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Estimates for Coefficients of Certain Analytic Functions

V. Ravichandran[a], **Shelly Verma**[a]

[a] *Department of Mathematics, University of Delhi, Delhi–110 007, India*

Abstract. For $-1 \leq B \leq 1$ and $A > B$, let $\mathcal{S}^*[A, B]$ denote the class of generalized Janowski starlike functions consisting of all normalized analytic functions f defined by the subordination $zf'(z)/f(z) \prec (1 + Az)/(1 + Bz)$ ($|z| < 1$). For $-1 \leq B \leq 1 < A$, we investigate the inverse coefficient problem for functions in the class $\mathcal{S}^*[A, B]$ and its meromorphic counter part. Also, for $-1 \leq B \leq 1 < A$, the sharp bounds for first five coefficients for inverse functions of generalized Janowski convex functions are determined. A simple and precise proof for inverse coefficient estimations for generalized Janowski convex functions is provided for the case $A = 2\beta - 1$ ($\beta > 1$) and $B = 1$. As an application, for $F := f^{-1}$, $A = 2\beta - 1$ ($\beta > 1$) and $B = 1$, the sharp coefficient bounds of F/F' are obtained when f is a generalized Janowski starlike or generalized Janowski convex function. Further, we provide the sharp coefficient estimates for inverse functions of normalized analytic functions f satisfying $f'(z) \prec (1 + z)/(1 + Bz)$ ($|z| < 1, -1 \leq B < 1$).

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Preliminaries

Let $\mathbb{D}$ denote the unit disc. Let $\mathcal{A}$ be the class of all normalized analytic functions $f : \mathbb{D} \to \mathbb{C}$ of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. An analytic function f is said to be subordinate to an analytic function g, written $f \prec g$, if $f = g \circ w$ for some analytic function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$. If g is univalent, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Let φ be an analytic univalent function with positive real part mapping $\mathbb{D}$ onto domains symmetric with respect to real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. Let $\mathcal{P}(\varphi)$ denote the class of all analytic functions $p : \mathbb{D} \to \mathbb{C}$ such that $p \prec \varphi$. For such φ, Ma and Minda [22] introduced the subclasses $\mathcal{S}^*(\varphi)$ ($\mathcal{K}(\varphi)$) of $\mathcal{S}$ consisting of functions $f \in \mathcal{S}$ such that $zf'(z)/f(z)$ $(1 + zf''(z)/f'(z)) \in \mathcal{P}(\varphi)$. For different choices of φ, several well-known classes can be easily obtained from these classes which were earlier considered and studied one by one for their geometric and analytic properties. For instance, $\mathcal{S}^*((1 + z)/(1 - z)) =: \mathcal{S}^*$ and $\mathcal{K}((1 + z)/(1 - z)) =: \mathcal{K}$, the usual classes of starlike and convex functions respectively; for $0 \leq \alpha < 1$, $\mathcal{S}^*((1 + (1 - 2\alpha)z)/(1 - z)) =: \mathcal{S}^*(\alpha)$ and $\mathcal{K}((1 + (1 - 2\alpha)z)/(1 - z)) =: \mathcal{K}(\alpha)$, the well-known classes of starlike and convex functions of order α, respectively introduced in [31]; for $0 < \alpha \leq 1$, $\mathcal{S}^*(((1 + z)/(1 - z))^\alpha) =: \mathcal{SS}^*(\alpha)$ is the well-known class of strongly starlike functions of order α introduced

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Email addresses: vravi68@gmail.com; vravi@maths.du.ac.in (V. Ravichandran), jmdsv.maths@gmail.com (Shelly Verma)

in [6]. In [22], the authors gave a unified treatment to the geometric as well as analytic properties of these well-known classes.

We observe that the distortion theorem, upper bound of $|f|$, rotation theorem, upper bound of Feketo-Szegö coefficient functional $|a_3 - \mu a_2^2|$ ($\mu \in \mathbb{R}$) for $f \in \mathcal{K}(\varphi)$ given in [22] still hold for a normalized locally univalent function f satisfying $1 + zf''(z)/f'(z) \prec \varphi(z)$ if we drop the condition that φ has positive real part. Consequently, the growth theorem and upper bound of Feketo-Szegö coefficient functional $|a_3 - \mu a_2^2|$ ($\mu \in \mathbb{R}$) follow for a normalized analytic function f satisfying $zf'(z)/f(z) \prec \varphi(z)$ even if φ does not have positive real part. This motivates one to consider the following subclasses of $\mathcal{A}$, for $-1 \leq B \leq 1$, $A > B$,

$$\mathcal{K}[A,B] = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}[A,B] \right\} \quad \text{and} \quad \mathcal{S}^*[A,B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \in \mathcal{P}[A,B] \right\}$$

where $\mathcal{P}[A,B] := \mathcal{P}((1+Az)/(1+Bz))$. For $-1 \leq B < A \leq 1$, $\mathcal{S}^*[A,B]$ is a subclass of $\mathcal{S}^*$ introduced by Janowski [11] and for particular values of A and B, it reduces to several known subclasses of $\mathcal{S}^*$. Precisely, $\mathcal{S}^*[1-2\alpha,-1] =: \mathcal{S}^*(\alpha)$ $(0 \leq \alpha < 1)$[31]; $\mathcal{S}^*[1, 1/M - 1] =: \mathcal{S}^*(M)$ $(M > 1/2)$[10]; $\mathcal{S}^*[\beta,-\beta] =: \mathcal{S}^{*(\beta)}$ $(0 < \beta \leq 1)$ [26]; $\mathcal{S}^*[1-\beta, 0] =: \mathcal{S}^*_{1-\beta}$ $(0 \leq \beta < 1)$ [34]. Note that, for $-1 \leq B \leq 1 < A$, the functions in the classes $\mathcal{K}[A,B]$ and $\mathcal{S}^*[A,B]$ may not be univalent but must be locally univalent in $\mathbb{D}$ and non-vanishing in $\mathbb{D} \setminus \{0\}$, respectively.

Recently, the classes $\mathcal{S}^*[2\beta-1, 1]$ and $\mathcal{K}[2\beta-1, 1]$ $(\beta > 1)$ have been studied by several authors, see [24, 25, 37]. Moreover, the upper bound of the Feketo-Szegö coefficient functional $|a_3 - \mu a_2^2|$ ($\mu \in \mathbb{C}$) for $f \in \mathcal{K}[2\beta-1, 1]$ or $f \in \mathcal{S}^*[2\beta-1, 1]$; the distortion theorem, upper bound of $|f|$, rotation theorem for $f \in \mathcal{K}[2\beta-1,1]$; and the growth theorem for $f \in \mathcal{S}^*[2\beta-1,1]$ are given in [1] which can actually be deduced, even for the functions in the generalized classes $\mathcal{S}^*[A,B]$ and $\mathcal{K}[A,B]$ $(-1 \leq B \leq 1 < A)$, from the results in [22] and the inequality (7) in [13, p. 10]. Also, for $-1 \leq B \leq 1$ and $A > B$, one can consider the meromorphic counter part of $\mathcal{S}^*[A,B]$, namely, the class $\Sigma^*[A,B]$ consisting of analytic functions of the form

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \tag{1}$$

defined on $\mathbb{C} \setminus \overline{\mathbb{D}}$ such that $zg'(z)/g(z) \in p_0(\mathbb{D})$ where $p_0 : \mathbb{D} \to \mathbb{C}$ is defined by $p_0(z) = (1+Az)/(1+Bz)$. For $-1 \leq B < A \leq 1$, the class $\Sigma^*[A,B]$ has been considered in [3] and the particular choices of A and B give the meromorphic counter parts of the classes corresponding to those of $S^*[A,B]$ such as $\Sigma^*[1-2\alpha,-1] =: \Sigma^*(\alpha)$ $(0 \leq \alpha < 1)$ [27]; $\Sigma^*[1, 1/M-1] =: \Sigma^*(M)$ $(M > 1/2)$[38]; $\Sigma^*[\beta,-\beta] =: \Sigma^{*(\beta)}$ $(0 < \beta \leq 1)$[26]; $\Sigma^*[1-\beta, 0] =: \Sigma^*_{1-\beta}$ $(0 \leq \beta < 1)$. Hallenbeck [8] considered the class $\mathfrak{I}$ consisting of functions $f \in \mathcal{S}$ such that $f' \in \mathcal{P}$, where $\mathcal{P} := \mathcal{P}\big((1+z)/(1-z)\big)$. Further, Libera and Złotkiewicz [16, 18] investigated the inverse coefficient problem of functions in the class $\mathfrak{I}$. For $-1 \leq B < A \leq 1$, let $\mathfrak{I}[A,B]$ denote the subclass of $\mathcal{S}$ consisting of functions $f \in \mathcal{S}$ such that $f' \in \mathcal{P}[A,B]$.

The problem of estimating the coefficients of inverse functions lay its origin in 1923 when Löwner [21] gave the sharp coefficient estimates for inverse function of $f \in \mathcal{S}$ along with the sharp coefficient estimation for the third coefficient of $f \in \mathcal{S}$. Later, several authors [5, 7, 28, 33] gave alternate proofs for the inverse coefficient problem for functions in the class $\mathcal{S}$ but the inverse coefficient problem is still an open problem even for the well-known classes $\mathcal{K}$ and $\mathcal{S}^*(\alpha)$ $(0 \leq \alpha < 1)$, although the sharp estimates for initial inverse coefficients are known for these classes, for details see [12, 14, 15]. This leads to several works related to the inverse coefficient problem for functions in certain subclasses of $\mathcal{S}$, see [2, 17, 19, 20, 23, 29, 35, 36]. Recently, the inverse coefficient problem is completely settled in [1] for functions in the classes $\mathcal{S}^*[2\beta-1, 1]$ or $\Sigma^*[2\beta-1, 1]$ or $\mathcal{K}[2\beta-1, 1]$, $\beta > 1$.

In this paper, we are mainly concerned about the determination of the sharp inverse coefficient bounds for functions in the classes $\mathcal{S}^*[A,B]$ or $\Sigma^*[A,B]$ $(-1 \leq B \leq 1 < A)$. Also, we are giving the sharp coefficient bounds for the inverse functions of functions in the class $\mathfrak{I}[1,B]$ $(-1 \leq B < 1)$ and the sharp first five coefficient bounds for the inverse functions of functions in the class $\mathcal{K}[A,B]$ for $-1 \leq B \leq 1 < A$. Apart from this, we present a slightly simpler proof than the proof given in [1] for the sharp inverse coefficient estimation for functions in the class $\mathcal{K}[2\beta-1, 1]$ $(\beta > 1)$. As an application, for $F := f^{-1}$ and $\beta > 1$, the sharp coefficient bounds of F/F' are obtained when $f \in \mathcal{S}^*[2\beta-1,1]$ or $f \in \mathcal{K}[2\beta-1,1]$. Further, under some conditions, the sharp coefficient estimates are determined for functions in the class $\Sigma^*[A,B]$ $(-1 \le B \le 1 < A)$.

We need the following lemmas to prove our results.

Lemma 1.1. *[9, Theorem II, p. 547] Let Ω be the family of functions f such that for $|z| < \rho$ with $\rho > 0$, $f(z) = \sum_{n=1}^{\infty} a_n z^n$ $(a_1 \neq 0)$. If $f \in \Omega$ and ϕ is the inverse function of f, then $\phi \in \Omega$. For any integer t, let $f(z)^t = \sum_{n=-\infty}^{\infty} a_n^{(t)} z^n$ and $\phi(w)^t = \sum_{n=-\infty}^{\infty} b_n^{(t)} w^n$ in some neighbourhoods of the origin, where $a_n^{(t)}$ and $b_n^{(t)}$ are zero for $n < t$. Then*

$$b_n^{(t)} = \frac{t}{n} a_{-t}^{(-n)}, \quad n \neq 0.$$

For $n = 0$, $b_0^{(t)}$ is defined by

$$\sum_{t=-\infty}^{\infty} b_0^{(t)} z^{-t-1} = \frac{f'(z)}{f(z)}.$$

Lemma 1.2. *[32, Theorem X, p. 70] Let $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ $(z \in \mathbb{D})$ be such that $f \prec g$. If g is univalent in $\mathbb{D}$ and $g(\mathbb{D})$ is convex, then $|a_n| \le |b_1|$.*

By using the above lemma, the following result is proved. This has been proved in [4] for the case $-1 \le B < A \le 1$.

Lemma 1.3. *If $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ is in $\mathcal{P}[A,B]$ $(-1 \le B \le 1, A > B)$ then $|c_n| \le A - B$. The bounds are sharp.*

Proof. Since $p \in \mathcal{P}[A,B]$, $p(z) \prec (1+Az)/(1+Bz)$. Let $g(z) := (1+Az)/(1+Bz)$. Clearly, g is univalent in $\mathbb{D}$. For $-1 < B < 1$, $g(\mathbb{D})$ is the disc $|w - (1-AB)/(1-B^2)| < (A-B)/(1-B^2)$. For $B = 1$ and $B = -1$, $g(\mathbb{D})$ is the left half plane $\operatorname{Re}(w) < (1+A)/2$ and the right half plane $\operatorname{Re}(w) > (1-A)/2$ respectively. Therefore, $g(\mathbb{D})$ is convex and hence by Lemma 1.2, $|c_n| \le A - B$ for each n. Define a function $p_n : \mathbb{D} \to \mathbb{C}$ as

$$p_n(z) = \frac{1+Az^n}{1+Bz^n} = 1 + (A-B)z^n - B(A-B)z^{2n} + \cdots.$$

Clearly, the result is sharp for the function p_n. □

The following lemma follows easily by induction on m and for $-1 \le B < A \le 1$, it is given in [4, Lemma 2, p. 737].

Lemma 1.4. *Let $A > B$, $-1 \le B \le 1$. Then for any integer t and $m \in \mathbb{N}$, we have*

$$m^2 \prod_{j=0}^{m-1} \left(\frac{(A-B)t + Bj}{j+1} \right)^2 = (A-B)^2 t^2 + \sum_{k=1}^{m-1} \left(\left((A-B)t + Bk \right)^2 - k^2 \right) \prod_{j=0}^{k-1} \left(\frac{(A-B)t + Bj}{j+1} \right)^2.$$

2. Main Results

The following theorem gives estimates for inverse coefficients of functions in the class $\mathcal{S}^*[A,B]$ $(-1 \le B \le 1 < A)$.

Theorem 2.1. *Let $f \in \mathcal{S}^*[A,B]$ $(-1 \le B \le 1 < A)$ and $f^{-1}(w) =: F(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ in some neighbourhood of the origin. Then for each $n \ge 2$,*

$$|\gamma_n| \le \frac{1}{n} \prod_{m=0}^{n-2} \left(\frac{n(A-B) + mB}{m+1} \right). \tag{2}$$

The result is sharp.

Proof. For any integer $t > 0$, let

$$g(z) := \left(\frac{f(z)}{z}\right)^{-t} = 1 + \sum_{j=1}^{\infty} a_j^{(-t)} z^j \quad (|z| < 1).$$

Then

$$-\frac{z}{t}\frac{g'(z)}{g(z)} = \frac{zf'(z)}{f(z)} - 1. \tag{3}$$

Since $f \in \mathcal{S}^*[A, B]$, we have

$$\frac{zf'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)} \tag{4}$$

for some analytic function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$. The equations (3) and (4) give

$$\sum_{j=1}^{\infty} ja_j^{(-t)} z^j = -w(z)\left((A - B)t + \sum_{j=1}^{\infty} \Big(B(j - t) + At\Big)a_j^{(-t)} z^j\right)$$

which can be rewritten as

$$\sum_{j=1}^{s} ja_j^{(-t)} z^j + \sum_{j=s+1}^{\infty} b_j^{(-t)} z^j = -w(z)\left((A - B)t + \sum_{j=1}^{s-1} \Big(B(j - t) + At\Big)a_j^{(-t)} z^j\right)$$

where

$$\sum_{j=s+1}^{\infty} b_j^{(-t)} z^j := \sum_{j=s+1}^{\infty} ja_j^{(-t)} z^j + w(z)\left(\sum_{j=s}^{\infty} \Big(B(j - t) + At\Big)a_j^{(-t)} z^j\right).$$

Since $|w(z)| < 1$ $(|z| < 1)$, squaring the moduli of both sides, we have

$$\left|\sum_{j=1}^{s} ja_j^{(-t)} z^j + \sum_{j=s+1}^{\infty} b_j^{(-t)} z^j\right|^2 < \left|(A - B)t + \sum_{j=1}^{s-1} \Big(B(j - t) + At\Big)a_j^{(-t)} z^j\right|^2.$$

Integrating along $|z| = r$, $0 < r < 1$ with respect to θ $(0 \le \theta \le 2\pi)$ and applying Parseval's identity that for an analytic function $g : \mathbb{D} \to \mathbb{C}$ of the form $g(z) = \sum_{n=0}^{\infty} A_n z^n$,

$$\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |A_n|^2 r^{2n} \quad (0 < r < 1)$$

we have

$$\sum_{j=1}^{s} |ja_j^{(-t)}|^2 r^{2j} + \sum_{j=s+1}^{\infty} |b_j^{(-t)}|^2 r^{2j} \le (A - B)^2 t^2 + \sum_{j=1}^{s-1} |B(j - t) + At|^2 |a_j^{(-t)}|^2 r^{2j}.$$

Letting $r \to 1$ yields

$$\sum_{j=1}^{s} |ja_j^{(-t)}|^2 \le (A - B)^2 t^2 + \sum_{j=1}^{s-1} |B(j - t) + At|^2 |a_j^{(-t)}|^2$$

and therefore,

$$|sa_s^{(-t)}|^2 \le (A - B)^2 t^2 + \sum_{j=1}^{s-1} \Big(\big((A - B)t + Bj\big)^2 - j^2\Big)|a_j^{(-t)}|^2. \tag{5}$$

We shall show that, for $-1 \leq B \leq 1$, $A > 1$, $t \geq (s-1)(1-B)/(A-B)$ and $s \geq 1$,

$$|a_s^{(-t)}| \leq \prod_{m=0}^{s-1} \left(\frac{(A-B)t + mB}{m+1} \right). \tag{6}$$

We proceed by induction on s. For $s = 1$, equation (5) gives

$$|a_1^{(-t)}| \leq (A-B)t.$$

Since $-1 \leq B \leq 1$ and $A > B$, for fixed $j \geq 1$, $\Big((A-B)t + Bj\Big)^2 - j^2 = \Big((A-B)t - j(1-B)\Big)\Big((A-B)t + j(1+B)\Big) \geq 0$ if $t \geq j(1-B)/(A-B)$. Assume that (6) holds for $s \leq q-1$ and $t \geq (q-1)(1-B)/(A-B)$. Then by using induction hypothesis and the equation (5) for $s = q$, we have

$$|qa_q^{(-t)}|^2 \leq (A-B)^2 t^2 + \sum_{j=1}^{q-1} \left(\Big((A-B)t + Bj\Big)^2 - j^2 \right) \prod_{m=0}^{j-1} \left(\frac{(A-B)t + mB}{m+1} \right)^2$$

which by using Lemma 1.4 gives

$$|a_q^{(-t)}| \leq \prod_{m=0}^{q-1} \left(\frac{(A-B)t + mB}{m+1} \right).$$

Thus, (6) holds for $s = q$ and hence by induction (6) holds for all $s \geq 1$. By applying Cauchy's integral formula for F', it can be easily seen that

$$\gamma_n = \frac{1}{n} a_{n-1}^{(-n)} \quad (n \geq 2). \tag{7}$$

Since $A > 1$, therefore $(n-2)(1-B)/(A-B) \leq n-2$ $(n \geq 2)$. So, for $t = n$ and $s = n-1$, the equation (6) gives

$$|\gamma_n| = \frac{1}{n} |a_{n-1}^{(-n)}| \leq \frac{1}{n} \prod_{m=0}^{n-2} \left(\frac{(A-B)n + mB}{m+1} \right).$$

Define a function $f_1 : \mathbb{D} \to \mathbb{C}$ by

$$f_1(z) = \begin{cases} z(1+Bz)^{(A-B)/B}, & B \neq 0 \\ ze^{Az}, & B = 0. \end{cases} \tag{8}$$

The result is sharp for the function f_1. □

For $A = 2\beta - 1$, $B = 1$ $(\beta > 1)$, the above theorem reduces to [1, Theorem 4.3, p. 14].

Corollary 2.2. *Let $f \in \mathcal{S}^*[2\beta - 1, 1]$ $(\beta > 1)$ and $f^{-1}(w) =: F(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ in some neighbourhood of the origin. If $F(w)/F'(w) = w + \sum_{n=2}^{\infty} \delta_n w^n$, then $|\delta_2| \leq 2(\beta - 1)$ and for $n > 2$,*

$$|\delta_n| \leq 2(\beta - 1) \prod_{j=2}^{n-1} \left(\frac{2(n-1)(\beta - 1) + j}{j} \right).$$

The result is sharp.

Proof. Since $f \in \mathcal{S}^*[2\beta - 1, 1]$ $(\beta > 1)$, $zf'(z)/f(z) \in \mathcal{P}[2\beta - 1, 1]$. This gives

$$\frac{zf'(z)}{f(z)} = p(z)$$

where $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}[2\beta - 1, 1]$. In terms of $F := f^{-1}$, the above equation becomes

$$\frac{F(w)}{F'(w)} = wp(F(w)).$$

Using power series expansions of F/F', p and F, we obtain

$$\sum_{n=2}^{\infty} \delta_n w^n = \sum_{n=2}^{\infty} \left(\sum_{j=1}^{n-1} c_j \gamma_{n-1,j} \right) w^n \tag{9}$$

where $\gamma_{n-1,j}$ denotes the coefficient of w^{n-1} in the expansion of $F(w)^j$. In fact, $\gamma_{n-1,j} = S_j(\gamma_2, \gamma_3, \ldots, \gamma_{n-2})$ is a polynomial in $\gamma_2, \gamma_3, \ldots, \gamma_{n-2}$ with non-negative coefficients and $\gamma_{n-1,n-1} = 1$. On comparing the coefficients of w^n, we have

$$\delta_n = \sum_{j=1}^{n-1} c_j \gamma_{n-1,j}.$$

An application of Lemma 1.3 gives

$$|\delta_n| \leq 2(\beta - 1) \sum_{j=1}^{n-1} S_j(|\gamma_2|, |\gamma_3|, \ldots, |\gamma_{n-2}|). \tag{10}$$

Define $g_1(z) := e^{-i\pi} f_1(e^{i\pi} z)$ where f_1 is given by (8) for $A = 2\beta - 1$ and $B = 1$. Clearly, $g_1 \in \mathcal{S}^*[2\beta - 1, 1]$. Then $G_1(w) := g_1^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n$ and $G_1(w)/G_1'(w) = w - \sum_{n=2}^{\infty} B_n w^n$ where w lies in some neighbourhood of the origin,

$$B_2 := 2(\beta - 1), \quad B_n := 2(\beta - 1) \prod_{j=2}^{n-1} \left(\frac{2(n-1)(\beta - 1) + j}{j} \right) \quad (n > 2)$$

and

$$A_n := \frac{1}{n} \prod_{m=0}^{n-2} \left(\frac{2n(\beta - 1) + m}{m + 1} \right) \ (n \geq 2).$$

Proceeding as in (9) for g_1 and then comparing the coefficients of w^n give

$$B_n = 2(\beta - 1) \sum_{j=1}^{n-1} S_j(A_2, A_3, \ldots, A_{n-2}) \ (n \geq 2). \tag{11}$$

Since $f \in \mathcal{S}^*[2\beta - 1, 1]$, applying Theorem 2.1 in (10) and using (11) give $|\delta_n| \leq B_n$. Clearly, the sharpness follows for the function g_1. □

Corollary 2.3. *Let g, given by* (1), *be in $\Sigma^*[A, B]$ $(-1 \leq B \leq 1 < A)$ and $n(1 - B) - (A - B) \leq 0$. Then for each $n \geq 0$,*

$$|b_n| \leq \prod_{m=0}^{n} \left(\frac{(A - B) + mB}{m + 1} \right).$$

The result is sharp.

Proof. It is easy to observe that for any $g \in \Sigma^*[A,B]$, there exists $f \in \mathcal{S}^*[A,B]$ such that for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, $g(z) = 1/f(1/z)$. Also, we note that the expansions of $f(z)^{-1}$ about the origin and $f(1/z)^{-1}$ about the infinity have same coefficients. Thus, if $z/f(z) = 1 + \sum_{n=1}^{\infty} a_n^{(-1)} z^n$ $(z \in \mathbb{D})$, then for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, we have

$$\frac{g(z)}{z} = \frac{1}{zf(1/z)} = 1 + \sum_{n=1}^{\infty} a_n^{(-1)} z^{-n}.$$

On comparing the coefficients, we obtain

$$b_n = a_{n+1}^{(-1)} \quad (n \geq 0). \tag{12}$$

An application of (6) for $t = 1$ and $s = n+1$ in the equation (12) gives the desired estimate. Define a function $g_1 : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C}$ by

$$g_1(z) = \frac{1}{f_1(1/z)} \tag{13}$$

where f_1 is given by (8). The result is sharp for the function g_1 given by (13). □

For $A = 2\beta - 1$, $B = 1$ $(\beta > 1)$, the above result is mentioned in [1, Theorem 4.5, p. 17]. Next, we prove the meromorphic counter part of the Theorem 2.1.

Theorem 2.4. *Let the function $g \in \Sigma^*[A,B]$ $(-1 \leq B \leq 1 < A)$ and $g^{-1}(w) = w + \sum_{n=0}^{\infty} \tilde{\gamma}_n w^{-n}$ in some neighbourhood of the infinity. Then $|\tilde{\gamma}_0| \leq A - B$ and*

$$|\tilde{\gamma}_n| \leq \frac{1}{n} \prod_{m=0}^{n} \left(\frac{(A-B)n + mB}{m+1} \right) \quad (n \geq 1).$$

The result is sharp.

Proof. Since $g \in \Sigma^*[A,B]$, there exists $f \in \mathcal{S}^*[A,B]$ such that for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, $g(z) = 1/f(1/z)$ and $g^{-1}(w) = 1/f^{-1}(1/w)$, see [28, Theorem 2.4, p. 459]. Therefore, for each $n \geq 0$,

$$|\tilde{\gamma}_n| = |\gamma_{n+1}^{(-1)}| \tag{14}$$

where $\gamma_{n+1}^{(-1)}$ is the coefficient of $w^{-(n+1)}$ in $1/(wf^{-1}(1/w)) = 1 + \sum_{n=1}^{\infty} \gamma_n^{(-1)} w^{-n}$.

Since $f \in \mathcal{S}^*[A,B]$, we have $zf'(z)/f(z) = q(z) \in \mathcal{P}[A,B]$. If $q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n$, then by applying Lemma 1.1, we have

$$\sum_{p=-\infty}^{\infty} \gamma_1^{(p)} z^{-p-1} = \frac{f'(z)}{f(z)} = \frac{q(z)}{z} = \frac{1}{z}\left(1 + \sum_{n=1}^{\infty} q_n z^n\right).$$

Therefore, in view of (14) and Lemma 1.3, $|\tilde{\gamma}_0| = |\gamma_1^{-1}| = |q_1| \leq A - B$. For $n \geq 1$, an application of Lemma 1.1 and the inequality (6) for $t = n$, $s = n+1$ in (14) gives

$$|\tilde{\gamma}_n| = |\gamma_{n+1}^{(-1)}| = \frac{1}{n} |a_{n+1}^{(-n)}| \leq \frac{1}{n} \prod_{m=0}^{n} \left(\frac{(A-B)n + mB}{m+1} \right).$$

The sharpness follows for the function g_1 given by (13). □

For $A = 2\beta - 1$, $B = 1$ $(\beta > 1)$, the above theorem reduces to [1, Theorem 4.8, p. 18]. Recall that for $-1 \leq B < A \leq 1$,

$$\Im[A,B] := \left\{ f \in \mathcal{S} : f'(z) \prec \frac{1+Az}{1+Bz} \right\}.$$

The following theorem gives the sharp inverse coefficient estimates for functions in the class $\Im[1,B]$ and its proof is based on the fact that if $p \in \mathcal{P}[A,B]$ $(-1 \leq B < A \leq 1)$, then $1/p \in \mathcal{P}[-B,-A]$ $(-1 \leq -A < -B \leq 1)$.

Theorem 2.5. *For $-1 \leq B < 1$, let $f \in \Im[1, B]$ and $g(z) = \int_0^z (1-t)/(1-Bt)\,dt$ ($|z| < 1$). If $f^{-1}(w) =: F(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ and $g^{-1}(w) =: G(w) = w + \sum_{n=2}^{\infty} A_n w^n$ where w lies in some neighbourhood of the origin, then for each $n \geq 2$, $|\gamma_n| \leq A_n$. The result is sharp.*

Proof. Since $f' \in \mathcal{P}[1, B]$, $f'(z) = p(z)$ for some $p \in \mathcal{P}[1, B]$. Let $w = f(z)$ then $f'(z)F'(w) = 1$ and so we have

$$F'(w) = P(F(w))$$

where $P(z) := 1/p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}[-B, -1]$. This gives

$$1 + \sum_{n=1}^{\infty} (n+1)\gamma_{n+1} w^n = 1 + \sum_{n=1}^{\infty} c_n F(w)^n.$$

On comparing the coefficients of w^n, we have

$$(n+1)\gamma_{n+1} = \sum_{i=1}^{n} c_i \gamma_{n,i} \quad (n \geq 1) \tag{15}$$

where $\gamma_{n,i}$ denotes the coefficient of w^n in the expansion of $F(w)^i$ and $\gamma_{n,n} = 1$. Since $g'(z) = (1-z)/(1-Bz) \in \mathcal{P}[1, B]$, proceeding as above, we have

$$G'(w) = \frac{1 - BG(w)}{1 - G(w)} \tag{16}$$

which gives

$$\sum_{n=1}^{\infty} (n+1) A_{n+1} w^n = \sum_{n=1}^{\infty} (1-B) G(w)^n.$$

Comparing the coefficients of w^{n-1}, we get

$$nA_n = (1-B) \sum_{i=1}^{n-1} A_{n-1,i} \quad (n \geq 2) \tag{17}$$

where $A_{n-1,i}$ denotes the coefficient of w^{n-1} in the expansion of $G(w)^i$ and $A_{n-1,n-1} = 1$. We first show that $A_n > 0$ for all $n \geq 2$. By using the power series expansion of G in (16) and on comparing the coefficients of both sides, we obtain

$$2A_2 = 1 - B, \quad 3A_3 = (1 - B + 2)A_2, \quad \text{and}$$
$$(n+1)A_{n+1} = (1 - B + n)A_n + \sum_{k=1}^{n-2} (k+1) A_{k+1} A_{n-k} \quad (n > 2).$$

Since $-1 \leq B < 1$, $A_2 = (1-B)/2 > 0$. By using induction on n, it can be easily seen from the above relations that $A_n > 0$ for all $n \geq 2$.

Next, we shall show that for all $n \geq 2$, $|\gamma_n| \leq A_n$. We proceed by induction on n. Since $P \in \mathcal{P}[-B, -1]$, by using Lemma 1.3, $|c_i| \leq 1 - B$ for each $i \geq 1$. Clearly, the result holds for $n = 2$. Assume that $|\gamma_i| \leq A_i$ for $i \leq n-1$. It is easy to observe that $\gamma_{n,i} = S_i(\gamma_2, \gamma_3, \ldots, \gamma_{n-1})$ is a polynomial in $\gamma_2, \gamma_3, \ldots, \gamma_{n-1}$ with non-negative coefficients and thus $|\gamma_{n,i}| \leq S_i(|\gamma_2|, |\gamma_3|, \ldots, |\gamma_{n-1}|) \leq S_i(A_2, A_3, \ldots, A_{n-1})$. Therefore, in view of (15) and (17), we have

$$n|\gamma_n| \leq \sum_{i=1}^{n-1} |c_i||\gamma_{n-1,i}| \leq (1-B) \sum_{i=1}^{n-1} S_i(A_2, A_3, \ldots, A_{n-2}) = (1-B) \sum_{i=1}^{n-1} A_{n-1,i} = nA_n$$

where $A_{n-1,i} = S_i(A_2, A_3, \ldots, A_{n-2})$ is the coefficient of w^{n-1} in the expansion of $G(w)^i$. □

For $B = -1$, the above theorem reduces to the theorem given in [18].

The following theorem has been proved in [1, Theorem 4.4, p. 14] by using the coefficient bounds of the functions in the class $\mathcal{P}$ but we are providing a slightly different proof by making use of the coefficient bounds of the functions in the class $\mathcal{P}[2\beta-1,1]$ ($\beta > 1$) which shortens the computations involved in the proof to some extent.

Theorem 2.6. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{K}[2\beta-1,1]$ ($\beta > 1$) and $f^{-1}(w) =: F(w) = w + \sum_{n=2}^{\infty}\gamma_n w^n$ in some neighbourhood of the origin. Then for $n \geq 2$,*

$$|\gamma_n| \leq \frac{1}{n}\prod_{m=0}^{n-2}\left(\frac{2(\beta-1)+m(2\beta-1)}{m+1}\right).$$

The result is sharp.

Proof. Since $f \in \mathcal{K}[2\beta-1,1]$, we have $1 + zf''(z)/f'(z) = p(z)$ where $p(z) = 1 + \sum_{i=1}^{\infty} c_i z^i \in \mathcal{P}[2\beta-1,1]$ and $\beta > 1$. This gives

$$\frac{d}{dw}\left(\frac{F(w)}{F'(w)}\right) = 1 - \frac{F(w)F''(w)}{(F'(w))^2} = p(F(w)) \tag{18}$$

where $w = f(z)$ lies in some disk around the origin. Integrate the equation (18) along the line segment $[0, w]$ and using the power series expansions of F and p, we have

$$\sum_{n=1}^{\infty}\gamma_n w^n = \sum_{n=1}^{\infty} n\gamma_n w^n + \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} k\gamma_k \sum_{j=1}^{n-k} c_j \frac{\gamma_{n-k,j}}{n-k+1}\right)w^n \tag{19}$$

where $\gamma_1 = 1$ and $\gamma_{n-k,j}$ denotes the coefficient of w^{n-k} in the expansion of $F(w)^j$ with $\gamma_{n-k,n-k} = 1$.

On comparing the coefficients of w^n, we have

$$-(n-1)\gamma_n = \sum_{k=1}^{n-1}\frac{k\gamma_k}{n-k+1}\sum_{j=1}^{n-k} c_j\gamma_{n-k,j} \quad (n \geq 2). \tag{20}$$

Define a function $f_1 : \mathbb{D} \to \mathbb{C}$ such that

$$f_1'(z) = (1-z)^{2(\beta-1)}. \tag{21}$$

Then $F_1(w) := f_1^{-1}(w) = w + A_2z^2 + A_3z^3 + \cdots$ where for $n \geq 2$,

$$A_n := \frac{1}{n}\prod_{m=0}^{n-2}\left(\frac{2(\beta-1)+m(2\beta-1)}{m+1}\right). \tag{22}$$

We shall show that for all $n \geq 2$, $|\gamma_n| \leq A_n$. We proceed by induction on n. Since $p \in \mathcal{P}[2\beta-1,1]$ ($\beta > 1$), an application of Lemma 1.3 gives $|c_j| \leq 2(\beta-1)$ for each $j \geq 1$. Therefore, the desired estimate holds for $n = 2$. Assume that the theorem is true for $j \leq n-1$ and thus we have $|\gamma_j| \leq A_j$ for $j \leq n-1$. Since $\gamma_{n,j} = S_j(\gamma_2, \gamma_3, \ldots, \gamma_{n-1})$ is a polynomial in γ_2, γ_3, ..., γ_{n-1} with non-negative coefficients, we have $|\gamma_{n,j}| \leq S_j(|\gamma_2|, |\gamma_3|, \ldots, |\gamma_{n-1}|) \leq S_j(A_2, A_3, \ldots, A_{n-1})$. An application of induction hypothesis and bounds of c_j in (20) gives

$$\begin{aligned}(n-1)|\gamma_n| &\leq 2(\beta-1)\sum_{k=1}^{n-1}\frac{k|\gamma_k|}{n-k+1}\sum_{j=1}^{n-k}|\gamma_{n-k,j}| \\ &\leq 2(\beta-1)\sum_{k=1}^{n-1}\frac{kA_k}{n-k+1}\sum_{j=1}^{n-k}S_j(A_2, A_3, \ldots, A_{n-k-1}) \\ &= 2(\beta-1)\sum_{k=1}^{n-1}\frac{kA_k}{n-k+1}\sum_{j=1}^{n-k}A_{n-k,j}\end{aligned} \tag{23}$$

where $A_1 = 1$ and $A_{n-k,j}$ denotes the coefficient of w^{n-k} in the expansion of $F_1(w)^j$ with $A_{n-k,n-k} = 1$. We now show that for each $n \geq 2$,

$$2(\beta-1)\sum_{k=1}^{n-1}\frac{kA_k}{n-k+1}\sum_{j=1}^{n-k}A_{n-k,j} = (n-1)A_n. \tag{24}$$

For f_1, given by (21), we have

$$1+\frac{zf_1''(z)}{f_1'(z)} = \frac{1-(2\beta-1)z}{1-z}.$$

In terms of $F_1 := f_1^{-1}$, the above equation can be rewritten as

$$\frac{d}{dw}\left(\frac{F_1(w)}{F_1'(w)}\right) = 1-\frac{F_1(w)F_1''(w)}{(F_1'(w))^2} = \frac{1-(2\beta-1)F_1(w)}{1-F_1(w)}.$$

By proceeding as in (19), we obtain

$$\sum_{n=1}^{\infty}A_nw^n = \sum_{n=1}^{\infty}nA_nw^n - 2(\beta-1)\sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1}kA_k\sum_{j=1}^{n-k}\frac{A_{n-k,j}}{n-k+1}\right)w^n.$$

On comparing the coefficients of w^n, we get

$$(n-1)A_n = 2(\beta-1)\sum_{k=1}^{n-1}\frac{kA_k}{n-k+1}\sum_{j=1}^{n-k}A_{n-k,j} \quad (n\geq 2).$$

This proves (24) and hence, in view of (23), we have $|\gamma_n| \leq A_n$. The sharpness follows for the function f_1, given in (21). □

Corollary 2.7. *Let $f \in \mathcal{K}[2\beta-1,1]$ ($\beta > 1$) and $f^{-1}(w) =: F(w) = w + \sum_{n=2}^{\infty}\gamma_n w^n$ in some neighbourhood of the origin. If $F(w)/F'(w) = w + \sum_{n=2}^{\infty}\delta_n w^n$, then $|\delta_2| \leq \beta - 1$ and for $n > 2$,*

$$|\delta_n| \leq \frac{2(\beta-1)}{n(n-1)}\prod_{m=0}^{n-3}\left(\frac{2\beta+m(2\beta-1)}{m+1}\right).$$

The result is sharp.

Proof. On integrating the equation (18) along the line segment $[0, w]$ and using the power series expansions of F/F', F and p, we have

$$w+\sum_{n=2}^{\infty}\delta_n w^n = w + \sum_{n=2}^{\infty}\sum_{j=1}^{n-1}c_j\frac{\gamma_{n-1,j}}{n}w^n \tag{25}$$

where $\gamma_{n-1,j}$ denotes the coefficient of w^{n-1} in the expansion of $F(w)^j$ with $\gamma_{n-1,n-1} = 1$. Note that $\gamma_{n-1,j} = S_j(\gamma_2,\gamma_3,\ldots,\gamma_{n-2})$ is a polynomial in γ_2, γ_3, …, γ_{n-2} with non-negative coefficients. On comparing the coefficients of w^n in (25) and using Lemma 1.3 and Theorem 2.6, we have

$$\begin{aligned}|\delta_n| &\leq \frac{2(\beta-1)}{n}\sum_{j=1}^{n-1}S_j(A_2,A_3,\ldots,A_{n-2})\\ &= \frac{2(\beta-1)}{n}\sum_{j=1}^{n-1}A_{n-1,j}\end{aligned} \tag{26}$$

where $A_{n-1,j} = S_j(A_2, A_3, \ldots, A_{n-2})$ denotes the coefficient of w^{n-1} in the expansion of $F_1(w)^j$ with $A_{n-1,n-1} = 1$ and F_1 is given by (22). Corresponding to F_1, $F_1(w)/F_1'(w) = w - \sum_{n=2}^{\infty} B_n w^n$ where

$$B_2 := (\beta - 1) \quad \text{and} \quad B_n := \frac{2(\beta-1)}{n(n-1)} \prod_{m=0}^{n-3} \left(\frac{2\beta + m(2\beta - 1)}{m+1} \right) \ (n > 2).$$

For f_1, given by (21), by proceeding as in (25), we have

$$w - \sum_{n=2}^{\infty} B_n w^n = w - \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} 2(\beta - 1) \frac{A_{n-1,j}}{n} w^n.$$

On comparing the coefficients of w^n, we obtain

$$B_n = \frac{2(\beta-1)}{n} \sum_{j=1}^{n-1} A_{n-1,j}. \tag{27}$$

In view of (26) and (27), the desired estimates follow. □

In the generalized class $\mathcal{K}[A, B]$ $(-1 \leq B \leq 1 < A)$, the technique used in the Theorem 2.6 does not hold true. However, we are able to give the sharp estimation for the initial inverse coefficients for functions in $\mathcal{K}[A, B]$.

Theorem 2.8. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{K}[A, B]$ $(-1 \leq B \leq 1 < A)$ and $f^{-1}(w) =: F(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ in some neighbourhood of the origin. Then for $n = 2, \ldots, 6$,*

$$|\gamma_n| \leq \frac{1}{n} \prod_{m=0}^{n-2} \left(\frac{(A - B) + mA}{m+1} \right).$$

The result is sharp.

Proof. Since $f \in \mathcal{K}[A, B]$, $1 + zf''(z)/f'(z) \prec (1 + Az)/(1 + Bz)$ which is equivalent to $1 + zf''(z)/f'(z) \prec (1 - Az)/(1 - Bz)$. Let $g(z) := zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$ and $p(z) := zg'(z)/g(z) = 1 + b_1 z + b_2 z^2 + \cdots$. Then $p(z) \prec (1 - Az)/(1 - Bz)$ and for $n > 1$, we have

$$(n-1) n a_n = \sum_{k=1}^{n-1} (n-k) b_k a_{n-k}. \tag{28}$$

It is easy to observe that if $p \prec \varphi$, then

$$p(z) = \varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1} \right), \quad p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}. \tag{29}$$

Using (28) and (29) for $\varphi = (1 - Az)/(1 - Bz)$, the coefficients a_i can be expressed in terms of c_i, A and B, see [30]. In particular, we have

$$\begin{aligned}
a_2 &= -\frac{1}{4}(A - B)c_1, \\
a_3 &= \frac{1}{24}(A - B)\Big((A - 2B + 1)c_1^2 - 2c_2\Big), \\
a_4 &= -\frac{1}{192}(A - B)\Big((A - 2B + 1)(A - 3B + 2)c_1^3 - 2(3A - 7B + 4)c_1 c_2 + 8c_3\Big), \\
a_5 &= \frac{1}{1920}(A - B)\Big(- 4(3A^2 - 17AB + 11A + 23B^2 - 29B + 9)c_1^2 c_2 \\
&\quad + (A - 2B + 1)(A - 3B + 2)(A - 4B + 3)c_1^4 + 16(2A - 5B + 3)c_1 c_3 \\
&\quad + 12(A - 3B + 2)c_2^2 - 48c_4\Big)
\end{aligned}$$

and

$$\begin{aligned} a_6 = \frac{1}{23040}&(A-B)\Big(-(A-5B+4)(A-4B+3)(A-3B+2)(A-2B+1)c_1^5 \\ &+4(5A^3-50A^2B+35A^2+160AB^2-220AB+75A-163B^3+329B^2 \\ &-219B+48)c_1^3c_2-16(5A^2-30AB+20A+43B^2-56B+18)c_1^2c_3 \\ &+32(5A-17B+12)c_2c_3-4(15A^2-100AB+70A+157B^2-214B+72)c_1c_2^2 \\ &+48(5A-13B+8)c_1c_4-384c_5\Big). \end{aligned}$$

Using power series expansions of f and f^{-1} in the relation $f(f^{-1}(w)) = w$, or

$$w = f^{-1}(w) + a_2(f^{-1}(w))^2 + \cdots,$$

we obtain

$$\begin{aligned} \gamma_2 &= -a_2, \\ \gamma_3 &= 2a_2^2 - a_3, \\ \gamma_4 &= -5a_2^3 + 5a_2a_3 - a_4, \\ \gamma_5 &= 14a_2^4 - 21a_2^2a_3 + 6a_2a_4 + 3a_3^2 - a_5 \end{aligned}$$

and

$$\gamma_6 = 7\Big(-6a_2^5 + 12a_2^3a_3 - 4a_2^2a_4 + a_2(a_5 - 4a_3^2) + a_3a_4\Big) - a_6.$$

Substituting the expressions of a_i in terms of c_i in the above expressions of γ_i, we have

$$\begin{aligned} \gamma_2 &= \frac{1}{4}(A-B)c_1, \\ \gamma_3 &= \frac{1}{24}(A-B)\left((2A-B-1)c_1^2+2c_2\right), \\ \gamma_4 &= \frac{1}{192}(A-B)\left((2A-B-1)(3A-B-2)c_1^3+2(7A-3B-4)c_1c_2+8c_3\right), \\ \gamma_5 &= \frac{1}{1920}(A-B)\Big(p(A,B)\,c_1^2c_2+(2A-B-1)(3A-B-2)(4A-B-3)c_1^4 \\ &\quad +8(11A-5B-6)c_1c_3+4(7A-B-6)c_2^2+48c_4\Big) \end{aligned}$$

and

$$\begin{aligned} \gamma_6 = \frac{1}{23040}&(A-B)\Big(q(A,B)\,c_1^2c_3+r(A,B)\,c_1c_2^2+384(2A-B-1)c_1c_4 \\ &+s(A,B)\,c_1^3c_2+(2A-B-1)(3A-B-2)(4A-B-3)(5A-B-4)c_1^5 \\ &+16(25A-B-24)c_2c_3+384c_5\Big) \end{aligned}$$

where

$$\begin{aligned} p(A,B) &:= 4(23A^2-17AB-29A+3B^2+11B+9), \\ q(A,B) &:= 8(101A^2-81AB-121A+16B^2+49B+36), \\ r(A,B) &:= 4(127A^2-58AB-196A+3B^2+52B+72) \end{aligned}$$

and

$$s(A,B) := 4(163A^3 - 160A^2B - 329A^2 + 50AB^2 + 220AB + 219A - 5B^3 - 35B^2 - 75B - 48).$$

Since $-1 \le B \le 1 < A$, we can easily see that

$$\frac{\partial p(A,B)}{\partial A} = 4\big(29(A-1) + 17(A-B)\big) > 0,$$
$$\frac{\partial q(A,B)}{\partial A} = 8\big(121(A-1) + 81(A-B)\big) > 0$$

and

$$\frac{\partial r(A,B)}{\partial A} = 4\big(196(A-1) + 58(A-B)\big) > 0.$$

Therefore, $p(A,B) > p(1,B) = 12(1-B)^2 \ge 0$; $q(A,B) > q(1,B) = 128(1-B)^2 \ge 0$ and $r(A,B) > r(1,B) = 12(1-B)^2 \ge 0$. Clearly,

$$\frac{\partial s(A,B)}{\partial A} = 4(489A^2 - 658A - 320AB + 219 + 220B + 50B^2)$$

and

$$\frac{\partial^2 s(A,B)}{\partial A^2} = 4\big(658(A-1) + 320(A-B)\big) > 0.$$

Therefore, $\partial s(A,B)/\partial A$ is a strictly increasing function of A and hence $\partial s(A,B)/\partial A > 200(1-B)^2 \ge 0$. Consequently, $s(A,B) > s(1,B) = 20(1-B)^3 \ge 0$.

Thus, for $n = 2,\ldots,6$, γ_n are polynomials in c_i $(i = 1,2,\ldots,5)$ with non-negative coefficients. Since $p_1 \in \mathcal{P}$, $|c_i| \le 2$ $(i = 1,2,\ldots)$ and therefore, the maximum of $|\gamma_n|$ would correspond to $|c_i| = 2$. On simplification, we get the desired estimates. Define a function $f_0 : \mathbb{D} \to \mathbb{C}$ such that

$$f_0'(z) = \begin{cases} (1-Bz)^{(A-B)/B}, & B \ne 0 \\ e^{-Az}, & B = 0. \end{cases}$$

The result is sharp for the function f_0. □

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