. Moreover, this proof is much shorter than [15]. As a result, this statement is more acceptable and applicable in applications.*

The following example illustrates this generalization to be genuine generalization.

Example 2.6. *Let $X = \{0, 1, 2\}$ be equipped with the following partial order:*

$$\preceq := \{(0,0), (1,1), (2,2), (0,1)\}.$$

Define a partial b-metric $p_b : X \times X \to [0, \infty)$ by

$$p_b(u,v) = \begin{cases} 0, & u = v; \\ (u+v)^2, & u \neq v. \end{cases} \tag{28}$$

It is easy to see that (X, p_b) is a p_b-complete partial b-metric space with $s = \frac{9}{5}$. Define self-maps $f = g$ with $f0 = g0 = 0$ and $f1 = f2 = g1 = g2 = 2$. Simple calculations show that f and g are weakly increasing mappings with respect to $\preceq$ and that f and g are continuous. In order to check that $(f, g) = (f, f)$ is a generalized $(\psi, \varphi)_{s,\varepsilon}$-contraction pair, only the case $u = 0, v = 1$ is nontrivial. For this case we arrive at

$$s^{\varepsilon} p_b(f0, g1) = \left(\frac{9}{5}\right)^{\varepsilon} p_b(f0, f1) = \left(\frac{9}{5}\right)^{\varepsilon} p_b(0, 2) = \left(\frac{9}{5}\right)^{\varepsilon} \cdot 4 \tag{29}$$

and

$$\begin{aligned} M_s^{f,g}(0,1) = M_s^{f,f}(0,1) &= \max\left\{p_b(0,1), p_b(0,f0), p_b(1,f1), \frac{p_b(0,f1) + p_b(1,f0)}{2 \cdot \frac{9}{5}}\right\} \\ &= \max\left\{1, 0, 9, \frac{25}{18}\right\} = 9. \end{aligned} \tag{30}$$

Now that $\left(\frac{9}{5}\right)^2 \cdot 4 > 9$, then the pair $(f, g) = (f, f)$ is not a generalized $(\psi, \varphi)_{\frac{9}{5},2}$-contractive for any altering functions ψ and φ. However, there exists $\varepsilon \in (1, 2)$ such that $\left(\frac{9}{5}\right)^{\varepsilon} \cdot 4 \leq 9$, that is, the pair $(f, g) = (f, f)$ satisfies the condition (11). Furthermore, there exist altering functions ψ and φ such that $(f, g) = (f, f)$ is a $(\psi, \varphi)_{\frac{9}{5},\varepsilon}$-contractive. Indeed, putting $\psi(t) = t$, we can find $\varphi(t) = kt, k \in (0, 1)$ such that

$$\left(\frac{9}{5}\right)^{\varepsilon} \cdot 4 \leq 9 - \varphi(9). \tag{31}$$

Hence, our Theorem 2.3 does be a real generalization compared with Theorems 3 and 4, Corollaries 3 and 4 of [32].

Finally, the generalization of Definition 1.8 and Theorem 1.9 is stated as follows.

Definition 2.7. *Let $(X, \preceq, p_b, s > 1)$ be an ordered partial b-metric space, and let ψ and φ be altering distance functions. A self-mapping $f : X \to X$ is called a generalized $(\psi, \varphi)_{s,\varepsilon}$-weakly contractive mapping if*

$$\psi\left(s^{\varepsilon} p_b(fu, fv)\right) \leq \psi\left(M_s^f(u,v)\right) - \varphi\left(M_s^f(u,v)\right) \tag{32}$$

for all comparable $u, v \in X$, where $\varepsilon > 1$ is a real constant.

Theorem 2.8. *Let $(X, \preceq, p_b, s > 1)$ be a p_b-complete ordered partial b-metric space. Let $f : X \to X$ be a nondecreasing, with respect to $\preceq$, continuous mapping (resp. $(X, \preceq, p_b, s > 1)$ is a regular). Suppose that f is a generalized $(\psi, \varphi)_{s,\varepsilon}$-weakly contractive mapping. If there exists $u_0 \in X$ such that $u_0 \preceq fu_0$, then f has a fixed point.*

Proof. First of all, the condition (32) implies

$$s^{\varepsilon} p_b(fu, fv) \leq M_s^f(u,v) \tag{33}$$

for all comparable $u, v \in X$. The rest of the proof is further similar to the proof of Theorem 2.3 so long as putting $g = f$ and therefore we omit it. □

Next one very important recent example is presented in the framework of partial b-metric spaces:

Example 2.9. *([48]). Let $X = \{x; y; z\}$ and put $b : X \times X \to [0; +\infty)$ as follows.*
1) $b(x;x) = b(z;z) = 1$ and $b(y;y) = 0.5$.
2) $b(x;z) = b(z;x) = 1.5$.
3) $b(y;z) = b(z;y) = 1$.
4) $b(x;y) = b(y;x) = 3$.
It is not difficult to check that $(X; b)$ is a partial b-metric space with coefficient $s = 3$.
For each $u \in X$ and each $\varepsilon > 0$, put $B(u,\varepsilon) = \{v \in X : b(u,v) < b(u,u) + \varepsilon\}$ and put $\mathcal{B} = \{B(u,\varepsilon) : u \in X \text{ and } \varepsilon > 0\}$. From [48] follows that $\mathcal{B}$ is not a base for any topology on X.

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