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Radius of Convexity of Partial Sums of Odd Functions in the Close-to-Convex Family

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Abstract. We consider the class of all analytic and locally univalent functions f of the form $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1}z^{2n-1}$, $|z| < 1$, satisfying the condition

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}.$$

We show that every section $s_{2n-1}(z) = z + \sum_{k=2}^{n} a_{2k-1}z^{2k-1}$, of f, is convex in the disk $|z| < \sqrt{2}/3$. We also prove that the radius $\sqrt{2}/3$ is best possible, i.e. the number $\sqrt{2}/3$ cannot be replaced by a larger one.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Main Result

Let $\mathcal{A}$ denote the class of all normalized analytic functions f in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, i.e. f has the Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

The Taylor polynomial $s_n(z) = s_n(f)(z)$ of f in $\mathcal{A}$, defined by,

$$s_n(z) = z + \sum_{k=2}^{n} a_k z^k$$

is called the n-th *section/partial sum* of f. Denote by $\mathcal{S}$, the class of *univalent* functions in $\mathcal{A}$. A function $f \in \mathcal{A}$ is said to be *locally univalent* at a point $z_0 \in D \subset \mathbb{C}$ if it is univalent in some neighborhood of z_0; equivalently $f'(z_0) \neq 0$. A function $f \in \mathcal{A}$ is called *convex* if $f(\mathbb{D})$ is a convex domain. The set of all convex functions are denoted by $\mathcal{C}$. The functions $f \in \mathcal{C}$ are characterized by the well-known fact

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad |z| < 1.$$

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In this article, we mainly focus on a class, denoted by $\mathcal{L}$, of all locally univalent *odd* functions f satisfying

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \quad z \in \mathbb{D}. \tag{2}$$

Clearly, a function $f \in \mathcal{L}$ will have the Taylor series expansion $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1}z^{2n-1}$. The function $f_0(z) = z/\sqrt{1-z^2}$ plays the role of an extremal function for $\mathcal{L}$; see for instance [16, p. 68, Theorem 2.6i]. This article is devoted to finding the largest disk $|z| < r$ in which every section $s_{2n-1}(z) = z + \sum_{k=2}^{n} a_{2k-1}z^{2k-1}$, of $f \in \mathcal{L}$, is convex; that is, s_{2n-1} satisfies

$$\operatorname{Re}\left(1+\frac{zs''_{2n-1}(z)}{s'_{2n-1}(z)}\right) > 0.$$

Our main objective in this article is to prove

Main Theorem. *Every section of a function in $\mathcal{L}$ is convex in the disk $|z| < \sqrt{2}/3$. The radius $\sqrt{2}/3$ cannot be replaced by a greater one.*

This observation is also explained geometrically in Figure 1 by considering the third partial sum, $s_{3,0}$, of the extremal function f_0. We next discuss some motivational background of our problem.

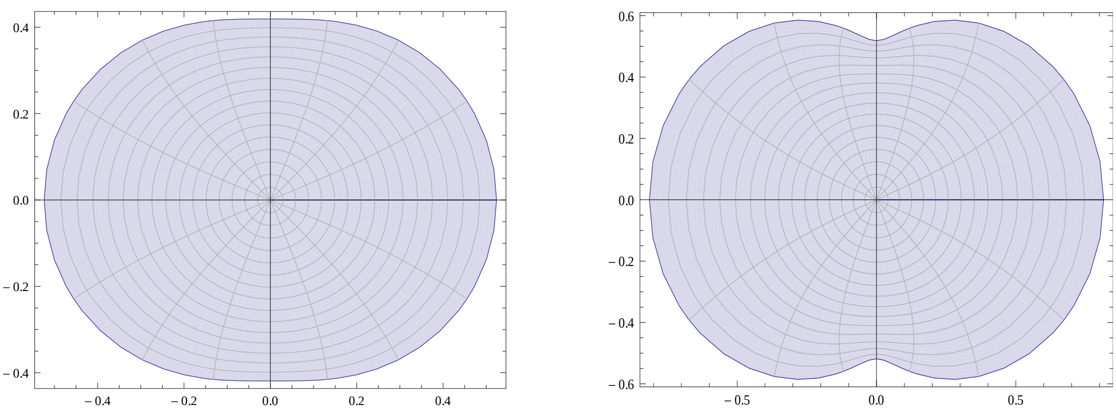

Figure 1: The first figure shows convexity of the image domain $s_{3,0}(z)$ for $|z| < \sqrt{2}/3$ and the second figure shows non-convexity of the image domain $s_{3,0}(z)$ for $|z| < 2/3 =: r_0$ ($r_0 > \sqrt{2}/3$).

Considering odd univalent functions and studying classical problems of univalent function theory such as (successive) coefficient bounds, inverse functions, etc. are quite interesting and found throughout the literature; see for instance [8, 12, 15, 35]. In fact, an application of the Cauchy-Schwarz inequality shows that the conjecture of Robertson: $1 + |c_3|^2 + |c_5|^2 + \cdots + |c_{2n-1}|^2 \leq n$, $n \geq 2$, for each odd function $f(z) = z + c_3z^3 + c_5z^5 + \cdots$ of $\mathcal{S}$, stated in 1936 implies the well-known Bieberbach conjecture [25]; see also [3]. In our knowledge, studying radius properties for sections of odd univalent functions are new (as we do not find in the literature).

Note that a subclass denoted by $\mathcal{F}$, of the class, $\mathcal{K}$, of close-to-convex functions, consisting of all locally univalent functions $f \in \mathcal{A}$ satisfying the condition (2) was considered in [22]. In this paper, we consider functions from $\mathcal{F}$ that have odd Taylor coefficients. Note that the following inclusion relations hold:

$$\mathcal{L} \subsetneq \mathcal{F} \subsetneq \mathcal{K} \subsetneq \mathcal{S}.$$

The fact that functions in $\mathcal{F}$ are close-to-convex may be obtained as a consequence of the result due to Kaplan (see [4, p. 48, Theorem 2.18]). In [22], Ponnusamy et. al. have shown that every section of a function in the class $\mathcal{F}$ is convex in the disk $|z| < 1/6$ and the radius $1/6$ is the best possible. They conjectured that every section of functions in the family $\mathcal{F}$ is univalent and close-to-convex in the disk $|z| < 1/3$. This conjecture has been recently settled by Bharanedhar and Ponnusamy in [1, Theorem 1].

The problem of finding the radius of univalence of sections of f in $\mathcal{S}$ was first initiated by Szegö in 1928. According to the Szegö theorem [4, Section 8.2, p. 243-246], every section $s_n(z)$ of a function $f \in \mathcal{S}$ is univalent in the disk $|z| < 1/4$; see [34] for the original paper. The radius 1/4 is best possible and can be verified from the second partial sum of the Koebe function $k(z) = z/(1-z)^2$. Determining the exact (largest) radius of univalence r_n of $s_n(z)$ ($f \in \mathcal{S}$) remains an open problem. However, many other related problems on sections have been solved for various geometric subclasses of $\mathcal{S}$, eg. the classes $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ of starlike, convex and close-to-convex functions, respectively (see Duren [4, §8.2, p.241–246], [5, 26, 27, 32] and the survey articles [6, 24]). In [13], MacGregor considered the class

$$\mathcal{R} = \{f \in \mathcal{A} : \operatorname{Re} f'(z) > 0, z \in \mathbb{D}\}$$

and proved that the partial sums $s_n(z)$ of $f \in \mathcal{R}$ are univalent in $|z| < 1/2$, where the radius 1/2 is best possible. On the other hand, in [30], Ram Singh obtained the best radius, $r = 1/4$, of convexity for sections of functions in the class $\mathcal{R}$. The reader can refer to [21] for related information. Radius of close-to-convexity of sections of close-to-convex functions is obtained in [14].

By the argument principle, it is clear that the n-th section $s_n(z)$ of an arbitrary function in $\mathcal{S}$ is univalent in each fixed compact subdisk $\overline{\mathbb{D}_r} := \{z \in \mathbb{D} : |z| \le r\} (r < 1)$ of $\mathbb{D}$ provided that n is sufficiently large. In this way one can get univalent polynomials in $\mathcal{S}$ by setting $p_n(z) = \frac{1}{r} s_n(rz)$. Consequently, the set of all univalent polynomials is dense in the topology of locally uniformly convergence in $\mathcal{S}$. The radius of starlikeness of the partial sums $s_n(z)$ of $f \in \mathcal{S}^*$ was obtained by Robertson in [26]; (see also [31, Theorem 2]) in the following form:

Theorem A. [26] *If $f \in \mathcal{S}$ is either starlike, convex, typically-real, or convex in the direction of imaginary axis, then there is an N such that, for $n \ge N$, the partial sum $s_n(z)$ has the same property in $\mathbb{D}_r := \{z \in \mathbb{D} : |z| < r\}$, where $r \ge 1 - 3(\log n)/n$.*

However, Ruscheweyh in [29] proved a stronger result by showing that the partial sums $s_n(z)$ of f are indeed starlike in $\mathbb{D}_{1/4}$ for functions f belonging not only to $\mathcal{S}$ but also to the closed convex hull of $\mathcal{S}$. Robertson [26] further showed that sections of the Koebe function $k(z)$ are univalent in the disk $|z| < 1 - 3n^{-1}\log n$ for $n \ge 5$, and that the constant 3 cannot be replaced by a smaller constant. However, Bshouty and Hengartner [2] pointed out that the Koebe function is not extremal for the radius of univalency of the partial sums of $f \in \mathcal{S}$. A well-known theorem by Ruscheweyh and Sheil-Small [28] on convolution allows us to conclude immediately that if f belongs to $\mathcal{C}, \mathcal{S}^*$, or $\mathcal{K}$, then its n-th section is respectively convex, starlike, or close-to-convex in the disk $|z| < 1 - 3n^{-1}\log n$, for $n \ge 5$. Silverman in [31] proved that the radius of starlikeness for sections of functions in the convex family $\mathcal{C}$ is $(1/2n)^{1/n}$ for all n. We suggest readers refer to [22, 27, 32, 34] and recent articles [17–20] for further interest on this topic. It is worth recalling that radius properties of harmonic sections have recently been studied in [7, 9–11, 23].

2. Preparatory Results

In this section we derive some useful results to prove our main theorem.

Lemma 2.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1} \in \mathcal{L}$, then the following estimates are obtained:*

(a) $|a_{2n-1}| \le \frac{(2n-2)!}{2^{2n-2}(n-1)!^2}$ *for $n \ge 2$. The equality holds for*

$$f_0(z) = \frac{z}{\sqrt{1-z^2}}$$

or its rotation.

(b) $\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{3r^2}{1-r^2}$ *for $|z| = r < 1$. The inequality is sharp.*

(c) $\frac{1}{(1+r^2)^{3/2}} \le |f'(z)| \le \frac{1}{(1-r^2)^{3/2}}$ *for* $|z| = r < 1$. *The inequality is sharp.*

(d) *If* $f(z) = s_{2n-1}(z) + \sigma_{2n-1}(z)$, *with* $\sigma_{2n-1}(z) = \sum_{k=n+1}^{\infty} a_{2k-1}z^{2k-1}$, *then for* $|z| = r < 1$ *we have*

$$|\sigma'_{2n-1}(z)| \le A(n,r) \quad \text{and} \quad |z\sigma''_{2n-1}(z)| \le B(n,r),$$

where

$$A(n,r) = \sum_{k=n+1}^{\infty} \frac{(2k-1)!}{2^{2k-2}(k-1)!^2} r^{2k-2} \quad \text{and} \quad B(n,r) = \sum_{k=n+1}^{\infty} \frac{(2k-2)(2k-1)!}{2^{2k-2}(k-1)!^2} r^{2k-2}.$$

The ratio test guarantees that both the series are convergent.

Proof. **(a)** Set

$$p(z) = 1 + \frac{2}{3}\left(\frac{zf''(z)}{f'(z)}\right). \tag{3}$$

Clearly, $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ is analytic in $\mathbb{D}$ and $\operatorname{Re} p(z) > 0$ there. So, by Carathéodory Lemma, we obtain that $|p_n| \le 2$ for all $n \ge 1$. Putting the series expansions for $f'(z)$, $f''(z)$ and $p(z)$ in (3) we get

$$\sum_{n=2}^{\infty}(2n-1)(2n-2)a_{2n-1}z^{2n-1} = \frac{3}{2}\sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} p_{2k-1}(2n-2k-1)a_{2n-2k-1}\right)z^{2n-2}$$
$$+\frac{3}{2}\sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} p_{2k}(2n-2k-1)a_{2n-2k-1}\right)z^{2n-1}.$$

Equating the coefficients of z^{2n-1} and z^{2n-2} on both sides, we obtain

$$\sum_{k=1}^{n-1} p_{2k-1}(2n-2k-1)a_{2n-2k-1} = 0$$

and

$$(2n-1)(2n-2)a_{2n-1} = \frac{3}{2}\sum_{k=1}^{n-1} p_{2k}(2n-2k-1)a_{2n-2k-1}, \quad \text{for all } n \ge 2. \tag{4}$$

Hence,

$$|a_{2n-1}| \le \frac{3}{(2n-1)(2n-2)}\sum_{k=1}^{n-1}(2k-1)|a_{2k-1}|. \tag{5}$$

For $n = 2$, we can easily see that $|a_3| \le 1/2$, and for $n = 3$, we have

$$|a_5| \le \frac{3}{20}(1 + 3|a_3|) \le \frac{3}{8}.$$

Now, we can complete the proof by method of induction. Therefore, if we assume $|a_{2k-1}| \le \frac{(2k-2)!}{2^{2k-2}(k-1)!^2}$ for $k = 2, 3, \ldots, n-1$, then we deduce from (5) that

$$|a_{2n-1}| \le \frac{3}{(2n-1)(2n-2)}\sum_{k=1}^{n-1}\frac{(2k-1)!}{2^{2k-2}(k-1)!^2}.$$

Induction principle tells us to show that

$$|a_{2n-1}| \leq \frac{(2n-2)!}{2^{2n-2}(n-1)!^2}.$$

It suffices to show that

$$\frac{3}{(2n-1)(2n-2)} \sum_{k=1}^{n-1} \frac{(2k-1)!}{2^{2k-2}(k-1)!^2} = \frac{(2n-2)!}{2^{2n-2}(n-1)!^2}$$

or,

$$\sum_{k=1}^{n-1} \frac{3(2k-1)!}{2^{2k-2}(k-1)!^2} = \frac{(2n-2)(2n-1)!}{2^{2n-2}(n-1)!^2}.$$

Again, we prove this by method of induction. It can easily be seen that for $k = 1$ it is true. Assume that it is true for $k = 2, 3, \ldots, n-1$, then we have to prove that

$$\sum_{k=1}^{n} \frac{3(2k-1)!}{2^{2k-2}(k-1)!^2} = \frac{(2n)(2n+1)!}{2^{2n}(n)!^2},$$

which is easy to see, since

$$\sum_{k=1}^{n} \frac{3(2k-1)!}{2^{2k-2}(k-1)!^2} = \frac{(2n-2)(2n-1)!}{2^{2n-2}(n-1)!^2} + \frac{3(2n-1)!}{2^{2n-2}(n-1)!^2} = \frac{(2n)(2n+1)!}{2^{2n}(n)!^2}.$$

Hence, the proof is complete. For equality, it can easily be seen that

$$f_0(z) = \frac{z}{\sqrt{1-z^2}} = z + \sum_{n=2}^{\infty} \frac{(2n-2)!}{2^{2n-2}(n-1)!^2} z^{2n-1}$$

belongs to $\mathcal{L}$.

The image of the unit disk $\mathbb{D}$ under f_0 is shown in Figure 2 which indicates that $f_0(\mathbb{D})$ is not convex.

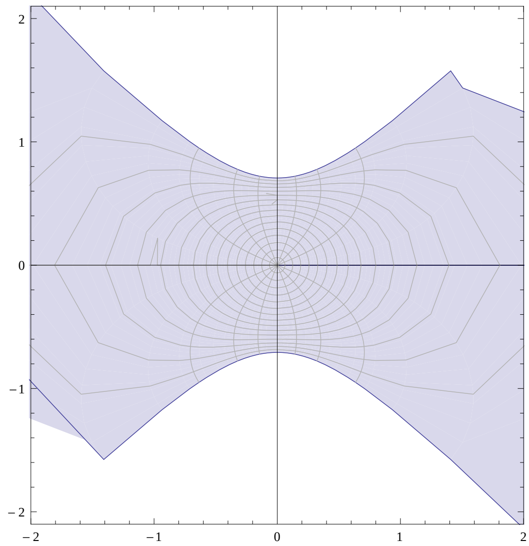

Figure 2: The image domain $f_0(\mathbb{D})$, where $f_0(z) = \frac{z}{\sqrt{1-z^2}}$.

(b) We see from the definition of $\mathcal{L}$ that

$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1+2z^2}{1-z^2}, \quad \text{i.e.,}\ \frac{zf''(z)}{f'(z)} \prec \frac{3z^2}{1-z^2} =: h(z),$$

where $\prec$ denotes the usual subordination. The poof of (b) now follows easily.

(c) Since

$$\frac{zf''(z)}{f'(z)} \prec h(z),$$

it follows by the well-known subordination result due to Suffridge [33] that

$$f'(z) \prec \exp\left(\int_0^z \frac{h(t)}{t}\mathrm{d}t\right) = \exp\left(3\int_0^z \frac{t}{1-t^2}\mathrm{d}t\right) = \frac{1}{(1-z^2)^{3/2}}.$$

Hence, the proof of (c) follows.

(d) By (a), we see that

$$|\sigma'_{2n-1}(z)| \le \sum_{k=n+1}^{\infty}(2k-1)|a_{2k-1}|r^{2k-2} \le A(n,r).$$

and

$$|z\sigma''_{2n-1}(z)| \le \sum_{k=n+1}^{\infty}(2k-1)(2k-2)|a_{2k-1}|r^{2k-2} \le B(n,r).$$

The proof of our lemma is complete. □

3. Proof of the Main Theorem

For an arbitrary $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1}z^{2n-1} \in \mathcal{L}$, we first consider its third section $s_3(z) = z + a_3z^3$ of f. Simple computation shows

$$1+\frac{zs_3''(z)}{s_3'(z)} = 1 + \frac{6a_3z^2}{1+3a_3z^2}.$$

By using Lemma 2.1(a), we have $|a_3| \le 1/2$ and hence

$$\mathrm{Re}\left(1+\frac{zs_3''(z)}{s_3'(z)}\right) \ge 1 - \frac{6|a_3||z|^2}{1-3|a_3||z|^2} \ge 1 - \frac{3|z|^2}{1-\frac{3}{2}|z|^2}$$

which is positive for $|z| < \sqrt{2}/3$. Thus, $s_3(z)$ is convex in the disk $|z| < \sqrt{2}/3$. To show that the constant $\sqrt{2}/3$ is best possible, we consider the function $f_0(z)$ defined by

$$f_0(z) = \frac{z}{\sqrt{1-z^2}}.$$

We denote by $s_{3,0}(z)$, the third partial sum $s_3(f_0)(z)$ of $f_0(z)$ so that $s_{3,0}(z) = z + (1/2)z^3$ and hence, we find

$$1+\frac{zs_{3,0}''(z)}{s_{3,0}'(z)} = \frac{2+9z^2}{2+3z^2}.$$

This shows that

$$\mathrm{Re}\left(1+\frac{zs_{3,0}''(z)}{s_{3,0}'(z)}\right) = 0$$

when $z^2 = (-2/9)$ or $(-2/3)$ i.e., when $|z|^2 = (2/9)$ or $(2/3)$. Hence, the equality occurs.

Next, let us consider the case $n = 3$. Our aim in this case is to show that

$$\mathrm{Re}\left(1 + \frac{zs_5''(z)}{s_5'(z)}\right) = \mathrm{Re}\left(\frac{1 + 9a_3z^2 + 25a_5z^4}{1 + 3a_3z^2 + 5a_5z^4}\right) > 0$$

for $|z| < \sqrt{2}/3$. Since the real part $\mathrm{Re}\,[(1 + 9a_3z^2 + 25a_5z^4)/(1 + 3a_3z^2 + 5a_5z^4)]$ is harmonic in $|z| \leq \sqrt{2}/3$, it suffices to check that

$$\mathrm{Re}\left(\frac{1 + 9a_3z^2 + 25a_5z^4}{1 + 3a_3z^2 + 5a_5z^4}\right) > 0$$

for $|z| = \sqrt{2}/3$. Also we see that

$$\mathrm{Re}\left(\frac{1 + 9a_3z^2 + 25a_5z^4}{1 + 3a_3z^2 + 5a_5z^4}\right) = 3 - \mathrm{Re}\left(\frac{2 - 10a_5z^4}{1 + 3a_3z^2 + 5a_5z^4}\right) \geq 3 - \left|\frac{2 - 10a_5z^4}{1 + 3a_3z^2 + 5a_5z^4}\right|$$

and, so by considering a suitable rotation of $f(z)$, the proof reduces to $z = \sqrt{2}/3$; this means that it is enough to prove

$$\frac{3}{2} > \left|\frac{81 - 20a_5}{81 + 54a_3 + 20a_5}\right|.$$

From (4), we have

$$a_3 = \frac{p_2}{4} \quad \text{and} \quad a_5 = \left(\frac{3}{40}\right)\left(\frac{3}{4}p_2^2 + p_4\right).$$

Since $|p_2| \leq 2$ and $|p_4| \leq 2$, it is convenient to rewrite the last two relations as

$$a_3 = \frac{\alpha}{2} \quad \text{and} \quad a_5 = \frac{3}{40}(3\alpha^2 + 2\beta)$$

for some $|\alpha| \leq 1$ and $|\beta| \leq 1$.

Substituting the values for a_3 and a_5, and applying the maximum principle in the last inequality, it suffices to show the inequality

$$\frac{3}{2}\left|81 + 27\alpha + \frac{9\alpha^2}{2} + 3\beta\right| > \left|81 - \frac{9\alpha^2}{2} - 3\beta\right|$$

for $|\alpha| = 1 = |\beta|$. Finally, by the triangle inequality, the last inequality follows if we can show that

$$9\left|9 + 3\alpha + \frac{\alpha^2}{2}\right| - 6\left|9 - \frac{\alpha^2}{2}\right| > 5$$

which is easily seen to be equivalent to

$$9\left|9\overline{\alpha} + 3 + \frac{\alpha}{2}\right| - 6\left|9\overline{\alpha} - \frac{\alpha}{2}\right| > 5$$

as $|\alpha| = 1$. Write $\mathrm{Re}\,(\alpha) = x$. It remains to show that

$$T(x) := 9\sqrt{18x^2 + 57x + \frac{325}{4}} - 6\sqrt{\frac{361}{4} - 18x^2} > 5$$

for $-1 \leq x \leq 1$.

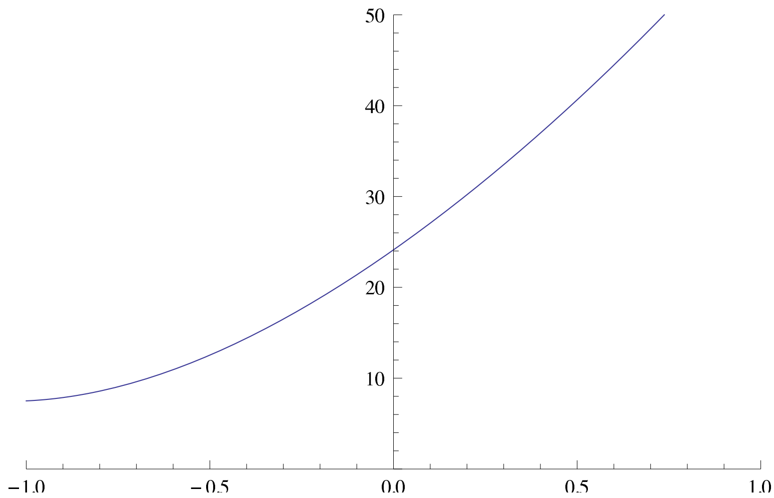

Figure 3: Graph of $T(x)$.

It suffices to show

$$9\sqrt{18x^2+57x+\frac{325}{4}} > 5+6\sqrt{\frac{361}{4}-18x^2}.$$

Squaring both sides we have

$$2106x^2+4617x+\frac{13229}{4} > 60\left(\sqrt{\frac{361}{4}-18x^2}\right).$$

Again by squaring both sides we have

$$\left(2106x^2+4617x+\frac{13229}{4}\right)^2 > 3600\left(\frac{361}{4}-18x^2\right).$$

After computing, it remains to show that $\phi(x)>0$, where

$$\phi(x)=ax^4+bx^3+cx^2+dx+e$$

and the coefficients are

$$a=4435236, b=19446804, c=35311626, d=30539146.5, e=10613002.5625.$$

Here we see that $\phi^{iv}(x)=24a>0$. Thus the function $\phi'''(x)$ is increasing in $-1\le x\le 1$ and hence $\phi'''(x)\ge\phi'''(-1)=10235160>0$. This implies $\phi''(x)$ is increasing. Hence $\phi''(x)\ge\phi''(-1)=7165260>0$. Consequently, $\phi'(x)$ is increasing and we have $\phi'(x)\ge\phi'(-1)=515362.5>0$. Finally we get, $\phi(x)$ is increasing and hence we have $\phi(x)>\phi(-1)=373914.0625>0$. This completes the proof for $n=3$.

We next consider the general case $n\ge 4$. It suffices to show that

$$\mathrm{Re}\left(1+\frac{zs''_{2n-1}}{s'_{2n-1}}\right)>0 \quad \text{for} \quad |z|=r$$

with $r=\sqrt{2}/3$ for all $n\ge 4$. From the maximum modulus principle, we shall then conclude that the last inequality holds for all $n\ge 4$

$$\mathrm{Re}\left(1+\frac{zs''_{2n-1}}{s'_{2n-1}}\right)>0$$

for $|z| < \sqrt{2}/3$. In other words, it remains to find the largest r so that the last inequality holds for all $n \geq 4$.

By the same setting of $f(z)$ as in Lemma 2.1(d), it follows easily that

$$1+\frac{zs''_{2n-1}}{s'_{2n-1}} = 1+\frac{z(f''(z)-\sigma''_{2n-1}(z))}{f'(z)-\sigma'_{2n-1}(z)} = 1+\frac{zf''(z)}{f'(z)}+\frac{\frac{zf''(z)}{f'(z)}\sigma'_{2n-1}(z)-z\sigma''_{2n-1}(z)}{f'(z)-\sigma'_{2n-1}(z)}$$

or,

$$\mathrm{Re}\left(1+\frac{zs''_{2n-1}}{s'_{2n-1}}\right) \geq 1-\left|\frac{zf''(z)}{f'(z)}\right| - \frac{\left|\frac{zf''(z)}{f'(z)}\right||\sigma'_{2n-1}(z)|+|z\sigma''_{2n-1}(z)|}{|f'(z)|-|\sigma'_{2n-1}(z)|}.$$

Then by using Lemma 2.1, we obtain

$$\mathrm{Re}\left(1+\frac{zs''_{2n-1}}{s'_{2n-1}}\right) \geq 1-\frac{3r^2}{1-r^2}-\frac{\left(\frac{3r^2}{1-r^2}\right)A(n,r)+B(n,r)}{\frac{1}{(1+r^2)^{(3/2)}}-A(n,r)}.$$

Thus, we conclude that

$$\mathrm{Re}\left(1+\frac{zs''_{2n-1}}{s'_{2n-1}}\right) > 0$$

provided

$$\frac{1-4r^2}{1-r^2}-\frac{(1+r^2)^{3/2}}{1-r^2}\left(\frac{3r^2A(n,r)+(1-r^2)B(n,r)}{1-(1+r^2)^{3/2}A(n,r)}\right) > 0,$$

or, equivalently

$$(1+r^2)^{3/2}\left(\frac{3r^2A(n,r)+(1-r^2)B(n,r)}{1-(1+r^2)^{3/2}A(n,r)}\right) < 1-4r^2.$$

We show that the above relation holds for all $n \geq 4$ with $r = \sqrt{2}/3$. The choice $r = \sqrt{2}/3$ brings the last inequality to the form

$$\left(\frac{11}{9}\right)^{3/2}\left(\frac{\frac{2}{3}A(n,\frac{\sqrt{2}}{3})+\frac{7}{9}B(n,\frac{\sqrt{2}}{3})}{1-(\frac{11}{9})^{3/2}A(n,\frac{\sqrt{2}}{3})}\right) < \frac{1}{9}.$$

Set

$$C\left(n,\frac{\sqrt{2}}{3}\right) := 1-\left(\frac{11}{9}\right)^{3/2}A\left(n,\frac{\sqrt{2}}{3}\right).$$

We shall prove that $C\left(n,\frac{\sqrt{2}}{3}\right) > 0$ for $n \geq 4$ i.e.,

$$A\left(n,\frac{\sqrt{2}}{3}\right) < \frac{27}{(11)^{3/2}}$$

and

$$A\left(n,\frac{\sqrt{2}}{3}\right)+B\left(n,\frac{\sqrt{2}}{3}\right) < \frac{27}{7\times(11)^{3/2}} \quad \text{for } n \geq 4.$$

If the last inequality is proved, then automatically the previous one follows. Hence, it is enough to prove the last inequality. Now,

$$\begin{aligned} A(n,r)+B(n,r) &= \sum_{k=n+1}^{\infty}\frac{(2k-1)(2k-1)!}{2^{2k-2}(k-1)!^2}(r^2)^{k-1} \\ &\leq \sum_{k=5}^{\infty}\frac{(2k-1)(2k-1)!}{2^{2k-2}(k-1)!^2}(r^2)^{k-1} \\ &= \sum_{k=1}^{\infty}\frac{(2k-1)(2k-1)!}{2^{2k-2}(k-1)!^2}(r^2)^{k-1} - \sum_{k=1}^{4}\frac{(2k-1)(2k-1)!}{2^{2k-2}(k-1)!^2}(r^2)^{k-1} \\ &= \frac{1+2r^2}{(1-r^2)^{5/2}} - \left(1+\frac{9}{2}r^2+\frac{75}{8}r^4+\frac{245}{16}r^6\right). \end{aligned}$$

Substituting the value $r = \sqrt{2}/3$, we obtain

$$A\left(n,\frac{\sqrt{2}}{3}\right)+B\left(n,\frac{\sqrt{2}}{3}\right) \leq 0.076\cdots < 0.105\cdots = \frac{27}{7\times(11)^{3/2}}.$$

This completes the proof of our main theorem. □

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