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Fixed Points and Continuity of Contractive Maps

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Abstract. The aim of this paper is to generalize celebrated results due to Boyd and Wong [2] and Matkowski [9] and also to provide yet new solutions to the once open problem on the existence of a contractive mapping which possesses a fixed point but is not continuous at the fixed point. Besides continuous mappings our results also apply to discontinuous mappings which include threshold operations that are integral part of many a phenomena.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

Discontinuities occur naturally in diverse biological, industrial and economic phenomena and these phenomena involve threshold operations which are discontinuous. For example, a neuron in a neural net either fires (corresponding to function value = 1) or does not fire (corresponding to function value =0) depending on whether the input crosses a certain threshold or not. Various industrial censors, band pass filters and the diode also work in this manner. Cromme and Diener [5] and Cromme [6] have proved results on approximate fixed points for such functions and have given applications of their results to neural nets, economic equilibria and analysis. We show that many a functions representing threshold operations satisfy weaker forms of continuity and new types of contractive conditions; and possess fixed points. Our results generalize results due to Boyd and Wong [2], Kannan [7, 8], Matkowski [9], Pant [11] and Bisht and Pant [1] and also provide yet new solutions to the once open problem on the existence of a contractive mapping which possesses a fixed point but is not continuous at the fixed point (see Rhoades [12], p.242). Moreover, our results apply to continuous as well as discontinuous mappings.

In 1971 Ciric [3] (see also [4]) introduced the notion of orbital continuity. If f is a self-mapping of a metric space (X, d) then the set $O(x, f) = \{f^n x : n = 0, 1, 2, \ldots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = lim_i f^{m_i} x$ implies $fu = lim_i f f^{m_i} x$.
Continuity of f obviously implies orbital continuity but not conversely [3]. We now define another weaker form of continuity.

Definition 1.1. *A self-mapping f of a metric space X will be called k-continuous, $k = 1, 2, 3, \ldots$, if $f^k x_n \to ft$ whenever $\{x_n\}$ is a sequence in X such that $f^{k-1} x_n \to t$.*

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Example 1.2. *Let $X = [0, 2]$ equipped with the usual metric and $f : X \to X$ be defined by*

$$fx = 1 \text{ if } 0 \leq x \leq 1, \qquad fx = 0 \text{ if } x > 1.$$

Then $fx_n \to t \Rightarrow f^2x_n \to t$ since $fx_n \to t$ implies $t = 0$ or $t = 1$ and $f^2x_n = 1$ for all n, that is, $f^2x_n \to 1 = ft$. Hence f is 2-continuous. However, f is discontinuous at $x = 1$.

Example 1.3. *Let $X = [0, 4]$ equipped with the usual metric. Define $f : X \to X$ by*

$$fx = 1 \text{ if } 0 \leq x \leq 1, \qquad fx = 0 \text{ if } 1 < x \leq 3, \qquad fx = x/3 \text{ if } 3 < x \leq 4.$$

Then $f^2x_n \to t \Rightarrow f^3x_n \to ft$ since $f^2x_n \to t$ implies $t = 0$ or $t = 1$ and $f^3x_n = 1 = ft$ for each n. Hence f is 3-continuous. However, $fx_n \to t$ does not imply $f^2x_n \to ft$, that is, f is not 2-continuous.

Example 1.4. *Let $X = [0, 2]$ and d be the usual metric. Define $f : X \to X$ by*

$$fx = (1 + x)/2 \text{ if } 0 \leq x \leq 1, \qquad fx = 0 \text{ if } x > 1.$$

Then it can be verified that f is 2-continuous but not continuous. Moreover, f^k is discontinuous for each positive integer k. Thus 2-continuity of f does not imply continuity of f^2. In general, k-continuity of f does not imply continuity of f^n.

Example 1.5. *Let $X = [0, 3] \cup (4, 5)$ equipped with usual metric and let $f : X \to X$ be defined by*

$$fx = 1 \text{ if } 0 \leq x \leq 1, \qquad fx = 0 \text{ if } 1 < x \leq 3, \qquad fx = x/4 \text{ if } 4 < x < 5.$$

Then f^2 is continuous but f is not 2-continuous. If we consider the sequence $\{x_n\}$ given by $x_n = 4 + 1/n$ then $fx_n \to 1$ but $f^2x_n \to 0 \neq f1$. Hence f is not 2-continuous.

The above examples show that continuity of f^k and k-continuity of f are independent conditions when $k > 1$. It is easy to see that 1- continuity is equivalent to continuity and

continuity $\Rightarrow$ 2-continuity $\Rightarrow$ 3-continuity $\Rightarrow \ldots,$

but not conversely.

We now prove a fixed point theorem by employing the above definition and a new type of $(\epsilon - \delta)$ condition.

2. Results

Theorem 2.1. *Let f be a self-mapping of a complete metric space (X, d) such that*

(i) $d(fx, fy) < \max\{d(x, fx), d(y, fy)\}$, whenever $\max\{d(x, fx), d(y, fy)\} > 0$,
(ii) Given $\epsilon > 0$ there exists a $\delta > 0$ such that $\epsilon < \max\{d(x, fx), d(y, fy)\} \leq \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon$.

If f is k-continuous or f^k is continuous for some $k \geq 1$ or f is orbitally continuous then f possesses a unique fixed point.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = fx_{n-1}$. If $x_n = x_{n+1}$ for some n then x_n is a fixed point of f. If $x_n \neq x_{n+1}$ for each n, then using (i) we get

$$d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) < \max\{d(x_{n-1}, fx_{n-1}), d(x_n, fx_n)\} = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n).$$

Thus $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence of positive real numbers and, hence, tends to a limit $r \geq 0$. Suppose $r > 0$. Then there exists a positive integer N such that

$$n \geq N \Rightarrow r < d(x_n, x_{n+1}) \leq r + \delta(r). \tag{1}$$

This yields $r < max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = max\{d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\} \leq r + \delta(r)$ which by virtue of (ii) yields $d(fx_n, fx_{n+1}) = d(x_{n+1}, x_{n+2}) \leq r$. This contradicts (1). Hence $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Now if p is any positive integer then

$$\begin{aligned} d(x_n, x_{n+p}) &= d(fx_{n-1}, fx_{n+p-1}) \\ &< max\{d(x_{n-1}, fx_{n-1}), d(x_{n+p-1}, fx_{n+p-1})\} \\ &= max\{d(x_{n-1}, x_n), d(x_{n+p-1}, x_{n+p})\} \\ &= d(x_{n-1}, x_n). \end{aligned}$$

This implies that $d(x_n, x_{n+p}) \to 0$ since $d(x_{n-1}, x_n) \to 0$. Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists t in X such that $x_n \to t$. Moreover, for each $k \geq 1$ we have, $f^k x_n \to t$.

Now suppose that f is k-continuous. Since $f^{k-1}x_n \to t$, k continuity of f implies that $f^k x_n \to ft$. Hence $t = ft$ as $f^k x_n \to t$. Therefore, t is fixed point of f.

Next suppose that f^k is continuous for some positive integer k. Then, $\lim_{n\to\infty} f^k x_n = f^k t$. This yields $f^k t = t$ as $f^k x_n \to t$. If $t \neq ft$ we get

$$\begin{aligned} d(t, ft) = d(f^k t, f^{k+1} t) &< max\{d(f^{k-1}t, f^k t), d(f^k t, f^{k+1} t)\} \\ &= d(f^{k-1}t, f^k t) < d(f^{k-2}t, f^{k-1}t) < \ldots < d(t, ft), \end{aligned}$$

a contradiction. Hence $t = ft$ and t is a fixed point of f.

Finally, suppose that f is orbitally continuous. Since $x_n \to t$, orbital continuity implies that $fx_n \to ft$. This gives $t = ft$ as $fx_n \to t$. Thus t is a fixed point of f. Uniqueness of the fixed point follows from (i). □

The following examples illustrate the above theorem.

Example 2.2. *Let $X = [0, 2]$ and d be the usual metric. Define $f : X \to X$ by*

$$fx = 1 \text{ if } 0 \leq x \leq 1, \qquad fx = 0 \text{ if } 1 < x \leq 2.$$

Then f satisfies all the conditions of the above theorem and has a unique fixed point $x = 1$; and f is discontinuous at the fixed point. The mapping f is 2-continuous, f^2 is continuous and f is also orbitally continuous. It can be easily verified that

$$\begin{aligned} d(fx, fy) = 0, &\quad 0 < max\{d(x, fx), d(y, fy)\} \leq 1 \text{ if } x, y \leq 1, \\ d(fx, fy) = 0, &\quad 1 < max\{d(x, fx), d(y, fy)\} \leq 2 \text{ if } x, y > 1, \\ \text{and } d(fx, fy) = 1, &\quad 1 < max\{d(x, fx), d(y, fy)\} \leq 2 \text{ if } x \leq 1, y > 1. \end{aligned}$$

Therefore, f satisfies condition (ii) with $\delta(\epsilon) = 1-\epsilon$ if $\epsilon < 1$ and $\delta(\epsilon) = 1$ for $\epsilon \geq 1$. The function f is, clearly, a threshold operation that models firing of a neuron, function of a diode, and also a low pass filter that allows low voltages to pass but not higher voltages (e.g. noise in music systems). It may be observed here that the $(\epsilon - \delta)$ condition (ii) used in the above theorem describes Example 2.2 better than the $(\epsilon - \delta)$ condition $\epsilon < max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon$ used in Pant [11] and Bisht and Pant [1] since $max\{d(x, fx), d(y, fy)\}$ can assume the value 1 when $x, y \leq 1$ in condition (ii) but not in the corresponding conditions used in Pant [11] or Bisht and Pant [1]. It may also be seen that the function f in this example does not satisfy the Meir and Keeler [10] type $(\epsilon - \delta)$ contractive condition

$$\epsilon \leq max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \Rightarrow d(fx, fy) < \epsilon.$$

Example 2.3. *Let $X = [-1, 3]$ and d be the usual metric. Define $f : X \to X$ by*

$$fx = 1/2 \text{ if } -1 \leq x < 0, \quad fx = 1 \text{ if } 0 \leq x \leq 1, \quad fx = 0 \text{ if } 1 < x \leq 2, \quad fx = -1 \text{ if } 2 < x \leq 3.$$

Then f satisfies the conditions of Theorem 2.1 and has a unique fixed point $x = 1$. It may be noted that f^3 is continuous since $f^3 x = 1$ for all x in X.

Remark 2.4. *Theorem 2.1 above generalizes Theorem 2.1 of Bisht and Pant [1] since the above theorem assumes continuity of f^k instead of assuming continuity of f^2. In Example 2.3, f^3 is continuous but not f^2. Similarly, Theorem 2.1 above generalizes Theorem 1 of Pant [11] since the condition $d(fx, fy) \leq \phi(max\{d(x, fx), d(y, fy)\})$ of Theorem 1 of Pant [11] ensures the existence of a fixed point and also implies orbital continuity of the mapping f. Our theorem also generalizes the results of Kannan [7, 8] since the Kannan contraction condition $d(fx, fy) \leq a[d(x, fx) + d(y, fy)], 0 \leq a < 1/2$, implies conditions (i) and (ii) of Theorem 2.1 but not conversely. Preliminary investigations by the first author suggest that fixed point property for mappings satisfying conditions (i) and (ii) of Theorem 2.1 have several equivalent characterizations including orbital continuity of the mapping. However, detailed investigation of the equivalent characterizations will be a work of different nature than the present work and, therefore, is not taken up here.*

In 1969 Boyd and Wong [2] obtained a remarkable generalization of the Banach contraction theorem by proving the following theorem:

Theorem 2.5 ([2]). *Let T be a mapping of a complete metric space (X, d) into itself. Suppose there exists a function ϕ, upper semi continuous from right from R_+ into itself such that $d(Tx, Ty) \leq \phi(d(x, y))$, for all x, y in X. If $\phi(t) < t$ for each $t > 0$, then t has a unique fixed point.*

Boyd and Wong theorem is a result of great significance and has found numerous applications. A mapping T satisfying $d(Tx, Ty) \leq \phi(d(x, y)), \phi(t) < t$ for each $t > 0$, may not possess a fixed point unless some additional condition is assumed on ϕ. Boyd and Wong [2] assumed ϕ to be upper semi-continuous from the right while Matkowski [9] assumed ϕ to be non-decreasing and $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$. However, often ϕ may be continuous or upper semi continuous only in the open interval $(0, d(f(X)))$ instead of in whole of R_+ , where $d(f(X))$ denotes the diameter of the set $f(X)$ - the range of f. Similarly, $\lim_{n\to\infty} \phi^n(t) = 0$ may hold for $t \in (0, d(f(X)))$ only. We now extend and generalize the scope of celebrated theorems by Boyd and Wong [2] and Matkowski [9] to include such possibilities and obtain a unified analogue of these theorems. In the following, $d(X)$ and $d(f(X))$ will respectively denote the diameter of X and the diameter of the range of f.

Theorem 2.6. *Let f be a self-mapping of a complete metric space (X, d) satisfying*

(iii) $d(fx, fy) \leq \phi(max\{d(x, fx), d(y, fy)\})$, for all x, y in X,

where the function $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$. If ϕ is upper semi-continuous in the open interval $(0, d(f^k(X)))$ or if ϕ is non-decreasing and $\lim_{n\to\infty} \phi^n(t) = 0$ for each t in $(0, d(f^k(X))), k = 0$ or 1, then f has a unique fixed point.

Proof. The case $k = 0$ is analogous to the Boyd and Wong [2] theorem and the Matkowski's [9] theorem. We, therefore, consider the case $k = 1$ only. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = fx_{n-1}$. We can assume that $x_n \neq x_{n+1}$ for each n otherwise x_n is a fixed point and the proof is complete. Then using (iii) we get

$$\begin{aligned} d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) &\leq \phi(max\{d(x_{n-1}, fx_{n-1}), d(x_n, fx_n)\}) \\ &= \phi(max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \phi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n). \end{aligned}$$

Thus $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence of positive real numbers and, hence, tends to a limit $r \geq 0$. Also, $d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) < d(x_n, x_{n-1}) = d(fx_{n-1}, fx_{n-2})$ implies that $r < d(f(X))$. Suppose that ϕ is upper semi continuous in $(0, d(f(X)))$. If possible, let $r > 0$. Then, since ϕ is upper semi continuous in $(0, d(f(X)))$ and $d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n))$, taking limit as $n \to \infty$ we get $r \leq \phi(r) < r$, a contradiction. Hence $r = 0$, that is, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Next, suppose that $\lim_{n\to\infty} \phi^n(t) = 0$ for each t in $(0, d(f(X)))$. Then, since ϕ is non-decreasing, we get $d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n)) \leq \phi^2(d(x_{n-2}, x_{n-1})) \leq \ldots \leq \phi^{n-2}(d(x_2, x_3))$ and $d(x_2, x_3) < d(x_1, x_2) = d(fx_0, fx_1) \leq$

$d(f(X))$. Taking limit we get $\lim_{n\to\infty} d(x_n, x_{n+1}) \le \lim_{n\to\infty} \phi^{n-2}(d(x_2, x_3)) = 0$. Thus in either case we get $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.
Now if p is any positive integer, we have

$$\begin{aligned} d(x_n, x_{n+p}) = d(fx_{n-1}, fx_{n+p-1}) &\le \phi(max\{d(x_{n-1}, fx_{n-1}), d(x_{n+p-1}, fx_{n+p-1})\}) \\ &= \phi(max\{d(x_{n-1}, x_n), d(x_{n+p-1}, x_{n+p})\}) < d(x_{n-1}, x_n). \end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, on taking limit as $n \to \infty$ the above inequality yields $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$. This shows that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point z in X such that $x_n \to z$. Also $fx_n \to z$. If $z \neq fz$, using (iii) for large values of n we get

$$d(fx_n, fz) \le \phi(max\{d(x_n, fx_n), d(z, fz)\} = \phi(d(z, fz)).$$

Taking limit as $n \to \infty$ the above inequality yields $d(z, fz) \le \phi(d(z, fz)) < d(z, fz)$, a contradiction. Hence $z = fz$ and z is fixed point of f. Uniqueness of the fixed point follows from (iii). □

The next example illustrates the above theorem.

Example 2.7. *Let $X = [0, 2]$ and d be the usual metric. Define $f : X \to X$ by*

$$fx = 1 \text{ if } 0 \le x \le 1, \qquad fx = 0 \text{ if } 1 < x \le 2.$$

Then f satisfies the conditions of Theorem 2.6 and has a unique fixed point $x = 2$. The function f satisfies condition (iii) with $\phi(t) = (1 + t)/2$ if $t > 1$ and $\phi(t) = t/2$ if $t \le 1$. It is easily seen that $d(f(X)) = 1$ and ϕ is continuous in the open interval $(0, 1)$ but f is discontinuous at the fixed point $x = 1$. It can also be seen that ϕ is non-decreasing and if $t > 1$ then $\lim_{n\to\infty} \phi^n(t) = \lim_{n\to\infty}(2^n - 1 + t)/2^n = 1$. Therefore, the function f does not satisfy Matkowski's [9] condition $\lim_{n\to\infty} \phi^n(t) = 0$. However, $\lim_{n\to\infty} \phi^n(t) = \lim_{n\to\infty} t/2^n = 0$ for each t in $(0, d(f(X))) = (0, 1)$. Thus Theorem 2.6 gives a proper generalization of the Boyd and Wong type theorems as well as Matkowski type theorems.

Remark 2.8. *Theorems 2.1 and 2.6 not only generalize the results mentioned above but also provide new solutions to the once open problem on the existence of a contractive mapping which possesses a fixed point but is not continuous at the fixed point (see [12], p. 242).*

In Theorem 2.1 the $(\epsilon - \delta)$ condition (ii) has been used with the contractive condition (i) because condition (ii) by itself does not imply a contractive condition. Mappings satisfying non-expansive type conditions may also satisfy condition (ii). We now prove a fixed point theorem for a non-expansive mapping satisfying $(\epsilon - \delta)$ condition. Such non-expansive mappings may be called $(\epsilon - \delta)$ non-expansive mappings. For the purpose of the next theorem we shall denote:

$$m(x, y) = max\{d(x, fx), d(y, fy)\} \text{ and } M = sup\{m(x, y) : x, y \in X\}.$$

Theorem 2.9. *Let f be a continuous self-mapping of a complete metric space (X, d) satisfying*

(iv) $d(fx, fy) \le m(x, y)$ *for all x, y in X and* $d(fx, f^2x) \le d(x, fx)$ *for all x in X,*

(v) given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon < m(x, y) \le \epsilon + \delta \Rightarrow d(fx, fy) \le \epsilon,$$

(vi) $inf\{m(x, y) : x, y \in X\} < M$, *that is, $m(x, y)$ is not constant on $X \times X$.*

If $\delta(\epsilon)$ is continuous in the open interval $(0, M)$ then f possesses a unique fixed point.

Proof. By virtue of (iv), for each y in X we have

$$d(f^n y, f^{n+1} y) \le d(f^{n-1} y, f^n y) \le d(f^{n-2} y, f^{n-1} y) \le \ldots \le d(y, fy), \tag{2}$$

that is, $\{d(f^n y, f^{n+1} y)\}$ is a non-increasing sequence of non-negative real numbers. Condition (vi) implies that there exist u, v in X such that $max\{d(u, fu), d(v, fv)\} < M$, that is, $d(u, fu) < M$ and $d(v, fv) < M$. By virtue of (2) we get

$$d(f^n u, f^{n+1} u) \le d(f^{n-1} u, f^n u) \le \ldots \le d(u, fu) < M. \tag{3}$$

If $f^n u = f^{n+1}u$ for some n then $f^n u$ is a fixed point of f and the proof is complete. We can therefore assume that $f^n u \neq f^{n+1}u$ for each n. Then $\{d(f^n u, f^{n+1}u)\}$ is a non-increasing sequence of positive real numbers and, hence, converges to a real number r such that $0 \leq r < M$. We assert that $r = 0$. If $r > 0$, continuity of $\delta(\epsilon)$ in $(0, M)$ implies that there exists $\epsilon_0 > 0$ such that

$$\epsilon_0 < r \leq \epsilon_0 + \delta(\epsilon_0).$$

Since $\lim_{n\to\infty} d(f^n u, f^{n+1}u) = r$ there exists a positive integer N such that

$$n \geq N \Rightarrow r \leq d(f^n u, f^{n+1}u) \leq \epsilon_0 + \delta(\epsilon_0), \tag{4}$$

that is, $\epsilon_0 < r \leq d(f^n u, f^{n+1}u) \leq \epsilon_0 + \delta(\epsilon_0)$. Since $d(f^n u, f^{n+1}u) = max\{d(f^n u, f^{n+1}u), d(f^{n+1}u, f^{n+2}u)\} = m(f^n u, f^{n+1}u)$, the last inequality yields $\epsilon_0 < m(f^n u, f^{n+1}u) \leq \epsilon_0 + \delta(\epsilon_0)$. This inequality, in view of (v) yields $d(f^{n+1}u, f^{n+2}u) \leq \epsilon_0 < r$. This contradicts (4). Hence $r = 0$, that is, $\lim_{n\to\infty} d(f^n u, f^{n+1}u) = 0$. If p is any positive integer, then using (iv) and (3) we have

$$d(f^n u, f^{n+p}u) \leq max\{d(f^n u, f^{n+1}u), d(f^{n+p}u, f^{n+p+1}u)\} = d(f^n u, f^{n+1}u).$$

Taking limit as $n \to \infty$ we get $\lim_{n\to\infty} d(f^n u, f^{n+p}u) = \lim_{n\to\infty} d(f^n u, f^{n+1}u) = 0$. Thus $\{f^n u\}$ is a Cauchy sequence. Since X is complete, there exists a point t in X such that $\lim_{n\to\infty} f^n u = t$. Continuity of f now implies that $\lim_{n\to\infty} f(f^n u) = ft$, that is, $\lim_{n\to\infty} f^n u = ft$. Hence $t = ft$ and t is a fixed point of f. Uniqueness of the fixed point follows from (iv). □

Example 2.10. *Let $X = [-1, 1]$ and d be the usual metric. Define $f : X \to X$ by*

$$fx = -|x|x, \text{ that is, } fx = -x^2 \text{ if } x \geq 0 \text{ and } fx = x^2 \text{ if } x < 0.$$

Then f satisfies all the conditions of Theorem 2.9 and has a unique fixed point $x = 0$. Also, f possesses two periodic points $x = 1$ and $x = -1$. It can be verified that f satisfies condition (iv) and satisfies condition (v) with $\delta(\epsilon) = (\sqrt{(\epsilon/2)}) - (\epsilon/2)$ if $\epsilon < 2$ and $\delta(2) = 2$. It is clear that $M = 2$ and $\delta(\epsilon)$ is continuous in the open interval $(0, M) = (0, 2)$. It may be noted that the function f does not satisfy any contractive condition. For example if we take $x = 1$ and $y = -1$ then $fx = -1, fy = 1, d(fx, fy) = 2, d(x, y) = 2, d(x, fx) = 2, d(y, fy) = 2, d(x, fy) = 0$ and $d(y, fx) = 0$. Therefore, f does not satisfy any contractive condition. It may be observed here that uniqueness of the fixed point in the above theorem is because of the particular form of the term $m(x, y)$. If we change the form of the term $m(x, y)$ then the fixed point need not be unique.

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