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Orbital Continuity and Fixed Points

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Abstract. The aim of the present paper is to show the significance of the concept of orbital continuity introduced by Ciric. We prove that orbital continuity of a pair of R-weak commuting self-mappings of type A_f or of type A_g of a complete metric space is equivalent to fixed point property under Jungck type contraction. We also establish a situation in which orbital continuity is a necessary and sufficient condition for the existence of a common fixed point of a pair of mappings yet the mappings are necessarily discontinuous at the fixed point.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

In 1971 Ciric [1] introduced the notion of orbital continuity. If f is a self-mapping of a metric space (X, d) then the set $O(x, f) = \{f^n x : n = 0, 1, 2, \ldots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_i f^{m_i} x$ implies $fu = \lim_i f f^{m_i} x$. Every continuous self-mapping is orbitally continuous but not conversely [1]. Shastri et al [11] defined the notion of orbital continuity for a pair of mappings. If f and g are self-mappings of a metric space (X, d) and if $\{x_n\}$ is a sequence in X such that $fx_n = gx_{n+1}, n = 0, 1, 2, \ldots,$ then the set $O(x_0, f, g) = \{fx_n : n = 0, 1, 2, \ldots\}$ is called the (f, g)-orbit at x_0 and g (or f) is called (f, g)-orbitally continuous if $\lim_n fx_n = u$ implies $\lim_n gfx_n = gu$ (or $\lim_n fx_n = u$ implies $\lim_n ffx_n = fu$). We now give some relevant definitions.

Definition 1.1 ([4]). *Two self-mappings f and g of a metric space (X, d) are called R- weakly commuting if there exists some real number $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$ for all x in X .The mappings f and g are called point-wise R-weakly commuting on X if given x in X there exists $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$ (see [5]). The notion of point-wise R-weak commuting implies commutativity at coincidence points and is, therefore, equivalent to the notion of weak compatibility.*

Definition 1.2 ([3]). *Two self-mappings f and g of a metric space (X, d) are called compatible if* $\lim_n d(fgx_n, gfx_n) = 0$ *whenever $\{x_n\}$ is a sequence in X such that* $\lim_n fx_n = \lim_n gx_n = t$ *for some t in X .*

Definition 1.3 ([9]). *Two self-mappings f and g of a metric space (X, d) are called R- weakly commuting of type A_g if there exists some real number $R > 0$ such that $d(ffx, gfx) \leq Rd(fx, gx)$ for all x in X. Similarly, the self-mappings f and g are called R- weakly commuting of type A_f if there exists some real number $R > 0$ such that $d(fgx, ggx) \leq Rd(fx, gx)$ for all x in X .*

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Definition 1.4 ([10]). *Two self-mappings f and g of a metric space (X, d) are called g-compatible or f-compatible according as* $\lim_n d(ffx_n, gfx_n) = 0$ *or* $\lim_n d(fgx_n, ggx_n) = 0$ *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_n fx_n = \lim_n gx_n = t$ *for some t in X.*

Definition 1.5 ([8]). *Two self-mappings f and g of a metric space (X, d) are called compatible of type (P) if* $\lim_n d(ffx_n, ggx_n) = 0$ *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_n fx_n = \lim_n gx_n = t$ *for some t in X.*

In a recent work [7], the authors introduced the following definitions:

Definition 1.6. *Two self-mappings f and g of a metric space (X, d) are called quasi R-commuting provided there exists a positive real number R such that given x in X we have $d(ffx, gfx) \leq Rd(fx, gx)$ or $d(fgx, gfx) \leq Rd(fx, gx)$ or $d(fgx, ggx) \leq Rd(fx, gx)$ or $d(ffx, ggx) \leq Rd(fx, gx)$.*

Definition 1.7. *Two self-mappings f and g of a metric space (X, d) are called quasi α - compatible provided every sequence $\{x_n\}$ in X satisfying $\lim_n fx_n = \lim_n gx_n = t$ for some t in X splits up in at most four sub-sequences such that any of these sub-sequences, say $\{x_{n_i}\}$, satisfies at least one of the four conditions $\lim_{n_i} d(ffx_{n_i}, gfx_{n_i}) = 0, \lim_{n_i} d(fgx_{n_i}, gfx_{n_i}) = 0, \lim_{n_i} d(fgx_{n_i}, ggx_{n_i}) = 0$, and $\lim_{n_i} d(ffx_{n_i}, ggx_{n_i}) = 0$.*

We now introduce the following notions:

Definition 1.8. *Two self-mappings f and g of a metric space (X, d) will be called semi R-commuting provided there exists $R > 0$ such that $d(ffx, gfx) \leq Rd(fx, gx)$ or $d(fgx, gfx) \leq Rd(fx, gx)$ or $d(fgx, ggx) \leq Rd(fx, gx)$ or $d(ffx, ggx) \leq Rd(fx, gx)$ is true for the set $\{x \in X : fx, gx \in f(X) \cap g(X)\}$.*

Definition 1.9. *Two self-mappings f and g of a metric space (X, d) will be called semi α-compatible provided every sequence $\{x_n\}$ in X satisfying $fx_n, gx_n \in f(X) \cap g(X)$ and $\lim_n fx_n = \lim_n gx_n = t$ for some t in X satisfies $\lim_n d(ffx_n, gfx_n) = 0$ or $\lim_n d(fgx_n, gfx_n) = 0$ or $\lim_n d(fgx_n, ggx_n) = 0$ or $\lim_n d(ffx_n, ggx_n) = 0$.*

It is easy to see that semi R-commuting implies semi α-compatible. It is also obvious that mappings which are compatible or f-compatible or g-compatible or compatible of type (P) are semi α-compatible.

Example 1.10. *Let $X = [0, \infty)$ and d be the usual metric. Define $f, g : X \to X$ by*

$$fx = x/2 \text{ for each } x \text{ in } X, \qquad gx = x \text{ for each } x \text{ in } X.$$

Then f and g are commuting, R-weakly commuting, R-weakly commuting of type A_f, R-weakly commuting of type A_g as well as semi R-commuting. It can also be verified that f and g are semi α-compatible.

Example 1.11. *Let $X = [2, 20]$ equipped with the Euclidean metric. Define $f, g : X \to X$ by*

$$f2 = 2, \quad fx = 3 \text{ if } 2 < x \leq 5, \quad fx = 3 \text{ if } x > 5,$$
$$g2 = 2, \quad gx = 12 \text{ if } 2 < x \leq 5, \quad gx = (x+1)/2 \text{ if } x > 5.$$

Then $d(fgx, gfx) \leq d(fx, gx)$ for all x satisfying $fx, gx \in f(X) \cap g(X)$, that is, f and g are semi R-commuting with $R = 1$. However, f and g are not quasi R-commuting. For example if we take $x_n = 5 + 1/n$ then $\lim_n fx_n = \lim_n gx_n = 3, \lim_n d(fx_n, gx_n) = 0, d(ffx_n, gfx_n) = 9, d(fgx_n, gfx_n) = 9, d(fgx_n, ggx_n) = 9, d(ffx_n, ggx_n) = 9$. Thus f and g fail to be quasi R-commuting. These computations also show that f and g are neither compatible, nor f-compatible, nor g-compatible, nor compatible of type (P). The notion of semi R-commuting is thus a proper generalization of these four conditions.

Example 1.12. *Let $X = [2, 11]$ equipped with the Euclidean metric. Define $f, g : X \to X$ by*

$$fx = (6-x)/2 \text{ if } x \leq 2, \quad fx = 3 \text{ if } 2 < x \leq 5, \quad fx = (11-x)/3 \text{ if } x > 5,$$
$$gx = x \text{ if } x \leq 2, \quad gx = 10 \text{ if } 2 < x \leq 5, \quad gx = (x+1)/3 \text{ if } x > 5.$$

Then for each $x \leq 2$ satisfying $fx, gx \in f(X) \cap g(X)$ we have $d(fgx, ggx) \leq d(fx, gx)$. On the other hand, for each x satisfying $5 < x \leq 8$ and $fx, gx \in f(X) \cap g(X)$ we have $d(ffx, gfx) \leq d(fx, gx)$. This shows that f and g are not semi R-commuting. However, f and g can be shown to be quasi R-commuting.

Examples 1.11 and 1.12 demonstrate that quasi R-commuting and semi R-commuting are independent notions. However, the notion of semi R-commuting is much easier to employ when both the conditions hold.

2. Main Results

Theorem 2.1. *Let f and g be R-weakly commuting self-mappings of type A_f or of type A_g of a complete metric space (X, d) such that $f(X) \subseteq g(X)$ and*

(i) $d(fx, fy) \leq hd(gx, gy), 0 \leq h < 1.$

Then f and g have a common fixed point if and only if f and g are (f, g)-orbitally continuous.

Proof. Let x_0 be any point in X. Define sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_n = fx_n = gx_{n+1}, \; n = 0, 1, 2, \ldots. \tag{1}$$

This can be done since $f(X) \subseteq g(X)$. Now using a standard argument and by virtue of (i) it follows easily that $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point t in X such that $y_n \to t$ as $n \to \infty$. Also, $\lim_n fx_n = t$ and $\lim_n gx_n = t$. Let us assume that f and g are orbitally continuous. Then

$$\lim_n fgx_n = \lim_n ffx_n = ft, \text{ and} \tag{2}$$

$$\lim_n ggx_n = \lim_n gfx_n = gt. \tag{3}$$

Suppose f and g are R-weakly commuting of type A_g. Then $d(ffx_n, gfx_n) \leq Rd(fx_n, gx_n)$. This, in view of (2) and (3) implies that $ft = gt$. Now if $t \neq ft$, using (i) we get

$$d(fx_n, ft) \leq hd(gx_n, gt).$$

On letting $n \to \infty$ this yields, $d(t, ft) \leq hd(t, gt) = hd(t, ft)$, that is, $t = ft = gt$. Hence t is a common fixed point of f and g. The proof is similar if f and g are R-weakly commuting of type A_f. Moreover, condition (i) implies uniqueness of the common fixed point.

Conversely let us assume that the mappings f and g satisfy (i) and possess a common fixed point, say z. Then $z = fz = gz$. Also, the (f, g)-orbit of any point x_0 defined by (1) converges to z, that is, $\lim_n fx_n = \lim_n gx_n = z$. Suppose that f and g are R-weakly commuting of type A_g. Then we have $d(ffx_n, gfx_n) \leq Rd(fx_n, gx_n)$. This implies

$$\lim_n d(ffx_n, gfx_n) = 0. \tag{4}$$

Now by virtue of (i) we have

$$\begin{aligned} d(ffx_n, fz) &\leq hd(gfx_n, gz) \\ &\leq h\{d(gfx_n, ffx_n) + d(ffx_n, gz)\} = h\{d(gfx_n, ffx_n) + d(ffx_n, fz)\}. \end{aligned}$$

This yields $(1 - h)d(ffx_n, fz) \leq hd(ffx_n, gfx_n)$ which, in view of (4), yields $\lim_n ffx_n = fz = z$. Hence f is (f, g)-orbitally continuous. Also $\lim_n d(ffx_n, gfx_n) = 0$ implies $\lim_n gfx_n = fz = gz$, that is, g is (f, g)-orbitally continuous. Similarly, f and g are orbitally continuous if f and g are assumed R- weakly commuting of type A_f. This establishes the theorem. □

The following examples illustrate the above theorem.

Example 2.2. *Let $X = [0, \infty)$ and d be the usual metric. Define $f, g : X \to X$ by*

$$fx = x/2 \text{ for each } x \text{ in } X, \qquad gx = x \text{ for each } x \text{ in } X.$$

Then it is easily seen that f and g satisfy all the conditions of the above theorem and have a unique common fixed point $x = 0$.

Example 2.3. *Let $X = [2, 20]$ and d be the usual metric. Define $f, g : X \to X$ by*

$$\begin{aligned} fx &= 2 \text{ if } x = 2 \text{ or } > 5, \quad fx = 6 \text{ if } 2 < x \leq 5, \\ g2 &= 2, \quad gx = 12 \text{ if } 2 < x \leq 5, \quad gx = (x+1)/3 \text{ if } x > 5. \end{aligned}$$

Then the mappings f and g are R-weakly commuting mappings of type A_g, $f(X) \subseteq g(X)$, $d(fx, fy) \leq (4/5)d(gx, gy)$, and $x = 2$ is the unique common fixed point of f and g. It is also easy to see that f and g are (f, g)-orbitally continuous.

Remark 2.4. *The mappings f and g in Example 2.3 are non-compatible. If we consider the sequence $\{x_n = 5 + 1/n : n \geq 1\}$ then $\lim_n fx_n = 2, \lim_n gx_n = 2, \lim_n fgx_n = 6$ and $\lim_n gfx_n = 2$. Hence f and g are non-compatible. In view of non-compatibility of f and g and following the proof of Theorem 2 in Pant [6] it follows that both f and g are discontinuous at the common fixed point $x = 2$, though both the mappings are orbitally continuous. The contraction condition (i) pertaining to a pair of mappings employed in the above theorem was introduced by Jungck [2] and is often referred to as Jungck contraction condition.*

Theorem 2.5. *Let f and g be orbitally continuous self-mappings of a complete metric space (X, d) such that $f(X) \subseteq g(X)$ and*

(ii) $d(fx, fy) \leq hd(gx, gy), 0 \leq h < 1.$

If f and g are semi R-commuting then f and g have a coincidence point which is their unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{y_n\}$ and $\{x_n\}$ in X as in (1) above. Then $\{y_n\}$ is a Cauchy sequence and there exists a point t in X such that $y_n \to t$ as $n \to \infty$ and $\lim_n fx_n = \lim_n gx_n = t$. Orbital continuity of f and g implies that (2) and (3) hold. Since the sequence $\{x_n\}$ satisfies $fx_n, gx_n \in f(X) \cap g(X)$ and $\lim_n fx_n = \lim_n gx_n = t$, semi R-commutativity of f and g implies that $d(ffx_n, gfx_n) \leq Rd(fx_n, gx_n)$ or $d(fgx_n, gfx_n) \leq Rd(fx_n, gx_n)$, or $d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n)$ or $d(ffx_n, ggx_n) \leq Rd(fx_n, gx_n)$. Suppose $d(ffx_n, gfx_n) \leq Rd(fx_n, gx_n)$ is satisfied. This implies (4), that is, $\lim_n d(ffx_n, gfx_n) = 0$. This, in view of (2) and (3) implies that $ft = gt$. Thus, semi R-commutativity in combination with orbital continuity implies that t is a coincidence point of f and g. It may be observed here that weak compatibility will not imply $ft = gt$ since weak compatibility does not imply (4). Now if $t \neq ft$, using (ii) we get

$$d(fx_n, ft) \leq hd(gx_n, gt).$$

This yields $t = ft = gt$. Hence t is a common fixed point of f and g. The proof follows on similar lines when $d(fgx_n, gfx_n) \leq Rd(fx_n, gx_n)$ or $d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n)$ or $d(ffx_n, ggx_n) \leq Rd(fx_n, gx_n)$. Uniqueness of the coincidence point or the common fixed point is a consequence of (ii). □

We now give an example to illustrate the above theorem:

Example 2.6. *Let $X = [0, 11]$ and d be the Euclidean metric. Define $f, g : X \to X$ by*

$$\begin{aligned} fx &= (6-x)/2 \text{ if } x \leq 2, \quad fx = 3 \text{ if } 2 < x \leq 5, \quad fx = 2 \text{ if } x > 5, \\ gx &= x \text{ if } x \leq 2, \quad gx = 10 \text{ if } 2 < x \leq 5, \quad gx = (x+1)/3 \text{ if } x > 5. \end{aligned}$$

Then f and g satisfy all the conditions of Theorem 2.5 and have a unique common fixed point $x = 2$. It can be seen in this example that $d(ffx, gfx) \leq d(fx, gx)$ whenever $fx, gx \in f(X) \cap g(X)$. Therefore the mappings f and g are semi R-commuting with $R = 1$. It can also be verified that f and g satisfy the contractive condition $d(fx, fy) \leq \frac{1}{2}d(gx, gy)$ for all x, y in X. Moreover, it is also easy to see that f and g are orbitally continuous mappings. It may be seen in this example that f and g are neither compatible, nor f-compatible, nor g-compatible nor compatible of type (P).

Remark 2.7. *It is worth noting that in Theorem 2.5 we cannot replace semi R-commuting by pointwise R-weak commuting (equivalently weak compatibility). This can be seen from the following example.*

Example 2.8. *Let $X = [2, 20]$ and d be the Euclidean metric. Define $f, g : X \to X$ by*

$$fx = 6 \text{ if } 2 \leq x \leq 5, \qquad fx = (x+7)/6 \text{ if } x > 5,$$
$$gx = 15 \text{ if } 2 \leq x \leq 5, \qquad gx = (x+1)/3 \text{ if } x > 5.$$

Then f and g satisfy the following conditions but do not have a common fixed point or a coincidence point:

a. $f(X) = (2, 9/2] \cup \{6\}$, $g(X) = (2, 7] \cup \{15\}$, $f(X) \subseteq g(X)$,
b. *f and g satisfy the contraction condition $d(fx, fy) \leq \frac{1}{2}d(gx, gy)$,*
c. *f and g are pointwise R-weakly commuting and vacuously weak compatible,*
d. *f and g are orbitally continuous. To see this, let $\{fx_n = gx_{n+1}, n = 0, 1, 2, \ldots\}$ be the (f, g)-orbit of some point x_0 in X. Then $x_n \to 5$ with $x_n > 5$, $\lim_n fx_n = \lim_n gx_n = 2, \lim_n ffx_n = \lim_n fgx_n = 6 = f2$, $\lim_n gfx_n = \lim_n ggx_n = 15 = g2$. Therefore f and g are orbitally continuous mappings.*

It may be observed that the mappings f and g in the above example are not semi R-commuting. This example and Theorem 2.5 very well demonstrate that while semi R-commuting condition is useful in establishing the existence of coincidence points and also implies commutativity at coincidence points, weak compatibility or pointwise R-weak commutativity may not ensure the existence of coincidence points. Proceeding on similar lines as in Theorem 2.5 we can prove the following:

Theorem 2.9. *Let f and g be (f, g)-orbitally continuous self-mappings of a complete metric space (X, d) such that $f(X) \subseteq g(X)$ and*

(iii) $d(fx, fy) \leq hd(gx, gy), 0 \leq h < 1.$

If f and g are semi α-compatible then f and g have a coincidence point which is their unique common fixed point.

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References

[1] Lj. B. Ciric, On contraction type mappings, Math. Balkanica 1(1971) 52-57.
[2] G. Jungck, Commuting mappings and common fixed points, Amer. Math. Monthly 83(1976) 261-263.
[3] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9(1986) 771-779.
[4] R. P. Pant, Common fixed points of noncommuting mappings, J. Math. Anal. Appl. 188(1994) 436-440.
[5] R. P. Pant, Common fixed points of contractive maps, J. Math. Anal. Appl. 226(1998) 251-258.
[6] R. P. Pant, Discontinuity and fixed points, J. Math. Anal. Appl. 240 (1999) 284-289.
[7] R.P. Pant and Abhijit Pant, Fixed point theorems under new commuting conditions, J. International Acad. Phys. Sci. 17-1(2013) 1-6.
[8] H.K. Pathak, Y.J. Cho, S.M. Kang and B.S. Lee, Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming, Le Matematiche Vol. L(1995) Fasc. I, 15-33.
[9] H.K. Pathak, Y.J. Cho and S.M. Kang, Remarks on R-weakly commuting mappings and common fixed point theorems, Bull. Korean Math. Soc. 34 (1997) 247-257.
[10] H.K. Pathak and M.S. Khan, A comparison of various types of compatible maps and common fixed points, Indian J. Pure Appl. Math. 28- 4(1997) 477-485.
[11] K.P.R. Shastri, S.V.R. Naidu, I.H.N. Rao and K.P.R. Rao, Common fixed points for asymptotically regular mappings, Indian J. Pure Appl. Math. 15-8(1984) 849-854.