Filomat 31:11 (2017), 3091–3093
https://doi.org/10.2298/FIL1711091I

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Note on Generalized Quasi-Contraction

Dejan Ilić[a], Darko Kocev[b]

[a]*University of Nis, Department of Mathematics,Faculty of Sciences and Mathematics, Visegradska 33, 18000 Nis, Serbia.*
[b]*University of Belgrade , Technical Faculty in Bor, Vojske Jugoslavije 12, 19210 Bor, Serbia.*

Abstract. In this paper we give a short proof of the main results of Kumam, Dung and Sitthithakerngkiet (P. Kumam, N.V. Dung, K. Sitthithakerngkiet, A Generalization of Ćirić Fixed Point Theorems, FILOMAT **29**:7 (2015), 1549–1556).

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Preliminaries

In 1974, Lj. Ćirić [2] introduced the concept of quasi-contraction and proved the following fundamental result:

Theorem 1.1. *Let (X, d) be a metric space. Suppose that $T : X \mapsto X$ is a quasi-contraction, i.e. that there exists $r \in [0, 1)$ such that for all $x, y \in X$ there holds*

$$d(Tx, Ty) \leq r \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

If X is $T-$orbitally complete, then

1. *T has a unique fixed point x^* in X.*
2. $\lim_{n\to\infty} T^n x = x^*$.
3. *$d(T^n x, x^*) \leq \frac{r^n}{1-r} d(x, Tx)$ for all $x \in X$ and $n \in \mathbb{N}$.*

In the paper of Kumam, Dung and Sitthithakerngkiet [7] the concept of quasi-contraction was generalized to that of a generalized quasi-contraction:

Definition 1.2. *Let $T : X \mapsto X$ be a mapping on a metric space X. The mapping T is said to be a generalized quasi-contraction iff there exists $r \in [0, 1)$ such that for all $x, y \in X$*

$$\begin{aligned} d(Tx, Ty) \leq r \cdot \max\{&d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), \\ &d(T^2x, x), d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty)\}. \end{aligned}$$

2010 *Mathematics Subject Classification*. Primary 47H10, 54H25.
Keywords. Generalized quasi-contraction; $w-$distance.
Received: 10 December 2016; Accepted: 15 March 2017
Communicated by Vladimir Rakočević
This paper is supported by the Ministry of Science of Republic of Serbia, Grant No. 174025.
Email addresses: ilicde@ptt.rs (Dejan Ilić), dkocev@tfbor.bg.ac.rs (Darko Kocev)

In [7], the following theorem is proved:

Theorem 1.3. *Let (X, d) be a metric space. Suppose that $T : X \mapsto X$ is a generalized quasi-contraction and X is $T-$orbitally complete. Then we have*

1. *T has a unique fixed point x^* in X.*
2. *$\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.*
3. *$d(T^n x, x^*) \leq \frac{r^n}{1-r} d(x, Tx)$ for all $x \in X$ and $n \in \mathbb{N}$.*

In this note we give a short proof of the main results of Kumam, Dung and Sitthithakerngkiet [7] using the notion of $w-$distance.

The notions and facts we give below are well-known.

Let X be a set endowed with a metric d. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$,

(2) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous,

(3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Let us recall that a real-valued function f defined on a metric space X is said to be lower semicontinuous at a point x_0 of X if either $\liminf_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf_{x \to x_0} f(x_n)$, whenever $x_n \in X$ and $x_n \to x_0$.

The concept of w-distance was introduced by Kada, Suzuki and Takahashi [6]. They gave several examples of w-distances and improved Caristi's fixed point theorem [1], Ekland's variational's principle [3] and the nonconvex minimization theorem according to Takahashi [8].

For more about $w-$distances the reader is referred to [4–6].

The proof of the next lemma and theorem which will be used in the proof of the main result can be found in [6]:

Lemma 1.4. *Let X be a metric space with metric d, let p be a $w-$distance on X and let α be a function from X into $[0, \infty)$. Then the function $q : X \times X \mapsto [0, \infty)$ given by*

$$q(x, y) = \max\{\alpha(x), p(x, y)\} \quad \text{for every } x, y \in X$$

is also a $w-$distance.

Theorem 1.5. *Let X be a complete metric space, let p be a $w-$distance on X and let T be a mapping from X into itself. Suppose that there exists $r \in [0, 1)$ such that*

$$p(Tx, T^2 x) \leq r \cdot p(x, Tx)$$

for every $x \in X$ and that

$$inf\{p(x, y) + p(x, Tx) : x \in X\} > 0$$

for every $y \in X$ with $y \neq Ty$. Then there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $p(v, v) = 0$.

2. Main Results

Theorem 2.1. *Let X be a complete metric space with metric d, and let T be a mapping from X into itself. Suppose T is a generalized quasi-contraction, i.e., there exists $r \in [0, 1)$ such that*

$$\begin{aligned} d(Tx, Ty) \leq r \cdot \max\{&d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), \\ &d(T^2 x, x), d(T^2 x, Tx), d(T^2 x, y), d(T^2 x, Ty)\} \end{aligned}$$

for every $x, y \in X$. Then T has a unique fixed point.

Proof. By Lemma 2 in [2], the set $\{x, Tx, T^2x, \dots\}$ is bounded for every $x \in X$. Hence we can define a function $p : X \times X \mapsto [0, \infty)$ by

$$p(x, y) = \max\{\operatorname{diam}\{x, Tx, T^2x, \dots\}, d(x, y)\}$$

for every $x, y \in X$. By Lemma 1.4, p is a $w-$distance on X. Let $x \in X$. Then we have, using Lemma 1 in [2],

$$\begin{aligned} p(Tx, T^2x) &= \operatorname{diam}\{Tx, T^2x, T^3x, \dots\} \\ &= \sup_{n\in\mathbb{N}} \operatorname{diam}\{Tx, T^2x, T^3x, \dots, T^nx\} \\ &\leq \sup_{n\in\mathbb{N}} r \cdot \operatorname{diam}\{x, Tx, T^2x, \dots, T^nx\} \\ &= r \cdot \operatorname{diam}\{x, Tx, T^2x, \dots\} \\ &= r \cdot p(x, Tx). \end{aligned}$$

Let $y \in X$ with $y \neq TY$ and assume that there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \{p(x_n, y) + p(x_n, Tx_n)\} = 0.$$

Then, we have

$$d(x_n, Tx_n) \leq \operatorname{diam}\{x_n, Tx_n, T^2x_n, \dots\} \leq p(x_n, y) \to 0$$

and

$$d(x_n, y) \leq p(x_n, y) \to 0.$$

So, both $\{x_n\}$ and $\{Tx_n\}$ converge to y. Since T is a generalized quasi-contraction,

$$\begin{aligned} d(Tx_n, Ty) \leq q \cdot \max\{&d(x_n, y), d(x_n, Tx_n), d(y, Ty), d(x_n, Ty), d(y, Tx_n), \\ &d(T^2x_n, x_n), d(T^2x_n, Tx_n), d(T^2x_n, y), d(T^2x_n, Ty)\} \end{aligned}$$

for any $n \in \mathbb{N}$ and hence

$$\begin{aligned} d(y, Ty) &\leq r \cdot \max\{d(y, y), d(y, y), d(y, Ty), d(y, Ty), d(y, y), d(Ty, y), d(Ty, y), d(Ty, y), d(Ty, Ty)\} \\ &\leq r \cdot d(y, Ty). \end{aligned}$$

This is a contradiction. Hence, if $y \neq Ty$,

$$\inf\{p(x, y) + p(x, Tx) : x \in X\} > 0.$$

By Theorem 1.5, there exists a fixed point z of T. Clearly, the fixed point is unique. □

References

[1] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions,* Trans. Amer. Math. Soc. **215** (1976), 241–251.

[2] Lj. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 267–273.

[3] I. Ekelend, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. **1** (1979), 443–474.

[4] D. Ilić, V. Rakočević, *Common fixed points for maps on metric space with w−distance,* Applied Mathematics and Computation **199** (2008), 599–610.

[5] D. Ilić, V. Rakočević, *Kontraktivna preslikavanja na metričkim prostorima i uopštenja*, Prirodno-matematički fakulet, Niš, (2015).

[6] O. Kada, T. Suzuki, W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japonica **40**(2) (1996), 381–391.

[7] P. Kumam, N.V. Dung, K. Sitthithakerngkiet, *A Generalization of Ćirić Fixed Point Theorems*, FILOMAT **29**:7 (2015), 1549–1556.

[8] W. Takahashi, *A convexity in metric space and nonexpansive mappings*, I. Kodai Math. Sem. Pep. **22** (1970), 142–149.