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C-Class Function on Khan Type Fixed Point Theorems in Generalized Metric Space

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Abstract. The aim of this article is to study common fixed point theorems in generalized metric spaces and obtain sufficient conditions for the existence of C-class function on Khan type common fixed points of a pair of mappings satisfying generalized contraction involving rational expressions. Also, some examples are obtained to support the obtained result. Also obtained results further generalize, extend and unify already existing results in literature.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Preliminaries

The Banach contraction principle [12] is very popular tool in solving existence problems in many branches of mathematical analysis. This famous theorem is stated as follows.

Theorem 1.1. *[12]. Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying:*

$$d(Tx, Ty) \leq \lambda d(x, y), \ \forall \, x, y \in X, \tag{1}$$

where $\lambda \in (0, 1)$. Then T has a unique fixed point $x^* \in X$.

There are a large number of generalizations of the Banach contraction principle (see [15], [11], [10], [16] and others) in literature. Some generalizations of the notion of a metric space have been proposed by some authors, such as, rectangular metric spaces, semi metric spaces, pseudo metric spaces, probabilistic metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, $D-$metric spaces, and cone metric spaces (see [1, 2, 5–7, 13, 18–20]).

First of all, we introduce some notations and definitions that will be used later. While, fixed points results dealing with general contractive conditions with rational expressions were appeared. One of the well-known results in this direction were established by Khan in [4]. Fisher in [3] revised Khan's result as follows:

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Definition 1.2. *[2] Let X be a nonempty set. Suppose that the mapping* $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ *be a mapping such that for all* $x, y \in X$ *and for all distinct points* $u, v \in X$*, which are different from x and y, satisfies*

G1 $d(x, y) = 0$ if and only if $x = y$,

G2 $d(x, y) = d(y, x)$ for all $x, y \in X$,

G3 $d(x, z) \le d(x, u) + d(u, v) + d(v, z)$, for all $x, y, z \in X$.
Then d is called generalized metric and the pair (X, d) is called generalized metric space (or shortly *GMS*).

The concepts of convergence, Cauchy sequence, completeness and continuity on a GMS are defined below.

Definition 1.3. *[2] Let* $\{x_n\}$ *be a sequence in a GMS,* (X, d)*;*

i) Sequence $\{x_n\}$ converges to some x, if $d(x_n, x) \to 0$ as $n \to \infty$.

ii) sequence $\{x_n\}$ is Cauchy, if for every $\epsilon > 0$ there exists positive integer $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $n > m > N(\epsilon)$.

iii) A GMS (X, d) is called complete if every GMS Cauchy sequence in X is GMS convergent.

iv) A mapping $T : X \to X$ is continuous if for any sequence $\{x_n\}$ in X such that

$$d(x_n, x) \to 0 \text{ as } n \to \infty,$$

we have

$$d(Tx_n, Tx) \to 0 \text{ as } n \to \infty.$$

Lemma 1.4. *([9]) Let* (X, d) *be a generalized metric space and* $\{x_n\}$ *be a Cauchy sequence in X such that* $x_m \neq x_n$ *whenever* $m \neq n$*. Then* $\{x_n\}$ *can converge to at most one point.*

Lemma 1.5. *([9]) Let* (X, d) *be a generalized metric space and* $\{x_n\}$ *be a sequence in X with distinct elements* ($x_n \neq x_m$ *for* $n \neq m$)*. Suppose that* $d(x_n, x_{n+1})$ *and* $d(x_n, x_{n+2})$ *tends to* 0 *as* $n \to \infty$ *and* $\{x_n\}$ *is not a Cauchy sequence. Then there exist* $\epsilon > 0$ *and two sequences* $\{m_k\}$ *and* $\{n_k\}$ *of positive integers such that* $n_k > m_k > k$ *and the following four sequences*

$$d(x_{m_k}, x_{n_k}), \quad d(x_{m_k}, x_{n_{k+1}}), \quad d(x_{m_{k-1}}, x_{n_k}), \quad d(x_{m_{k-1}}, x_{n_{k+1}}) \tag{2}$$

tends to ϵ *as* $k \to \infty$*.*

Theorem 1.6. *[17]Let* (X, d) *be a complete GMS and suppose that* $T : X \to X$ *be a self-mapping such that*

$$d(Tx, Ty) \le \begin{cases} \gamma \max\{d(x, y), \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(Tx,y)\}}\} & \text{if } \max\{d(x, Ty), d(Tx, y)\} \neq 0, \\ 0 & \text{otherwise} \max\{d(x, Ty), d(Tx, y)\} = 0, \end{cases}$$

for all $x, y \in X$ *and* $x \neq y$*, and for* $\gamma \in (0, 1)$ *Then T has a unique fixed point* $x^* \in X$*. Moreover, for all* $x \in X$*, the sequence* $\{T^n x\}$ *converges to* x^**.*

Proposition 1.7. *([10]) Suppose that* $\{x_n\}$ *is a Cauchy sequence in a GMS* (X, d) *with* $\lim_{n\to\infty} d(x_n, u) = 0$*, where* $u \in X$*. Then* $\lim_{n\to\infty} d(x_n, z) = d(u, z)$ *for all* $z \in X$*.*

In 2014, A.H.Ansari [8] introduced the concept of a C-class functions which covers a large class of contractive conditions.

Definition 1.8. *[8] A continuous function $F : [0,\infty)^2 \to \mathbb{R}$ is called C-class function if for any $s,t \in [0,\infty)$, the following conditions hold:*

c1 $F(s,t) \le s$;

c2 $F(s,t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on F that $f(0,0) = 0$ could be imposed in some cases if required. The letter C will denote the class of all C- functions.

Example 1.9. *[8] The following examples shows that the class C is nonempty:*

1. $F(s,t) = s - t$.
2. $F(s,t) = ms$, for some $m \in (0,1)$.
3. $f(s,t) = \frac{s}{(1+t)^r}$ for some $r \in (0,\infty)$.
4. $f(s,t) = \log(t + a^s)/(1+t)$, for some $a > 1$.

Let Φ_u denote the class of the functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfy the following conditions:

a) φ is continuous ;

b) $\varphi(t) > 0, t > 0$ and $\varphi(0) \ge 0$.

Definition 1.10. *([11])A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:*

i) ψ is non-decreasing and continuous,

ii) $\psi(t) = 0$ if and only if $t = 0$.

Let us suppose that Ψ denote the class of the altering distance functions.

Definition 1.11. *A tripled (ψ,φ,F) where $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in C$ is said to be a monotone if for any $x,y \in [0,\infty)$,*

$$x \leqslant y \Longrightarrow F(\psi(x),\varphi(x)) \leqslant F(\psi(y),\varphi(y)).$$

An example is provided to show that the class of monotone triplet (ψ,φ,F) is nonempty.

Example 1.12. *Let us consider a $C-$ class function $F(s,t) = s - t$, $\phi(x) = \sqrt{x}$ and*

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \le x \le 1, \\ x^2, & \text{if } x > 1, \end{cases}$$

then (ψ,ϕ,F) is monotone.

Now, we are going to show that the triplet (ψ,φ,F) is not monotone for every ψ,φ and F;

Example 1.13. *Let us suppose that $F(s,t) = s - t$, $\phi(x) = x^2$ and*

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \le x \le 1, \\ x^2, & \text{if } x > 1, \end{cases}$$

then it can be easily verify that (ψ,ϕ,F) is not monotone.

2. Main Results

Theorem 2.1. *Let (X, d) be a complete GMS and suppose that $T : X \to X$ be a self-mapping such that*

$$\psi(d(Tx,Ty)) \leq \begin{cases} F(\psi(m(x,y)),\varphi(m(x,y))) & \text{if } \max\{d(x,Ty),d(Tx,y)\} \neq 0, \\ 0 & \text{otherwise} \max\{d(x,Ty),d(Tx,y)\} = 0, \end{cases} \tag{3}$$

for all $x, y \in X$ and $x \neq y$, where $F \in \mathcal{C}, \psi \in \Psi, \varphi \in \Phi_u$ such that (ψ, ϕ, F) is monotone and

$$m(x,y) = \max\{d(x,y), \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(Tx,y)\}}\}.$$

Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*.

Proof. Let us suppose that $x_0 \in X$, construct a sequence $\{x_n\}$ as $x_{n+1} = Tx_n = T^n x_0$, for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then $x^* = x_n$ is a fixed point of T and we are done. If not, then assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. We shall divide the proof into two cases:

Case 2.2. *1: Assume that, if*

$$\max\{d(x_n, Tx_m), d(Tx_n, x_m)\} \neq 0,$$

for all $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then from (3), we have

$$\begin{aligned}
& \psi(d(x_n,x_{n+1})) = \psi(d(Tx_{n-1},Tx_n)) \\
\leq\ & F(\psi(\max\{d(x_{n-1},x_n), \frac{d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n) + d(x_n,Tx_n)d(x_n,Tx_{n-1})}{\max\{d(x_{n-1},Tx_n),d(Tx_{n-1},x_n)\}}\}), \\
& \varphi(\max\{d(x_{n-1},x_n), \frac{d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n) + d(x_n,Tx_n)d(x_n,Tx_{n-1})}{\max\{d(x_{n-1},Tx_n),d(Tx_{n-1},x_n)\}}\})) \\
=\ & F(\psi(\max\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1}) + d(x_n,x_{n+1})d(x_n,x_n)}{\max\{d(x_{n-1},x_{n+1}),d(x_n,x_n)\}}\}), \\
& \varphi(\max\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1}) + d(x_n,x_{n+1})d(x_n,x_n)}{\max\{d(x_{n-1},x_{n+1}),d(x_n,x_n)\}}\})) \\
=\ & F(\psi(d(x_{n-1},x_n)),\varphi(d(x_{n-1},x_n))) \leq \psi(d(x_{n-1},x_n)),
\end{aligned}$$

implies

$$\psi(d(x_n,x_{n+1})) \leq \psi(d(x_{n-1},x_n)). \tag{4}$$

Since ψ is nondecreasing, so from (4), implies that

$$d(x_n,x_{n+1}) \leq d(x_{n-1},x_n),$$

for each $n \in \mathbb{N}$. Thus, we conclude that the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing. As a result, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We claim that $r = 0$. On contrary, suppose that $r > 0$. Then, from (4), we have

$$\psi(r) \leq F(\psi(r),\varphi(r)),$$

which yields that either $\psi(r) = 0$ or $\varphi(r) = 0$. We have

$$r = \lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{5}$$

Analogously, we can prove that $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$, by substituting $x = x_{n-1}$ and $y = x_{n+1}$ in (3), also note that $\frac{ac+bd}{\max\{a,b\}} \leq c + d$, hence

$$\begin{aligned}
&\psi(d(x_n, x_{n+2})) = \psi(d(Tx_{n-1}, Tx_{n+1}))\\
\leq\ & F(\psi(\max\{d(x_{n-1}, x_{n+1}), \frac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_{n+1}) + d(x_{n+1}, Tx_{n+1})d(x_{n+1}, Tx_{n-1})}{\max\{d(x_{n-1}, Tx_{n+1}), d(Tx_{n-1}, x_{n+1})\}}\}),\\
&\varphi(\max\{d(x_{n-1}, x_{n+1}), \frac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_{n+1}) + d(x_{n+1}, Tx_{n+1})d(x_{n+1}, Tx_{n-1})}{\max\{d(x_{n-1}, Tx_{n+1}), d(Tx_{n-1}, x_{n+1})\}}\}))\\
=\ & F(\psi(\max\{d(x_{n-1}, x_{n+1}), \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+2}) + d(x_{n+1}, x_{n+2})d(x_{n+1}, x_n)}{\max\{d(x_{n-1}, x_{n+2}), d(x_n, x_{n+1})\}}\}),\\
&\varphi(\max\{d(x_{n-1}, x_{n+1}), \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+2}) + d(x_{n+1}, x_{n+2})d(x_{n+1}, x_n)}{\max\{d(x_{n-1}, x_{n+2}), d(x_n, x_{n+1})\}}\}))\\
\leq\ & F(\psi(\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})),\\
&\varphi(\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})))\\
=\ & F(\psi(\max\{d(x_{n-1}, x_n) + d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2}), d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})\}),\\
&\varphi(\max\{d(x_{n-1}, x_n) + d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2}), d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})\}))\\
=\ & F(\psi(d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})),\\
&\varphi(d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}))).
\end{aligned}$$

Apply limit $n \to \infty$ on both sides of above inequality and use (5), we deduce that

$$\psi(\lim_{n\to\infty} d(x_n, x_{n+2})) \leq F(\psi(\lim_{n\to\infty} d(x_n, x_{n+2})), \varphi(\lim_{n\to\infty} d(x_n, x_{n+2}))),$$

which yields either $\psi(\lim_{n\to\infty} d(x_n, x_{n+2})) = 0$ or $\varphi(\lim_{n\to\infty} d(x_n, x_{n+2})) = 0$. Which further implies that

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 0 = \lim_{n\to\infty} d(x_n, x_{n+2}). \tag{6}$$

Now to prove that $\{x_n\}$ is a Cauchy sequence in (X, d). On contrary, suppose that $\{x_n\}$ is not a Cauchy sequence. Then, by Lemma 1.5 there exists an $\varepsilon > 0$ and two subsequences $\left\{x_{m(k)}\right\}$ and $\left\{x_{n(k)}\right\}$ of $\{x_n\}$ with $m(k) > n(k) > k$, such that $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)}, x_{2n(k)-2}) < \varepsilon$ and

$$\begin{aligned}
\lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)}\right) &= \lim_{k\to\infty} d\left(x_{n(k)+1}, x_{m(k)}\right) = \\
\lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)-1}\right) &= \lim_{k\to\infty} d\left(x_{n(k)+1}, x_{m(k)+1}\right) = \varepsilon.
\end{aligned} \tag{7}$$

By substituting $x = x_{n(k)}$ and $y = x_{m(k)}$ in (3)

$$\begin{aligned}
&\psi(d(x_{n(k)+1}, x_{m(k)+1})) = \psi(d(Tx_{n(k)}, Tx_{m(k)}))\\
\leq\ & F(\psi(\max\{d(x_{n(k)}, x_{m(k)}), \frac{d(x_{n(k)}, Tx_{n(k)})d(x_{n(k)}, Tx_{m(k)}) + d(x_{m(k)}, Tx_{m(k)})d(x_{m(k)}, Tx_{n(k)})}{\max\{d(x_{n(k)}, Tx_{m(k)}), d(Tx_{n(k)}, x_{m(k)})\}}\}),\\
&\varphi(\max\{d(x_{n(k)}, x_{m(k)}), \frac{d(x_{n(k)}, Tx_{n(k)})d(x_{n(k)}, Tx_{m(k)}) + d(x_{m(k)}, Tx_{m(k)})d(x_{m(k)}, Tx_{n(k)})}{\max\{d(x_{n(k)}, Tx_{m(k)}), d(Tx_{n(k)}, x_{m(k)})\}}\}))\\
=\ & F(\psi(\max\{d(x_{n(k)}, x_{m(k)}), \frac{d(x_{n(k)}, x_{n(k)+1})d(x_{n(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{m(k)+1})d(x_{m(k)}, x_{n(k)+1})}{\max\{d(x_{n(k)}, x_{m(k)+1}), d(x_{n(k)+1}, x_{m(k)})\}}\}),\\
&\varphi(\max\{d(x_{n(k)}, x_{m(k)}), \frac{d(x_{n(k)}, x_{n(k)+1})d(x_{n(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{m(k)+1})d(x_{m(k)}, x_{n(k)+1})}{\max\{d(x_{n(k)}, x_{m(k)+1}), d(x_{n(k)+1}, x_{m(k)})\}}\}))\\
\leq\ & F(\psi(\max\{d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}) + d(x_{m(k)}, x_{m(k)+1})),\\
&\varphi(\max\{d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}) + d(x_{m(k)}, x_{m(k)+1}))
\end{aligned}$$

By applying limit $k \to \infty$ and use (5), (7) we have

$$\psi(\varepsilon) \leq F(\psi(\varepsilon), \varphi(\varepsilon)),$$

which implies that either $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$, a contradiction. So, we conclude that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete generalized metric space, so there exist $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{8}$$

Also

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + d(x_{n+1}, x_n) + d(x_{n+1}, x_n) + d(x^*, x_{n+1}) + d(x^*, Tx^*) \\ &= 2d(x^*, x_{n+1}) + 2d(x_{n+1}, x_n) + d(x^*, Tx^*). \end{aligned}$$

It follows from (5) and (8), that is

$$\lim_{n\to\infty} d(x_n, Tx^*) = d(x^*, Tx^*). \tag{9}$$

On the other hand, from (3), we have

$$\begin{aligned} &\psi(d(x_{n+1}, Tx^*)) = \psi(d(Tx_n, Tx^*)) \\ \leq\ & F(\psi(\max\{d(x_n, x^*), \frac{d(x_n, Tx_n)d(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, Tx_n)}{\max\{d(x_n, Tx^*), d(Tx_n, x^*)\}}\}), \\ & \varphi(\max\{d(x_n, x^*), \frac{d(x_n, Tx_n)d(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, Tx_n)}{\max\{d(x_n, Tx^*), d(Tx_n, x^*)\}}\})) \\ =\ & F(\psi(\max\{d(x_n, x^*), \frac{d(x_n, x_{n+1})d(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, x_{n+1})}{\max\{d(x_n, Tx^*), d(x_{n+1}, x^*)\}}\}), \\ & \varphi(\max\{d(x_n, x^*), \frac{d(x_n, x_{n+1})d(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, x_{n+1})}{\max\{d(x_n, Tx^*), d(x_{n+1}, x^*)\}}\})). \end{aligned} \tag{10}$$

So from (5), (8), (9) and taking limits as $n \to \infty$ on both side of (10), we have $d(x^*, Tx^*) = 0$. Now, we have to show that T has a unique fixed point. For this, assume that y^* is another fixed point of T in X such that $x^* \neq y^*$.

So from (3), we get

$$\begin{aligned} &\psi(d(y^*, x^*)) = \psi(d(Ty^*, Tx^*)) \\ \leq\ & F(\psi(\max\{d(y^*, x^*), \frac{d(y^*, Ty^*)d(y^*, Tx^*) + d(x^*, Tx^*)d(x^*, Ty^*)}{\max\{d(y^*, Tx^*), d(Ty^*, x^*)\}}\}), \\ & \varphi(\max\{d(y^*, x^*), \frac{d(y^*, Ty^*)d(y^*, Tx^*) + d(x^*, Tx^*)d(x^*, Ty^*)}{\max\{d(y^*, Tx^*), d(Ty^*, x^*)\}}\})) \\ =\ & F(\psi(\max\{d(y^*, x^*), \frac{d(y^*, y^*)d(y^*, x^*) + d(x^*, x^*)d(x^*, y^*)}{\max\{d(y^*, x^*), d(y^*, x^*)\}}\}), \\ & \varphi(\max\{d(y^*, x^*), \frac{d(y^*, y^*)d(y^*, x^*) + d(x^*, x^*)d(x^*, y^*)}{\max\{d(y^*, x^*), d(y^*, x^*)\}}\})) \\ =\ & F(\psi(d(y^*, x^*), \varphi(d(y^*, x^*))), \end{aligned}$$

which further implies that either $\psi(d(y^*, x^*)) = 0$ or $\varphi(d(y^*, x^*)) = 0$, therefore $d(y^*, x^*) = 0$ implies $y^* = x^*$, a contradiction, hence fixed point of T is unique. □

To support the result we have the following example as:

Example 2.3. *Let $X = \{\alpha, \beta, \gamma, \delta\}$ and define $d : X \times X \to [0, \infty)$ as follows:*

$$\begin{aligned} d(\alpha,\alpha) &= d(\beta,\beta) = d(\gamma,\gamma) = d(\delta,\delta) = 0, \\ d(\alpha,\delta) &= d(\delta,\alpha) = d(\gamma,\delta) = d(\delta,\gamma) = d(\beta,\gamma) = d(\gamma,\beta) = 2, \\ d(\alpha,\gamma) &= d(\gamma,\alpha) = d(\beta,\delta) = d(\beta,\delta) = 4.1, \\ d(\alpha,\beta) &= d(\beta,\alpha) = 2.1. \end{aligned}$$

Then it can be easily checked that (X, d) is a generalized metric space. Note that

$$4.1 = d(\alpha,\gamma) \leq d(\alpha,\beta) + d(\beta,\gamma) = 4$$

is not true. Hence (X, d) defined above is generalized metric space but is not a metric space. Also, define $T : X \to X$ such that

$$T(x) = \begin{cases} \beta & \text{if } x \neq \gamma, \\ \alpha & \text{if } x = \gamma, \end{cases}$$

and

$$F(s,t) = s - t, \text{ for all } s, t \geq 0.$$

Case 2.4. *Also, we define $\psi \in \Psi$ and $\varphi \in \Phi$ by*

$$\psi(t) = \frac{t}{4} \text{ and } \varphi(t) = \frac{t}{10} \text{ for all } t \in [0, \infty).$$

First have to show that (ψ, φ, F) is monotone, let $x, y \in X$ and $x \leq y$ (suppose) then

$$F(\psi(x), \varphi(x)) \leq F(\psi(y), \varphi(y))$$

must hold true. Since

$$\begin{aligned} F(\psi(x), \varphi(x)) &= \psi(x) - \varphi(x) \\ &= \frac{x}{4} - \frac{x}{10} \\ &= \frac{3x}{20} \end{aligned}$$

Since $x \leq y$ implies $\frac{3x}{20} \leq \frac{3y}{20}$, so above equation further implies that

$$\begin{aligned} F(\psi(x), \varphi(x)) &= \frac{3x}{20} \leq \frac{3y}{20} \\ &= \frac{y}{4} - \frac{y}{10} \\ &= \psi(y) - \varphi(y) \\ &= F(\psi(y), \varphi(y)) \end{aligned}$$

Hence $F(\psi(x), \varphi(x)) \leq F(\psi(y), \varphi(y))$, and the triplet (ψ, φ, F) is monotone. Next we have some auxalіry calcuations for all $x \neq y$:

$$\begin{aligned} m(\beta,\alpha) &= m(\alpha,\beta) = \max\{d(\alpha,\beta), \frac{d(\alpha,T\alpha)d(\alpha,T\beta) + d(\beta,T\beta)d(\beta,T\alpha)}{\max\{d(\alpha,T\beta), d(\beta,T\alpha)\}} \\ &= \max\{d(\alpha,\beta), \frac{d(\alpha,\beta)d(\alpha,\beta) + d(\beta,\beta)d(\beta,\beta)}{\max\{d(\alpha,\beta), d(\beta,\beta)\}}\} \\ &= \max\{2.1, \frac{(2.1)(2.1) + 0}{\max\{2.1, 0\}}\} = \max\{2.1, 2.1\} \\ &= 2.1 \end{aligned}$$

$$\begin{aligned}
m(\gamma,\alpha) &= m(\alpha,\gamma) = \max\{d(\alpha,\gamma), \frac{d(\alpha,T\alpha)d(\alpha,T\gamma)+d(\gamma,T\gamma)d(\gamma,T\alpha)}{\max\{d(\alpha,T\gamma),d(\gamma,T\alpha)\}}\\
&= \max\{d(\alpha,\gamma), \frac{d(\alpha,\beta)d(\alpha,\alpha)+d(\gamma,\alpha)d(\gamma,\beta)}{\max\{d(\alpha,\alpha),d(\gamma,\beta)\}}\}\\
&= \max\{4.1, \frac{0+(4.1)(2)}{\max\{0,2\}}\} = \max\{4.1,4.1\}\\
&= 4.1
\end{aligned}$$

$$\begin{aligned}
m(\delta,\alpha) &= m(\alpha,\delta) = \max\{d(\alpha,\delta), \frac{d(\alpha,T\alpha)d(\alpha,T\delta)+d(\delta,T\delta)d(\delta,T\alpha)}{\max\{d(\alpha,T\delta),d(\delta,T\alpha)\}}\\
&= \max\{d(\alpha,\delta), \frac{d(\alpha,\beta)d(\alpha,\beta)+d(\delta,\beta)d(\delta,\beta)}{\max\{d(\alpha,\beta),d(\delta,\beta)\}}\}\\
&= \max\{2, \frac{(2.1)(2.1)+(4.1)(4.1)}{\max\{2.1,4.1\}}\} = \max\{2, \frac{1061}{205}\}\\
&= \frac{1061}{205}
\end{aligned}$$

$$\begin{aligned}
m(\beta,\gamma) &= m(\gamma,\beta) = \max\{d(\gamma,\beta), \frac{d(\gamma,T\gamma)d(\gamma,T\beta)+d(\beta,T\beta)d(\beta,T\gamma)}{\max\{d(\gamma,T\beta),d(\beta,T\gamma)\}}\\
&= \max\{d(\gamma,\beta), \frac{d(\gamma,\alpha)d(\gamma,\beta)+d(\beta,\beta)d(\beta,\alpha)}{\max\{d(\gamma,\beta),d(\beta,\alpha)\}}\}\\
&= \max\{2, \frac{(4.1)(2)+0}{\max\{2,2.1\}}\} = \max\{2, \frac{82}{21}\}\\
&= \frac{82}{21}
\end{aligned}$$

$$\begin{aligned}
m(\beta,\delta) &= m(\delta,\beta) = \max\{d(\delta,\beta), \frac{d(\delta,T\delta)d(\delta,T\beta)+d(\beta,T\beta)d(\beta,T\delta)}{\max\{d(\delta,T\beta),d(\beta,T\delta)\}}\\
&= \max\{d(\delta,\beta), \frac{d(\delta,\beta)d(\delta,\beta)+d(\beta,\beta)d(\beta,\beta)}{\max\{d(\delta,\beta),d(\beta,\beta)\}}\}\\
&= \max\{4.1, \frac{(4.1)(4.1)+0}{\max\{4.1,0\}}\} = \max\{4.1,4.1\}\\
&= 4.1
\end{aligned}$$

$$\begin{aligned}
m(\delta,\gamma) &= m(\gamma,\delta) = \max\{d(\gamma,\delta), \frac{d(\gamma,T\gamma)d(\gamma,T\delta)+d(\delta,T\delta)d(\delta,T\gamma)}{\max\{d(\gamma,T\delta),d(\delta,T\gamma)\}}\\
&= \max\{d(\gamma,\delta), \frac{d(\gamma,\alpha)d(\gamma,\beta)+d(\delta,\beta)d(\delta,\alpha)}{\max\{d(\gamma,\beta),d(\delta,\alpha)\}}\}\\
&= \max\{2, \frac{(4.1)(2)+(4.1)(2)}{\max\{2,2\}}\} = \max\{2,8.2\}\\
&= 8.2
\end{aligned}$$

Let $x, y \in X$ with $x \neq y$ and consider the following possible cases:
Case 1. If $x, y \in \{\alpha, \beta, \delta\}$. Then $d(Tx, Ty) = d(\beta, \beta) = 0$ and hence 3 trivially holds.

Case 2. If $x = \gamma$, $y \in \{\alpha, \beta, \delta\}$. Then $d(Tx, Ty) = d(\alpha, \beta) = 2.1$.
For fixed $x = \gamma$ and $y = \alpha$,β and δ, then

$$F(\psi(m(x,y)), \varphi(m(x,y))) - \psi(d(Tx,Ty)) > 0,$$

then inequality 3 holds true for each $y \in \{\alpha, \beta, \delta\}$ and for $x = \gamma$.

Case 3. If $x \in \{\alpha, \beta, \delta\}, y = \gamma$. Since d is symmetric, therefore 3 holds by case 2.

So, T is contraction type map. Hence all the condition of Theorem 2.1 are satisfied and consequently β is the unique fixed point of mapping T.

If we assume $F(s,t) = ks$ in theorem 2.1, then the results follows:

Corollary 2.5. *Let (X,d) be a complete GMS and let $T : X \to X$ be a self-mapping such that*

$$\psi(d(Tx,Ty)) \leq \begin{cases} k\psi(m(x,y)) & \text{if } \max\{d(x,Ty), d(Tx,y)\} \neq 0; \\ 0 & \text{otherwise} \max\{d(x,Ty), d(Tx,y)\} = 0; \end{cases}$$

for all $x, y \in X$, and $x \neq y$, where $\psi \in \Psi, k \in (0,1)$ and

$$m(x,y) = \max\{d(x,y), \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{\max\{d(x,Ty), d(Tx,y)\}}\}$$

Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*

If we have $F(s,t) = ks$, $\psi(t) = t$ in theorem 2.1 or $\psi(t) = t$ in Corollary2.5, then we have the following corollary:

Corollary 2.6. *Let (X,d) be a complete GMS and let $T : X \to X$ be a self-mapping such that*

$$d(Tx,Ty) \leq \begin{cases} km(x,y) & \text{if } \max\{d(x,Ty), d(Tx,y)\} \neq 0; \\ 0 & \text{otherwise} \max\{d(x,Ty), d(Tx,y)\} = 0; \end{cases}$$

for all $x, y \in X$, and $x \neq y$, where $k \in (0,1)$,

$$m(x,y) = \max\{d(x,y), \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{\max\{d(x,Ty), d(Tx,y)\}}\}$$

Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*

Theorem 2.7. *Let (X,d) be a complete GMS and suppose that $T : X \to X$ be a self-mapping such that*

$$\psi(d(Tx,Ty)) \leq \begin{cases} F(\psi(M(x,y)), \varphi(M(x,y))) & \text{if } \max\{d(x,Ty), d(Tx,y)\} \neq 0; \\ 0 & \text{otherwise} \max\{d(x,Ty), d(Tx,y)\} = 0; \end{cases}$$

for all $x, y \in X$ and $x \neq y$, where $F \in C, \psi \in \Psi, \varphi \in \Phi_u$ such that (ψ, ϕ, F) is monotone and

$$M(x,y) = \frac{1}{\delta + \zeta}[\delta d(x,y) + \zeta \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{\max\{d(x,Ty), d(Tx,y)\}}]$$

for some $\delta, \zeta \in [0, \infty)$ such that $\delta + \zeta > 0$. Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*.

Proof. Since

$$\begin{aligned} & \frac{1}{\delta + \zeta}[\delta d(x,y) + \zeta \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{\max\{d(x,Ty), d(Tx,y)\}}] \\ \leq \quad & \max\{d(x,y), \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{\max\{d(x,Ty), d(Tx,y)\}}\}. \end{aligned}$$

So from theorem 2.1, the proof is complete. □

By taking $F(s,t) = ks$ in theorem 2.7, we can obtained the following result.

Corollary 2.8. *Let (X,d) be a complete GMS and let $T : X \to X$ be a self-mapping such that*

$$\psi(d(Tx,Ty)) \leq \begin{cases} k\psi(M(x,y)) & \text{if } \max\{d(x,Ty),d(Tx,y)\} \neq 0; \\ 0 & \text{otherwise} \max\{d(x,Ty),d(Tx,y)\} = 0; \end{cases}$$

for all $x,y \in X$, and $x \neq y$, where $\psi \in \Psi$, and

$$M(x,y) = \frac{1}{\delta+\zeta}[\delta d(x,y) + \zeta \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(Tx,y)\}}]$$

for some $\delta,\zeta \in [0,\infty)$ such tha $\delta+\zeta > 0$ .Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*

Let us consider $F(s,t) = ks$, $k = \delta + \zeta < 1$, $\delta,\zeta \in [0,1)$, $\psi(t) = t$ in theorem 2.7, then the result follows:

Corollary 2.9. *Let (X,d) be a complete GMS and let $T : X \to X$ be a self-mapping such that*

$$d(Tx,Ty)) \leq \begin{cases} N(x,y) & \text{if } \max\{d(x,Ty),d(Tx,y)\} \neq 0; \\ 0 & \text{otherwise} \max\{d(x,Ty),d(Tx,y)\} = 0; \end{cases}$$

for all $x,y \in X$, and $x \neq y$, where $\psi \in \Psi$, and

$$N(x,y) = \delta d(x,y) + \zeta \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(Tx,y)\}}$$

for some $\delta,\zeta \in [0,1)$ such tha $\delta+\zeta < 1$ .Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*,

Example 2.10. *Let $X = A \cup B$, where $A = \{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5}\}$ and $B = [\frac{3}{4},1]$. Define the function $d : X \times X \to \mathbb{R}$ as follows:*

$$\begin{aligned} d\left(\frac{1}{2},\frac{1}{3}\right) &= \frac{1}{2}, \\ d\left(\frac{1}{2},\frac{1}{4}\right) &= d\left(\frac{1}{3},\frac{1}{4}\right) = \frac{1}{5}, \\ d\left(\frac{1}{4},\frac{1}{5}\right) &= d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{3},\frac{1}{5}\right) = \frac{1}{6}, \\ d(x,x) &= 0 \text{ for all } x \in A, \\ d(x,y) &= d(y,x) \text{ for all } x,y \in A, \\ d(x,y) &= \left|x-y\right| \quad (x \in A, y \in B) \text{ or } (x \in B, y \in A) \text{ or } (x,y \in B). \end{aligned}$$

It is easy to check that (X,d) is generalized metric space. Let $T : X \to X$ be a mapping defined by

$$T(x) = \begin{cases} \frac{4x+1}{x+4} & \text{if } x \in B, \\ \frac{3}{4} & \text{otherwise.} \end{cases}$$

Put $\psi(t) = t$, $\varphi(t) = \frac{1}{1000}$ and $F(s,t) = \frac{s}{1+t}$. So

$$
\begin{aligned}
M(x,y) &= \max\{d(x,y), d(x,Tx), d(y,Ty)\} = \frac{9}{11} \quad x,y \in X \\
d(x,Tx) &\leq \max\{\frac{1}{4}, \frac{5}{12}, \frac{1}{2}, \frac{11}{20}\} = \frac{11}{20} \quad x \in A \\
d(y,Ty) &= \left|\frac{y^2-1}{y+4}\right| \leq \frac{1}{4} \quad y \in B \\
d(x,y) &\leq \frac{1}{2} \quad x,y \in A \\
d(x,y) &\leq \frac{1}{4} \quad x,y \in B \\
d(Tx,Ty) &= \left|\frac{3}{4} - \frac{4y+1}{y+4}\right| \leq \frac{4}{5} \quad (x \in A, y \in B) \text{ and similar } (x \in B, y \in A) \\
d(Tx,Ty) &= \left|\frac{4x+1}{x+4} - \frac{4y+1}{y+4}\right| \leq \frac{3}{19} \quad x,y \in B
\end{aligned}
$$

Note that

$$\max\{\frac{4x+1}{x+4} : x \in B\} = \{4 - \frac{15}{x+4} : x \in B\} = 1,$$

and

$$\min\{\frac{4x+1}{x+4} : x \in B\} = \{4 - \frac{15}{x+4} : x \in B\} = \frac{16}{19}.$$

The condition (3) is clear, for example, when $x, y \in B$ we have

$$\psi(d(Tx,Ty)) = \frac{3}{19} \leq F(\psi(\frac{16}{19}), \varphi(\frac{16}{19})) = \frac{\frac{16}{19}}{1+\frac{1}{100}}.$$

Now theorem 2.7 guarantees that $1 \in B$ is unique fixed point of T.

References

[1] J. Ahmad, M. Arshad and C. Vetro, On a Theorem of Khan in a Generalized Metric Space. Int. J. Anal., 2013.
[2] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen, 57, 31-37, 2000.
[3] B. Fisher, On a theorem of Khan, Riv. Math. Univ. Parma., 4, 135-137, 1978.
[4] M. S. Khan, A fixed point theorem for metric spaces, Rend. Inst. Math. Univ. Trieste, 8, 69-72, 1976.
[5] B. Samet, Discussion on a fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces. Branciari, Publ. Math. Debrecen, 76, 493-494, 2010.
[6] I. R. Sarma, J. M. Rao and S. S. Rao, Contractions over generalized metric spaces. J. Nonlinear Sci. Appl. 2, 180-182, 2009.
[7] H. Piri, S. Rahrovi and P. Kumam, Khan type fixed point theorems in a generalized metric space, J. Math. Computer Sci. 16, 211-217, 2016.
[8] A. H.Ansari, Note on "φ-ψ-contractive type mappings and related fixed point", The 2nd regional conference on mathematics and applications, Payame Noor University, 377-380, 2014.
[9] Z. Kadelburg and S. Radenović, Fixed point results in generalized metric spaces without Hausdorff property, Math. Sci. 8:125, 2014.
[10] W. A. Kirk and N. Shahzad, Generalized metrics and Caristi's theorem, Fixed Point Theory and Appl. 129, 2013.
[11] M. S. Khan, M. Swaleh and S. Sessa, Fixed point theorems by altering distances between the points, Bulletin of the Australian Mathematical Society, 30,1, 1–9, 1984.
[12] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrals, Fundamenta Mathematicae 3 133-181, 1922.
[13] M. Abbas, N. Saleem and M. De la Sen, Optimal coincidence point results in partially ordered non-Archimedean fuzzy metric spaces, Fixed Point Theory and Applications, 1, 1–18, 2016.

[14] Lj. Ciric: A generalization of Banach's contraction principle. Proceedings of the American Mathematical society 45, 267-273, 1974.
[15] K. Fan: Extensions of two fixed point Theorems of F. E. Browder. Math. Z. 112, 234–240, 1969.
[16] M. Ozavsar, A. C. Cevikel, Fixed points of multiplicative contraction mappings on multiplicative metric spaces, arXiv, 2012.
[17] H. Piri, S. Rahrovi and P. Kumam, Khan type fixed point theorems in a generalized metric space, Journal of Mathematics and computer science, 16, 211-217, 2016.
[18] N. Saleem, B. Ali, M. Abbas and Z. Raza, Fixed points of Suzuki type generalized multivalued mappings in fuzzy metric spaces with applications, Fixed Point Theory and Applications, 1,1-18, 2015.
[19] N. Saleem, M. Abbas and Z. Raza, Optimal coincidence best approximation solution in non-Archimedean Fuzzy Metric Spaces, Iranian Journal of Fuzzy Systems, 13(3), 113-124, 2016.
[20] Z. Raza, N. Saleem and M. Abbas, Optimal coincidence points of proximal quasi-contraction mappings in non-Archimedean fuzzy metric spaces, Journal of nonlinear science and application, 9, 3787–3801, 2016.