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Dynamics of a Family of Orbitally Continuous Mappings

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Abstract. The aim of the present paper is to study the dynamics of a class of orbitally continuous non-linear mappings defined on the set of real numbers and to apply the results on dynamics of functions to obtain tests of divisibility. We show that this class of mappings contains chaotic mappings. We also draw Julia sets of certain iterations related to multiple lowering mappings and employ the variations in the complexity of Julia sets to illustrate the results on the quotient and remainder. The notion of orbital continuity was introduced by Lj. B. Ciric and is an important tool in establishing existence of fixed points.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Definitions and Preliminaries

A discrete dynamical system consists of a function and its iterates. We call the behaviour of points under iteration of a function the dynamics of the function. We first give some well known definitions relevant to our work.

Definition 1.1. *If f is a function, the orbit of a point x is the set of points $x, f(x), f^2(x), f^3(x), f^4(x), \ldots$ where f^2x denotes $f(f(x))$, $f^3(x)$ denotes $f(f(f(x)))$ etc.*

Orbits can be quite complicated sets even for simple mappings. However, orbits associated with fixed points and periodic points are especially simple and these play a central role in the study of an entire system.

Definition 1.2. *The point x is called a fixed point of f if $f(x) = x$. A point x is called a periodic point of period n if $f^n(x) = x$. The least positive integer n for which $f^n(x) = x$ is called the prime period of x. The prime period need not be a prime number (see Example 1.4 given below). The set of all iterates of a periodic point forms a periodic orbit.*

Definition 1.3. *A point x is an eventually fixed point of f if there exists an integer $N > 0$ such that $f^{(n+1)}(x) = f^n(x)$ for $n \geq N$. A point x is eventually periodic of period k if there exists $N > 0$ such that $f^{(n+k)}(x) = f^n(x)$ for $n \geq N$.*

Example 1.4 ([2, 3]). *Let S^1 denote the unit circle in the xy-plane and identify each point on the circle by the radian measure in a counter-clockwise direction of the angle between the positive x-axis and the ray beginning at the origin*

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and passing through the point. We, thus, denote a point in S^1 by its angle θ measured in radians and the point is determined by any angle of the form $\theta + 2k\pi$ for an integer k. Let $f(\theta) = 2\theta$. Then $f(\theta + 2k\pi) = f(\theta)$ and $f^n(\theta) = 2^n\theta$, so that θ is periodic point of period n if and only if $2^n\theta = \theta + 2k\pi$ for some integer k, i.e., if and only if $\theta = 2k\pi/(2^n - 1)$. Hence the periodic points of order n for f are the $(2^n - 1)^{th}$ roots of unity. For example, the periodic points of prime period 4 are given by $\theta = 2k\pi/15$. It follows that the set of periodic points is dense in S^1. It may be noted that $f(0) = 0$ is fixed while if $\theta = 2k\pi/2^n$, then $f^n(\theta) = 2k\pi$, so that θ is eventually a fixed point. It follows that the set of eventually fixed points is dense in S^1. This mapping is referred to as the doubling map on S^1 ([3], p.124).

Definition 1.5. *Let p be a periodic point of period n. A point x is forward asymptotic to p if $\lim_{k\to\infty} f^{kn}(x) = p$. The stable set of p denoted by $W^s(p)$, consists of all points forward asymptotic to p. If p is non-periodic, we define x to be forward asymptotic to p if $|f^i(x) - f^i(p)| \to 0$ as $i \to \infty$. If the sequence $|x|, |f(x)|, |f^2(x)|, |f^3(x)|, \ldots$ grows without bound, then x is defined to be forward asymptotic to ∞. The stable set of ∞, denoted by $W^s(\infty)$, consists of all points which are forward asymptotic to ∞.*

Definition 1.6. *Let p be a periodic point of prime period n. The point p is called hyperbolic if, $|(f^n)'(p)| \neq 1$ where $(f^n)'(p)$ denotes the derivative of $f^n(x)$ at $x = p$.*

The following Theorem tells about the behaviour under iteration of the points in the neighbourhood of a hyperbolic fixed point.

Theorem 1.7 ([2, 3]). *Let p be a hyperbolic fixed point of f and $f'(p)$ be the derivative of $f(x)$ at $x = p$. If $|f'(p)| < 1$, then there is an open interval U containing p such that if $x \in U$ then $f^n(x) \to p$ as $n \to \infty$. If $|f'(p)| > 1$, then there is an open interval U containing p such that if $x \in U$, $x \neq p$, then there exists $k > 0$ such that $f^k(x) \notin U$.*

Definition 1.8. *Let p be a periodic point of f with prime period n. If $|(f^n)'(p)| < 1$, then p is called an attracting periodic point (an attractor) or a sink. If $|(f^n)'(p)| > 1$, then p is called a repelling periodic point (a repeller) of f or a source.*

Most maps have only hyperbolic periodic points. However, non-hyperbolic periodic points often occur in families of maps. When this happens, the periodic point structure often undergoes a bifurcation.

Definition 1.9. *Let $f_c(x)$ be a parametrized family of functions. Then there is a bifurcation at c_0 if there exists $\epsilon > 0$ such that whenever a and b satisfy $c_0 - \epsilon < a < c_0$ and $c_0 < b < c_0 + \epsilon$, then the dynamics of $f_a(x)$ is different from the dynamics of $f_b(x)$.*

In other words, the dynamics of functions changes when the parameter value crosses through the point c_0.

Definition 1.10. *A mapping $f : J \to J$, J being an open interval, is said to be topologically transitive if for any pair of open sets $U, V \subset J$ there exists $k > 0$ such that $f^k(U) \cap V \neq \phi$.*

Intuitively, a topologically transitive map has points, which eventually move under iteration from one arbitrarily small neighbourhood to any other.

Definition 1.11. *A mapping $f : J \to J$, J an interval, is said to have sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in J$ and any neighbourhood N of x, there exist y in N and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.*

Definition 1.12. *Let V be a set. Then $f : V \to V$ is said to be chaotic on V, in the sense of Devaney ([2]), if*

a. f has sensitive dependence on initial conditions
b. f is topologically transitive
c. periodic points of f are dense in V.

Example 1.13 ([2, 3]). *If S^1 is the unit circle in the plane then the doubling map $f : S^1 \to S^1$ of Example 1.4 defined by $f(\theta) = 2\theta$ is chaotic. Since the angle between two points is doubled upon iteration, f is sensitive to initial conditions. Topological transitivity also follows from this observation since any small arc in S^1 is eventually expanded by some f^k to cover all of S^1, and in particular, any arc in S^1. The density of periodic points has already been observed in Example 1.4 above. For more details on the chaotic behaviour of the doubling map one can refer to Holmgren ([3], p. 124).*

Definition 1.14. *Let $f : A \to A$ and $g : B \to B$ be two maps. Then f and g are said to be topologically conjugate if there exists a homeomorphism $h : A \to B$ such that $hof = goh$. The homeomorphism h is called a topological conjugacy.*

Mappings which are topologically conjugate are completely equivalent in terms of their dynamics.

Definition 1.15 ([1]). *If f is a self-mapping of a metric space (X, d), then the set $O(x, f) = \{f^n x : n = 0, 1, 2, \dots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_i f^{m_i} x$ implies $fu = \lim_i f f^{m_i} x$.*

The notion of orbital continuity was introduced by Ciric ([1]) and is an important tool in establishing the existence of fixed points. Continuity of f obviously implies orbital continuity but not conversely ([1]).

2. Results

Let X denote the set of real numbers equipped with the Euclidean metric. In the sequel, for applying our results to divisibility of polynomials by polynomials of the form $z^3 \pm q$, we shall write the positive (resp. negative) numbers in X either in the form $10^3 x + y$ (resp. - $(10^3 x + y)$) with x a non-negative integer and $0 \le y < 10^3$ or in the form $10^3 a + 10^2 b + 10c + d$ (resp. - $(10^3 a + 10^2 b + 10c + d)$) where $0 \le d < 10$ and a, b, c are non-negative integers with $b, c < 10$. Given a positive real number $x = 10^3 p + q$ in X consider the self-mapping f_x on X induced by x under which the image of a positive number $y = 10^3 a_3 + 10^2 a_2 + 10a_1 + a_0$ in X is given by:

$$f_x(y) = f_x(10^3 a_3 + 10^2 a_2 + 10a_1 + a_0) = (10^2 a_2 + 10a_1 + a_0)p - a_3 q, \tag{1}$$

and the image of $y = -(10^3 a_3 + 10^2 a_2 + 10a_1 + a_0)$ is given by

$$f_x(y) = f_x(-(10^3 a_3 + 10^2 a_2 + 10a_1 + a_0)) = -((10^2 a_2 + 10a_1 + a_0)p - a_3 q), \tag{2}$$

where p, a_1, a_2, a_3 are non-negative integers, $0 \le a_0, a_1, a_2 < 10$ and $0 \le q < 10^3$. In view of a property of these mappings to be observed in Theorem 2.1 below, we refer to such selfmappings as multiple-lowering mappings (see [8]) and mappings defined by (1) and (2) are called multiple-lowering mappings of order 3. It follows from (1) and (2) that

$$\begin{aligned} |f_x(y)| &= |f_x(10^3 a_3 + 10^2 a_2 + 10a_1 + a_0)| = |(10^2 a_2 + 10a_1 + a_0)p - a_3 q| \\ &< max\{10^3 p + q, 10^3 a_3 + 10^2 a_2 + 10a_1 + a_0\} \\ &= max\{x, y\}, \end{aligned} \tag{3}$$

and

$$\begin{aligned} |f_x(y)| &= |f_x(-(10^3 a_3 + 10^2 a_2 + 10a_1 + a_0))| = |-((10^2 a_2 + 10a_1 + a_0)p - a_3 q)| \\ &< max\{10^3 p + q, 10^3 a_3 + 10^2 a_2 + 10a_1 + a_0\} \\ &= max\{x, |y|\}. \end{aligned} \tag{4}$$

Pant [5–8] studied multiple-lowering mappings and their applications to divisibility of polynomials. However, the mappings studied in [5–7] were defined on the set of nonnegative integers and such mappings have a relatively simple dynamics and they do not exhibit the phenomena of chaos and bifurcation. In [8], multiple-lowering mappings of order 7 defined on the set of non-negative real numbers were studied and it was proved that such mappings exhibit the phenomena of chaos and bifurcation. The aim of the present paper is to generalize the multiple lowering mappings of order 3 studied in [7] by extending the domain of their definition from the set of non-negative integers to the set of real numbers and to study the dynamics of these mappings. The next theorem describes the orbits of integers under iteration of such mappings.

Theorem 2.1. *Let X be the set of real numbers, $p \geq 1$ an integer, and $0 \leq q < 10^3$. Let $x = 10^3p + q$ and $f = f_x$ be the multiple-lowering mapping on X induced by x and defined by (1), (2). Then every integral multiple of $10^3p + q$ is an eventually fixed point of f. If q is also an integer then every integer in X that is not an integral multiple of 10^3p+q is either an eventually fixed point or an eventually periodic point of f. Moreover, f is a nonlinear mapping.*

Proof. Suppose that $y = \pm(10^3a_3 + 10^2a_2 + 10a_1 + a_0)$ is a non-zero integral multiple of $10^3p + q$ where a_0, a_1, a_2, a_3 are non-negative integers such that $a_0, a_1, a_2 < 10$. Then there exists an integer $k \geq 1$ such that $10^3a_3 + 10^2a_2 + 10a_1 + a_0 = k(10^3p + q)$ and $-(10^3a_3 + 10^2a_2 + 10a_1 + a_0) = -k(10^3p + q)$. Then $f(10^3a_3 + 10^2a_2 + 10a_1 + a_0) = 10^2a_2p + 10a_1p + a_0p - a_3q = (pk - a_3)(10^3p + q)$ and $f(-(10^3a_3 + 10^2a_2 + 10a_1 + a_0)) = a_3q - (10^2a_2p + 10a_1p + a_0p) = (a_3 - pk)(10^3p + q)$, that is, $f(y)$ is an integral multiple of $10^3p + q$ since a_3, p, k are integers. Further, since $k \geq 1, max\{10^3p+q, 10^3a_3+10^2a_2+10a_1+a_0\} = 10^3a_3+10^2a_2+10a_1+a_0$. By virtue of (3) and (4) we get,$|f(y)| < y = 10^3a_3+10^2a_2+10a_1+a_0$. Therefore, the f-image of an integral multiple of 10^3p+q is a numerically smaller integral multiple of 10^3p+q. But there are exactly $2k-1$ integral multiples, corresponding to $n = 0, \pm1, \pm2, , \pm(k-1)$, of 10^3p+q which are numerically smaller than $10^3a_3+10^2a_2+10a_1+a_0 = k(10^3p+q)$ or numerically smaller than $-(10^3a_3 + 10^2a_2 + 10a_1 + a_0) = -k(10^3p + q)$. Moreover, $f(10^3p + q) = |qp - pq| = 0$ and $f(0) = f(10^3.0 + 10^2.0 + 10.0 + 0) = 0$. Thus, if $y = \pm(10^3a_3 + 10^2a_2 + 10a_1 + a_0) = \pm k(10^3p + q), k \geq 1$, then $f^n(y) = 0$ for some $n \leq 2k - 1$ and $f^m(y) = 0$ for each $m \geq n$ and $f(0) = 0$. In other words, 0 is a fixed point of f and every other integral multiple of $10^3p + q$ is an eventually fixed point of f.

Now suppose that q is also an integer and a given integer $y = \pm(10^3a_3+10^2a_2+10a_1+a_0)$ is not an integral multiple of $x = 10^3p + q$. Then, by virtue of (3) and (4), there exists $n > 0$ such that $0 \leq |f^n(y)| < x = 10^3p + q$. Moreover, for each $i > 0$ we get $0 \leq |f^{n+i}(y)| < 10^3p + q$. If $0 < |f^{n+i}(y)| < 10^3p + q$, then the points $\{f^{n+i}(y) : i \geq 0\}$ form a periodic orbit and $y = \pm(10^3a_3 + 10^2a_2 + 10a_1 + a_0)$ is an eventually periodic point of f, otherwise y is an eventually fixed point of f.

It is also clear from the above computations that irrespective of whether y is a multiple of $x = 10^3p + q$ or not there exists some positive integer n such that for each integer $m \geq n$ we have

$$0 \leq |f^m(y)| < x = 10^3p + q. \tag{5}$$

To show that the mapping f is nonlinear, let us consider the case when $p = 2$ and $q = 4$, that is, $x = 10^3p + q = 2004$. Then $f(1503) = f_{2004}(1503) = 1002$ and $f(1503 + 1503) = f(3006) = 0$, that is, $f(1503 + 1503) \neq f(1503) + f(1503)$. This shows that f is not linear. □

Remark 2.2. *We thus see that when q is an integer, the orbits of integers under iterations of f_x are relatively simple sets since the integers are either eventually fixed points or eventually periodic points. However, the orbits of other real numbers in X may be much more complicated sets if the mapping f_x induced by $x = 10^3p + q$ is chaotic. We now demonstrate the presence of chaotic maps in the set of multiple-lowering mappings.*

Theorem 2.3. *Suppose f is the mapping defined by (1), (2) and induced by $x = 1000k$ (corresponding to $q = 0, p = k, k = 2, 3, 4, \ldots$). Then the restriction of f on the closed interval $[0, 1000k]$ is topologically conjugate to the mapping g on the unit circle S^1 defined by $g(\theta) = k\theta$, that is, f is chaotic on $[0, 1000k]$. Further, multiple lowering mappings induced by two distinct integers of the form $1000k$ are not conjugate.*

Proof. It is sufficient to prove the theorem for the case $k = 2$, in other cases the proof follows on similar lines. On $[0, 2000]$ the mapping f induced by $x = 2000$ (corresponding to $q = 0, p = 2$) can equivalently be defined by:

$$f(y) = 2y \text{ if } y < 1000, f(y) = 2y - 2000 \text{ if } 1000 \leq y < 2000,$$

$$f(y) = 2y - 4000 \text{ if } y = 2000.$$

Similarly, $g : S^1 \to S^1$ given by $g(\theta) = 2\theta$ can equivalently be represented as:

$$g(\theta) = 2\theta \text{ if } \theta < \pi, g(\theta) = 2\theta - 2\pi \text{ if } \pi \leq \theta < 2\pi, g(\theta) = 2\theta - 4\pi \text{ if } \theta = 2\pi.$$

Now define a mapping $h : S^1 \to [0, 2000]$, that is, $h : [0, 2\pi] \to [0, 2000]$ by

$$h(\theta) = 1000\theta/\pi.$$

We now show that f and g are topologically conjugate and h is a topological conjugacy by showing that h is a homeomorphism and $hog = foh$. The mapping h is clearly a one-one mapping of $[0, 2\pi]$ onto $[0, 2000]$ and h is invertible with the inverse h^{-1} being given by $h^{-1}(y) = \pi y/1000$ for each y in $[0, 2000]$. In view of their simple definitions it is easy to see that both h and h^{-1} are continuous functions. Hence h is a homeomorphism. The following computations show that $hog = foh$:

$$\begin{aligned}
(hog)(\theta) &= 2000\theta/\pi \text{ if } \theta < \pi,\\
(hog)(\theta) &= (2000\theta/\pi) - 2000 \text{ if } \pi \leq \theta < 2\pi,\\
(hog)(\theta) &= 0 \text{ if } \theta = 2\pi,\\
(foh)(\theta) &= 2000\theta/\pi \text{ if } \theta < \pi,\\
(foh)(\theta) &= (2000\theta/\pi) - 2000 \text{ if } \pi \leq \theta < 2\pi,\\
(foh)(\theta) &= 0 \text{ if } \theta = 2\pi.
\end{aligned}$$

Therefore, $hog = foh$. Hence f and g are topologically conjugate and h is a topological conjugacy. Since the doubling map is chaotic we conclude that the multiple-lowering map $f = f_x$ is chaotic when $x = 2000$. Moreover, the mapping $g(\theta) = k\theta, k = 3, 4, 5, \ldots,$ can be shown to be chaotic by using a method analogous to that used in proving $g(\theta) = 2\theta$ to be chaotic and, then, the theorem can be proved for the cases $k = 3, 4, 5, \ldots,$ on similar lines as for the case $k = 2$. It is also easy to see that multiple lowering mappings induced by $1000k_1$ and $1000k_2$, $k_1 \neq k_2$, are not conjugate. For example if we consider multiple lowering mappings induced respectively by $x = 2000$ and $x = 3000$ then the first mapping has only one fixed point $y = 0$ while the latter has two fixed points $y = 0$ and $y = 1500$. Therefore, these two mappings fail to be conjugate.
It is clear from the above computations that the mapping f induced by $x = 2000$ is discontinuous at $y = 1000$ and $y = 2000$. In fact f is discontinuous at $y = 1000k, k = 1, 2, 3, \ldots$. However, f is orbitally continuous since for the chaotic mapping f, $u = \lim_i f^{m_i}t$ implies that t is either a periodic point or an eventually periodic point and $fu = \lim_i ff^{m_i}t$. Therefore, f is orbitally continuous. □

Graphical Interpretation of Conjugacy: Let us consider the family of circles centered on the origin and having radius ≤ 2000. On an arbitrary circle of radius ≤ 2000 centered on the origin in the complex plane C, the doubling map can be defined by $f(z) = z^2/|z|$. Similarly, corresponding to an arbitrary radius vector of length 2000 in the complex plane C, the multiple lowering mapping corresponding to $p = 2$ and $q = 0$ can be defined as:

$$f(z) = 2z \text{ if } |z| < 1000, f(z) = 2(|z| - 1000)z/|z| \text{ if } 1000 \leq |z| < 2000, f(z) = 0 \text{ if } z = 2000.$$

Let us consider the periodic points of period 2 for the two mappings. The periodic points of period 2 for the doubling map on each circle are given by $\theta = 2k\pi/3$, that is,

$$\theta = 0, \theta = 2\pi/3, \theta = 4\pi/3.$$

These periodic points for the family of circles have been graphically depicted in Figure 2.4 by three radii vectors at an angle of $2\pi/3$ with each other. The periodic points of period 2 for the multiple lowering map on a radius vector z are given by $kz/3, k = 0, 1, 2$, that is, $z_1 = 0, z_2 = z/3, z_3 = 2z/3$. These periodic points for the family of radii vectors of length 2000 have been depicted in Figure 2.5 by two inner circles and the center.

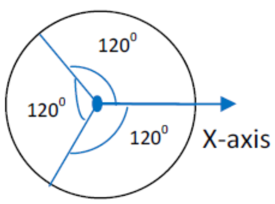


Figure 2.4

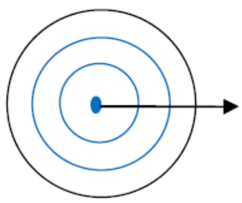


Figure 2.5

We see that the periodic points for the doubling map on the family of circles are given by three radii vectors whose arguments are in arithmetic progression while the periodic points for multiple lowering mapping on the radii vectors are given by three concentric circles whose radii are in arithmetic progression. We also see that there is a one-one correspondence between the periodic points of the two mappings. A similar correspondence can be graphically depicted for periodic points of any order for the two mappings.

Remark 2.4. *By employing a proof similar to that used in the proof of Theorem 3 in [8] it can be shown that (i) bifurcation occurs in the family of multiple lowering mappings f_x defined by (1) at $x = 2000$, (ii) for parameter values x satisfying $1000 \le x < 2000$ the mappings f_x possess non-hyperbolic fixed points, and (iii) for $x = 2000$ each periodic point is a repelling periodic point and the value of $|(f^n)'(p)|$ equals 2^n at each periodic point of period n.*

3. Applicatios to Divisibility

We now give a number theoretic application of the Theorem 2.1, which is a fixed point theorem, to obtain a test of divisibility of one integer by another integer. The question of divisibility of one integer by another or of one polynomial by another is an important question. However, before [5–8] the only important known result on the divisibility of one polynomial by another is the well known Factor Theorem which states that a linear polynomial $x - a$ divides a polynomial $f(x)$ if and only if $f(a) = 0$. Likewise, the general tests of divisibility given in [5–8] are perhaps the only results of their type.

In proving the divisibility theorems we shall use the elementary result that if the integers a, b are relatively prime and if a divides the product of the integers b and c then a divides c. In the sequel, we shall use the notation $x|y$ to show that x divides y.

Theorem 3.1. *Let $x = 10^3p + q$ and $y = 10^3a_3 + 10^2a_2 + 10a_1 + a_0$ be given integers where $p > 0, q > 0, a_0, a_1, a_2, a_3$ are integers and p, q are relatively prime. If f is the multiple lowering mapping induced by $x = 10^3p + q$ then*

$$10^3p + q|10^3a_3 + 10^2a_2 + 10a_1 + a_0 \Leftrightarrow 10^3p + q|10^2a_2p + 10a_1p + a_0p - a_3q, \tag{6}$$

or equivalently, $x|y \Leftrightarrow x|f(y)$.

Proof. Let $x|f(y)$, that is, $10^3p + q|(10^2a_2p + 10a_1p + a_0p - a_3q)$. Then there exists some integer k such that $10^2a_2p+10a_1p+a_0p-a_3q = k(10^3p+q)$. Then $q(a_3+k) = p(10^2a_2+10a_1+a_0-10^3k)$. This implies that $p|q(a_3+k)$, that is, $p|(a_3+k)$ since p and q are relatively prime. Also, we get $10^3a_3+10^2a_2+10a_1+a_0 = ((a_3+k)/p)(10^3p+q)$. This means that $10^3p + q$ divides $y = 10^3a_3 + 10^2a_2 + 10a_1 + a_0$ since $p|(a_3 + k)$ and $(a_3 + k)/p$ is an integer. Conversely, suppose that $10^3p+q|10^3a_3+10^2a_2+10a_1+a_0$, that is, $x|y$. Then $10^3a_3+10^2a_2+10a_1+a_0 = k(10^3p+q)$ for some integer k. This yields $10^2a_2p + 10a_1p + a_0p - a_3q = (pk - a_3)(10^3p + q)$. Therefore,$10^3p + q|(10^2a_2p + 10a_1p + a_0p - a_3q)$, that is, $x|f(y)$. This establishes the theorem and (6) is proved. □

In a similar manner, using mathematical induction we can prove the following:

Theorem 3.2. *Let $x = 10^3p + q$ and $y = 10^{3n+2}a_{3n+2} + 10^{3n+1}a_{3n+1} + \ldots + 10a_1 + a_0$ be given integers where $p > 0, q > 0, a_0, a_1, a_2, \ldots, a_{3n+2}$ are integers and p, q are relatively prime. If f is the multiple lowering mapping induced by $x = 10^3p + q$ then*

$$\begin{aligned} 10^3p \quad + \quad & q|10^{3n+2}a_{3n+2} + 10^{3n+1}a_{3n+1} + \ldots + 10a_1 + a_0 \\ \Leftrightarrow \qquad & 10^3p + q|\{10^2(a_2p^n - a_5p^{n-1}q + \ldots + (-1)^n a_{3n+2}q^n) + 10(a_1p^n - a_4p^{n-1}q + \ldots + \\ & (-1)^n a_{3n+1}q^n) + a_0p^n - a_3p^{n-1}q + \ldots + (-1)^n a_{3n}q^n\}, \end{aligned}$$

or equivalently, $x|y \Leftrightarrow x|f^n(y)$.

We now generalize the above results on divisibility of numbers to obtain conditions of divisibility of polynomials by cubic polynomials of the form $z^3 + q$ in a manner analogous to that employed in obtaining conditions of divisibility of numbers by a number of the form $10^3p + q$. In the process of generalization, without loss of generality, we can take $p = 1$ and can represent a polynomial $P(z)$ in the form $P(z) = a_3(z)z^3 + a_2z^2 + a_1z + a_0$ where $a_3(z)$ is a polynomial. In the sequel, C will denote the field of complex numbers. As done in (1), we can define the multiple-lowering mapping f induced by $z^3 + q$ as $f(P(z)) = a_2z^2 + a_1z + a_0 - a_3(z)q$.

Theorem 3.3. *Let $Q(z) = z^3 + q$ and $P(z) = a_3(z)z^3 + a_2z^2 + a_1z + a_0$ be polynomials over C such that $a_3(z)$ is a polynomial over C. If f is the multiple-lowering mapping induced by $z^3 + q$ then*

$$z^3 + q | a_3(z)z^3 + a_2z^2 + a_1z + a_0 \Leftrightarrow z^3 + q | a_2z^2 + a_1z + a_0 - a_3(z)q.$$

or equivalently, $Q(z)|P(z) \Leftrightarrow Q(z)|f(P(z))$.

Proof. Suppose that $z^3 + q|a_3(z)z^3 + a_2z^2 + a_1z + a_0$. Then there exists some polynomial $k(z)$ over C such that

$$a_3(z)z^3 + a_2z^2 + a_1z + a_0 = k(z)(z^3 + q).$$

Then, $a_0 = k(z)(z^3 + q) - a_3(z)z^3 - a_2z^2 - a_1z$ and $a_2z^2 + a_1z + a_0 - a_3(z)q = (z^3 + q)(k(z) - a_3(z))$. This implies that $z^3 + q$ divides $a_2z^2 + a_1z + a_0 - a_3(z)q$.

Conversely, suppose that $z^3 + q|a_2z^2 + a_1z + a_0 - a_3(z)q$. Then there exists some polynomial $h(z)$ over C such that $a_2z^2 + a_1z + a_0 - a_3(z)q = h(z)(z^3 + q)$. Then $a_0 = h(z)(z^3 + q) - a_2z^2 - a_1z + a_3(z)q$ and

$$a_3(z)z^3 + a_2z^2 + a_1z + a_0 = (a_3(z) + h(z))(z^3 + q)$$

This implies that $z^3 + q$ divides $a_3(z)z^3 + a_2z^2 + a_1z + a_0$. This establishes the theorem. □

Theorem 3.3 can be generalized by using mathematical induction to obtain the following:

Theorem 3.4. *Let $Q(z) = z^3 + q$ and $P(z) = a_{3n+2}z^{3n+2} + \ldots + a_1z + a_0$ be polynomials over C. If f is the multiple-lowering mapping induced by $z^3 + q$ then*

$$\begin{aligned} f^k(P(z)) = a_0 - a_3q \quad + \quad & \ldots + (-q)^k a_{3k} + (a_1 - a_4q + \ldots + (-q)^k a_{3k+1})z + (a_2 - a_5q + \ldots + (-q)^k a_{3k+2})z^2 \\ & +(-q)^k(a_{3k+3}z^3 + a_{3k+4}z^4 + \ldots + a_{3n+2}z^{3n+2-3k}), 1 \leq k \leq n. \end{aligned}$$

Moreover,

$$f^n((P(z)) = (a_2 - a_5q + \ldots + (-q)^n a_{3n+2})z^2 + (a_1 - a_4q + \ldots + (-q)^n a_{3n+1})z + a_0 - a_3q + \ldots + (-q)^n a_{3n} \tag{7}$$

is the remainder on dividing $P(z)$ by $Q(z)$ and is a fixed point of f, and

$$\begin{aligned} z^3 \quad + \quad & q|a_{3n+2}z^{3n+2} + a_{3n+1}z^{3n+1} + \ldots + a_1z + a_0 \\ \Leftrightarrow \quad & z^3 + q|\{(a_2 - a_5q + \ldots + (-q)^n a_{3n+2})z^2 + (a_1 - a_4q + \ldots + (-q)^n a_{3n+1})z + a_0 - a_3q + \ldots + (-q)^n a_{3n}\}, \end{aligned}$$

or equivalently, $Q(z)|P(z) \Leftrightarrow Q(z)|f^n(P(z))$.

It may be observed that $f^n(P(z))$ is a polynomial of degree less than 3 and is a fixed point of f. In Theorem 3.4 if we take $a_1 = a_4 = \ldots = a_{3n-2} = a_{3n+1} = 0, a_2 = a_5 = \ldots = a_{3n-1} = a_{3n+2} = 0$, write b_n for a_{3n} and write y for z^3 then we obtain the well-known Factor Theorem of algebra as a corollary of Theorem 3.4:

Corollary 3.5. *If $y - q$ and $P(y) = b_ny^n + \ldots + b_1y + b_0$ are polynomials over C then*

$$\begin{aligned} y - q|b_ny^n + b_{n-1}y^{n-1} + \ldots + b_1y + b_0 \quad & \Leftrightarrow \quad y - q|b_nq^n + b_{n-1}q^{n-1} + \ldots + b_1q + b_0 \\ & \Leftrightarrow \quad b_nq^n + b_{n-1}q^{n-1} + \ldots + b_1q + b_0 = P(q) = 0. \end{aligned}$$

We thus see that Theorem 3.4 is a generalization of the Factor Theorem. With the help of Theorem 3.4 we can compute the quotient obtained on dividing a polynomial $P(z)$ by $z^3 + q$.

We do this in the next corollary which may be called as the Quotient Theorem.

Corollary 3.6 (The Quotient Theorem). *The quotient on dividing the polynomial $P(z) = a_{3n+2}z^{3n+2} + a_{3n+1}z^{3n+1} + \ldots + a_1z + a_0$ by $Q(z) = z^3 + q$ is given by*

$$\sum_{r=3}^{3n+2} (a_r(z^{r-3} - qz^{r-6} + q^2z^{r-9} - \ldots + (-1)^{[r/3]-1}q^{[r/3]-1}z^{r-3[r/3]})), \tag{8}$$

where $[k]$ denotes the greatest integer not exceeding the number k.

Proof. Consider the division of $P(z) = a_{3n+2}z^{3n+2} + a_{3n+1}z^{3n+1} + \ldots + a_1 z + a_0$ by the polynomial $z^3 + q$. Then, by virtue of Theorem 3.4, we get

$$P(z) - \text{remainder} = (z^3 + q)\{a_3 + a_4 z + a_5 z^2 + a_6(z^3 - q) + a_7(z^4 - qz) + a_8(z^5 - qz^2) + a_9(z^6 - qz^3 + q^2) + a_{10}(z^7 - qz^4 + q^2 z) + a_{11}(z^8 - qz^5 + q^2 z^2) + a_{12}(z^9 - qz^6 + q^2 z^3 - q^3) + \ldots\}.$$

This proves the theorem. □

4. Julia Sets Associated with Multiple-Lowering Mappings

Among the Julia sets, Julia sets of rational functions occupy the most prominent position and their study is presently an area of intense research activity. Their beauty and complexity is fascinating. Suppose $Q(z) = z^3 + q$ is a given polynomial, $P(z) = a_{3n+2}z^{3n+2} + \ldots + a_1 z + a_0$ is any polynomial over the field C of complex numbers, and f is the multiple-lowering mapping induced by $Q(z)$. Then, by (7) and Theorem 3.4, $f^n(P(z)) = (a_2 - a_5 q + \ldots + (-1)^n a_{3n+2}q^n)z^2 + (a_1 - a_4 q + \ldots + (-1)^n a_{3n+1}q^n)z + a_0 - a_3 q + \ldots + (-1)^n a_{3n}q^n$ is the remainder on dividing $P(z)$ by $Q(z)$ and $f^n(P(z))$ is a fixed point of f. Moreover, $P(z)$ is forward asymptotic to the fixed point $f^n(P(z))$. Similarly, by (8) and the Quotient Theorem, the quotient on dividing $P(z)$ by $Q(z)$ is given by:

$$\text{Quotient}((P(z)/Q(z)) = \sum_{r=3}^{3n+2}(a_r(z^{r-3} - qz^{r-6} + q^2 z^{r-9} - \ldots + (-1)^{[r/3]-1}q^{[r/3]-1}z^{r-3[r/3]})).$$

Let us now consider the iterations:

$$z \to P(z)/Q(z), z \to \text{ Quotient } (P(z)/Q(z)) \text{ and } z \to f^n(P(z))/Q(z).$$

We draw some Julia sets of these iterations of rational functions and demonstrate that the Julia sets have simplest shapes when $P(z)$ is forward asymptotic to the zero polynomial and the complexity and beauty of these sets increases with variations in $P(z)$. The variations in the complexity of Julia sets are, in turn, employed to illustrate our results on the value of quotient and remainder. We will be specially interested in the Julia sets of the iteration $z \to f^n(P(z))/Q(z)$ as this iteration is defined by a multiple-lowering mapping and is a new area for study. To fix the ideas let $Q(z) = z^3 + 1$ and $P(z) = z^6 + 2z^3 + a_0$ where a_0 is a parameter. Then $a_6 = 1, a_5 = a_4 = 0, a_3 = 2, a_2 = a_1 = 0$ and, using the Quotient Theorem, the quotient equals $a_3 z^0 + a_6(z^3 - z^0) = 2 + z^3 - 1 = z^3 + 1$. Also, the remainder $f^n(P(z)), n \geq 2$, equals $a_0 - 1$ and $f^n(P(z))/Q(z)$ equals $(a_0 - 1)/(z^3 + 1)$. Thus, $P(z)$ is forward asymptotic to the zero degree polynomial $a_0 - 1$. It is obvious that $P(z)/Q(z)$ and $f^n(P(z))/Q(z), n \geq 2$, will have simplest form for the values of a_0 that imply divisibility of $P(z)$ by $Q(z)$, and $P(z)$ is forward asymptotic to the zero polynomial for such values of a_0. When the value of a_0 is such that $P(z)$ is divisible by $Q(z)$, the remainder $f^2(P(z))$ will be zero and the Julia set of the iteration $z \to f^2(P(z))/Q(z)$ will be the empty set. Similarly, $P(z)/Q(z)$ will equal the quotient in such a case and the Julia set of the iteration $z \to (P(z)/Q(z))$ should be identical with that of the iteration $z \to$ Quotient $(P(z)/Q(z))$. If the parameter a_0 moves farther and farther away from its value corresponding to divisibility of $P(z)$ by $Q(z)$, the modulus of the remainder will increase which should result in a corresponding increase in the complexity and beauty of the Julia sets of the iterations $z \to P(z)/Q(z)$ and $z \to f^2(P(z))/Q(z)$. Our results on the remainder and the quotient as given by Theorem 3.4 and the Quotient Theorem will stand verified via the Julia sets if

(i) the Julia sets of the iterations $z \to$ [Quotient$(P(z)/Q(z)) + f^2(P(z))/Q(z)$] and $z \to P(z)/Q(z)$ are identical,
(ii) the Julia sets of the iteration $z \to f^2(P(z))/Q(z)$ are empty for a value of a_0 which implies divisibility of $P(z)$ by $Q(z)$, and
(iii) the Julia set of the iteration $z \to$ Quotient $(P(z)/Q(z))$ are respectively identical with the Julia set of the iteration $z \to P(z)/Q(z)$ when the value of a_0 is such that $f^2(P(z)) = 0$.

Fig. 4.1 gives the Julia sets of the iteration $z \to f^2(P(z))/Q(z)$ for seven values of a_0 for the polynomial $P(z) = z^6 + 2z^3 + a_0$. Here $f^2(P(z))/Q(z)$ equals $(a_0 - 1)/(z^3 + 1)$.
The Julia sets in Fig. 4.1 exhibit a three-fold symmetry (see [4]). Periodicity in the Julia sets is also evident, an identifiable copy of the set can be obtained by rotation through an angle $2\pi/3$. It is obvious that $a_0 = 1$ implies divisibility of $P(z)$ by $Q(z)$ and the corresponding Julia set is empty. As a_0 moves farther and farther away from the value 1 the complexity and beauty of the Julia sets goes on increasing.

Fig. 4.2 shows the Julia set of the iteration $z \to \text{Quotient}(P(z)/Q(z))$ for the already chosen seven values of a_0 for $P(z) = z^6 + 2z^3 + a_0$. The Julia set is same in each case as the quotient is same in each of these cases and equals $z^3 + 1$. It is also clear that the Julia set of $z \to z^3 + 1$ exhibits three-fold symmetry and periodicity (see [4]).
Fig.4.3 shows the Julia sets of the iteration $z \to [\text{Quotient}(P(z)/Q(z)) + f^2(P(z))/Q(z)]$ for the same values of a_0 as in Figs. 4.1 and 4.2.

The Julia sets of the iteration $z \to P(z)/Q(z)$ for the chosen values of a_0 are found identical with those in Fig. 4.3 and, hence, have not been given separately. Identical Julia sets for the iterations $z \to P(z)/Q(z)$ and $z \to [\text{Quotient}(P(z)/Q(z)) + f^2(P(z))/Q(z)]$ provide illustration for our results on remainder and quotient. Figures 4.2 and 4.3 verify the quotient theorem since the Julia set of the iteration $z \to \text{Quotient}(P(z)/Q(z))$ in Fig. 4.2 is identical with the Julia set for $a_0 = 1$ in Fig.4.3.

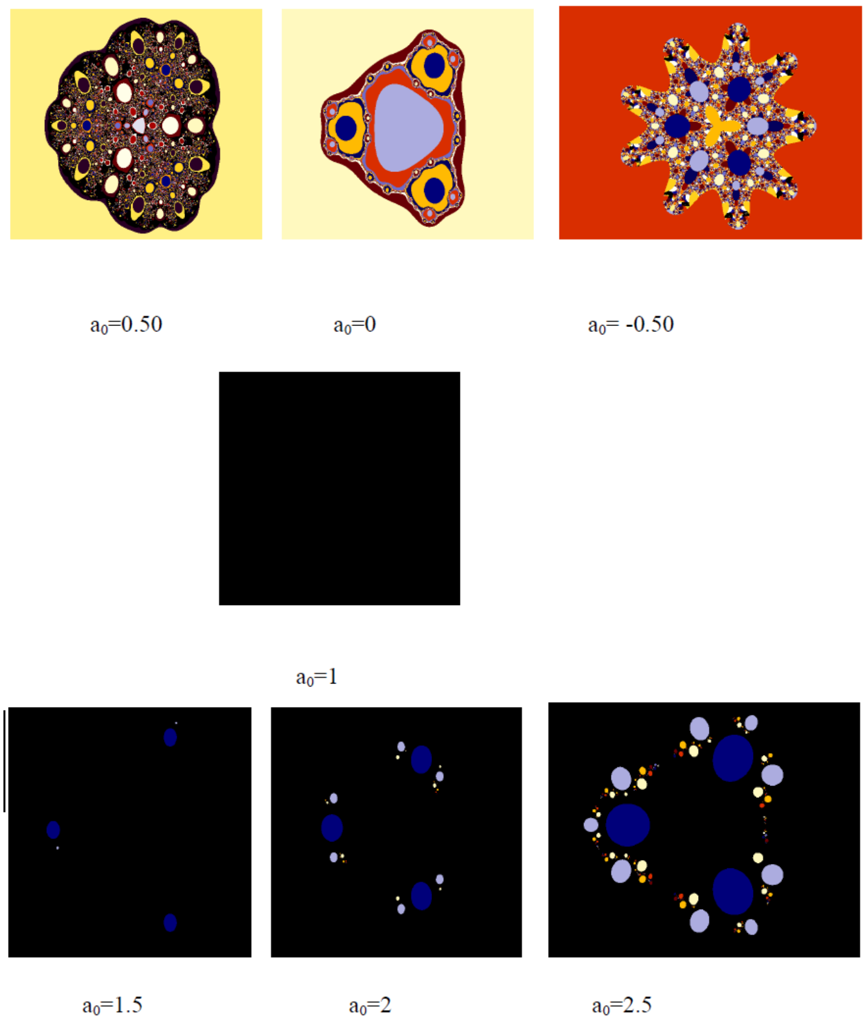


Fig. 4.1

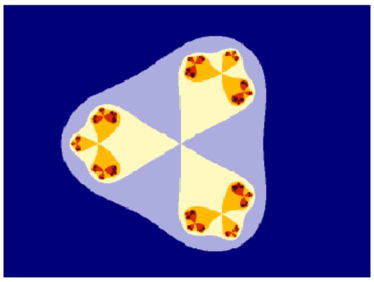
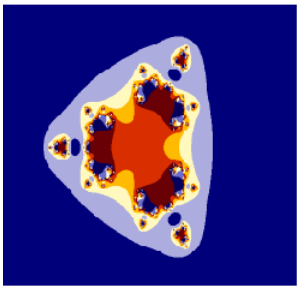
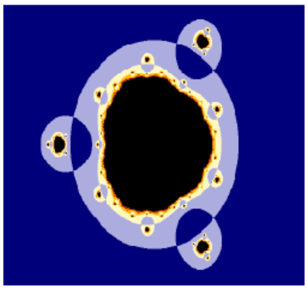


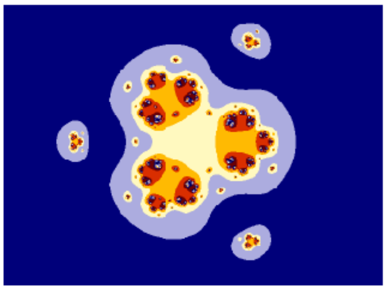
Fig. 4.2



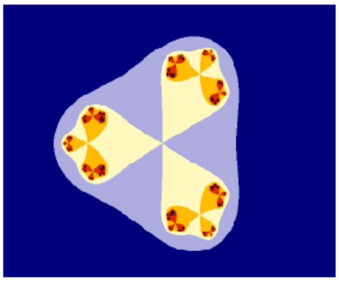
a_0=0.50



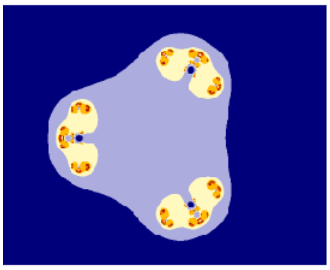
a_0=0



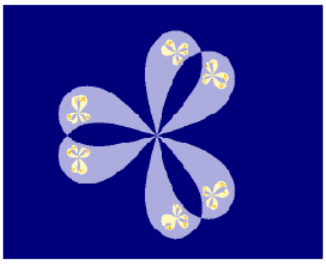
a_0= -0.50



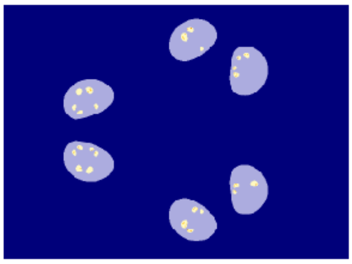
a_0=1



a_0=1.5



a_0=2



a_0=2.5

Fig. 4.3

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