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A Fixed Point Theorem for Mappings on the ℓ_∞-Sum of a Metric Space and its Application

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Abstract. The aim of this paper is to prove a counterpart of the Banach fixed point principle for mappings $f : \ell_\infty(X) \to X$, where X is a metric space and $\ell_\infty(X)$ is the space of all bounded sequences of elements from X. Our result generalizes the theorem obtained by Miculescu and Mihail in 2008, who proved a counterpart of the Banach principle for mappings $f : X^m \to X$, where X^m is the Cartesian product of m copies of X. We also compare our result with a recent one due to Secelean, who obtained a weaker assertion under less restrictive assumptions. We illustrate our result with several examples and give an application.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

If (X, d) is a metric space and $m \in \mathbb{N}$, then by X^m we denote the Cartesian product of m copies of X. We endow X^m with the maximum metric:

$$d_m((x_0, ..., x_{m-1}), (y_0, ..., y_{m-1})) := \max\{d(x_0, y_0), ..., d(x_{m-1}, y_{m-1})\}.$$

Miculescu and Mihail in [6] and [7] obtained an interesting generalization of the Banach principle for mappings defined on X^m. Namely, they proved the following

Theorem 1.1. *Assume that (X, d) is a complete metric space and $g : X^m \to X$ is such that the Lipschitz constant $Lip(g) < 1$. Then there exists a unique point $x_* \in X$ such that $g(x_*, ..., x_*) = x_*$. Moreover, for every $x_0, ..., x_{m-1} \in X$, the sequence (x_k) defined by*

$$x_{m+k} = g(x_{k+m-1}, ..., x_k), \quad k \geq 0, \tag{1}$$

converges to x_*.

A point $x_* \in X$ which satisfies the equality $g(x_*, ..., x_*) = x_*$ is called *a generalized fixed point of g*.
An interesting study of such fixed points can also be found in the paper [1] of Professor Ljubomir B. Ćirić and S.B. Prešić.

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Theorem 1.1 gave a background for a version of the Hutchinson–Barnsley fractals theory for such mappings defined on finite Cartesian products – see the above mentioned papers and the references therein. Also, note that the above theorem can be extended to mappings which satisfy weaker contractive conditions – see, e.g., [9] and [10].

The next step was done by Secelean [8]. Denote by $\ell_\infty(X)$ the ℓ_∞-sum of a metric space X, that is, the set of all bounded sequences of elements of X:

$$\ell_\infty(X) := \{(x_k) \subset X : (x_k) \text{ is bounded}\}.$$

Endow $\ell_\infty(X)$ with the supremum metric:

$$d_s((x_n),(y_n)) := \sup\{d(x_n,y_n) : n \in \mathbb{N}^*\}, \tag{2}$$

where $\mathbb{N}^* := \{0,1,2,...\}$ (throughout the paper we enumerate sequences by nonnegative integers).

Remark 1.2. Let us notice that the notion of the $\ell_\infty(X)$-sum of a family of spaces originates from functional analysis; see, e.g., [4, p. xii].

Remark 1.3. It is also worth to observe that if X is bounded, then $\ell_\infty(X)$ is exactly the product of countably many copies of X, that is, $\ell_\infty(X) = \prod_{k=0}^{\infty} X$. On the other hand, if X is unbounded, then $\ell_\infty(X)$ is a proper subspace of $\prod_{k=0}^{\infty} X$.

If $f : \ell_\infty(X) \to X$, then we define $f_s : X \to X$ by

$$f_s(x) := f(x,x,...), \quad x \in X. \tag{3}$$

A point $x_* \in X$ is called *a generalized fixed point of f*, if x_* is a fixed point of f_s, i.e., if x_* satisfies:

$$f(x_*,x_*,...) = x_*.$$

Secelean [8, Theorem 3.1] proved the following fixed point theorem:

Theorem 1.4. *Assume that X is a complete metric space and $f : \ell_\infty(X) \to X$ is such that $Lip(f) < 1$. Then there exists a unique generalized fixed point x_* of f. Moreover, for every $x = (x_n) \in \ell_\infty(X)$, the sequence (y_k) defined by*

$$y_k := f\left(f_s^k(x_0), f_s^k(x_1), f_s^k(x_2), ...\right), \; k \geq 0, \tag{4}$$

converges to x_*. More precisely, for every $k \in \mathbb{N}^*$,

$$d(x_*, y_k) \leq \frac{\mathrm{Lip}(f)^{k+1}}{1 - \mathrm{Lip}(f)} \sup\{d(f_s(x_i), x_i) : i \in \mathbb{N}^*\}.$$

Remark 1.5. In fact, Secelean formulated his result in a more general way. Firstly, he considered also weaker contractive conditions and secondly, he studied also mappings defined on a finite product of spaces. However, the idea of dealing with weaker contractive conditions is relatively similar (but much more technically complicated), and also we will not be interested in the case of finite products here.

Remark 1.6. Theorem 1.4 can be viewed as a generalization of the Banach fixed point theorem or Theorem 1.1. However, it seems that the iteration procedure (4) is not a very natural counterpart of (1). It is rather closer to iterating map f_s.

We are going to show that under more restrictive (yet still natural) contractive conditions, we can obtain a stronger thesis. In particular, our result will imply the whole Theorem 1.1. Also, we will present examples that our assumptions are essential for the thesis and, in particular, that Theorem 1.4 is too weak to obtain our assertion.
Finally, we will present an application.

2. Other Metrics on $\ell_\infty(X)$

2.1. Metrics $d_{s,(a_n)}$ and $d_{p,(a_n)}$

Let (X,d) be a metric space. We start with defining other metrics on space $\ell_\infty(X)$. If (a_n) is a sequence of reals, then set:

$$d_{s,(a_n)}(x,y) := \sup\{a_n d(x_n,y_n) : n \in \mathbb{N}^*\} \quad \text{for any } x = (x_n), y = (y_n) \in \ell_\infty(X)$$

and, if additionally $a_n \geq 0$, $n \in \mathbb{N}^*$, and $p \in [1,\infty)$, then set:

$$d_{p,(a_n)}(x,y) := \left(\sum_{n=0}^{\infty} a_n d^p(x_n,y_n)\right)^{1/p} \quad \text{for any } x = (x_n), y = (y_n) \in \ell_\infty(X).$$

It turns out that under natural assumptions on a sequence (a_n), functions $d_{s,(a_n)}$ and $d_{p,(a_n)}$ are metrics with good properties:

Proposition 2.1. *Let (X,d) be a metric space such that X is not a singleton and (a_n) be a sequence of reals. The following statements are equivalent:*

(i) $d_{s,(a_n)}$ is a metric on $\ell_\infty(X)$;

(ii) $a_n > 0$ for any $n \in \mathbb{N}^*$ and $(a_n) \in l_\infty$.

Moreover, if (a_n) is as in (ii), then the convergence with respect to $d_{s,(a_n)}$ implies the convergence in the Tychonoff product topology (when considering $\ell_\infty(X)$ as a subspace of $\prod_{k=0}^{\infty} X$).

Proof. $(i) \Rightarrow (ii)$: Suppose, on the contrary, that $a_p \leq 0$ for some $p \in \mathbb{N}^*$. By hypothesis, there exist $x,y \in X$ such that $x \neq y$. Define $\mathbf{x} := (x,x,...)$ and $\mathbf{y} = (x,...,x,y,x,...)$, where the p-th coordinate of $\mathbf{y}$ is equal to y. Then

$$0 < d_{s,(a_n)}(\mathbf{x},\mathbf{y}) = \max\{0, a_p d(x,y)\} = 0,$$

which yields a contradiction. Thus $a_n > 0$ for any $n \in \mathbb{N}^*$.
We show that $(a_n) \in l_\infty$. Take again $x,y \in X$ with $x \neq y$ and define $\mathbf{x} := (x,x,...)$ and $\mathbf{y} = (y,y,...)$. Then $d_{s,(a_n)}(\mathbf{x},\mathbf{y}) = \sup_{n\in\mathbb{N}^*} a_n d(x,y) = d(x,y)\sup_{n\in\mathbb{N}^*} a_n$, so $\sup_{n\in\mathbb{N}^*} a_n = \frac{d_{s,(a_n)}(\mathbf{x},\mathbf{y})}{d(x,y)} < \infty$. Thus (a_n) is bounded.

The proof of $(ii) \Rightarrow (i)$ is standard and we leave it to the reader.

Now assume that $d_{s,(a_n)}(x^k,x) \to 0$, where $x^k = (x_i^k)_{i\in\mathbb{N}^*}$ and $x = (x_i)_{i\in\mathbb{N}^*}$. Then for any $i \in \mathbb{N}^*$,

$$0 \leq a_i d(x_i^k, x_i) \leq d_{s,(a_n)}(x^k, \mathbf{x}),$$

which implies that $\lim_{k\to\infty} d(x_i^k,x_i) = 0$, i.e., (x^k) converges to x in the Tychonoff topology. □

Proposition 2.2. *Let (X,d) be a metric space such that X is not a singleton and (a_n) be a bounded sequence of positive reals. Let τ_T denote the Tychonoff product topology on $\ell_\infty(X)$ and $\tau_{d_{s,(a_n)}}$ be the topology induced by metric $d_{s,(a_n)}$. The following statements are equivalent:*

(i) $\tau_T = \tau_{d_{s,(a_n)}}$;

(ii) $(a_n) \in c_0$ and (X,d) is bounded.

Proof. $(i) \Rightarrow (ii)$: Suppose, on the contrary, that $(a_n) \notin c_0$. Then there exist $\varepsilon_0 > 0$ and a subsequence (a_{n_j}) such that $a_{n_j} \geq \varepsilon_0$ for any $j \in \mathbb{N}^*$. Take $x,y \in X$ with $x \neq y$, and define $\mathbf{x} = (x,x,...)$ and $x^k = (x_i^k)_{i\in\mathbb{N}^*}$, where

$$x_i^k := \begin{cases} x & \text{if} \quad i \leq k, \\ y & \text{if} \quad i > k. \end{cases}$$

Clearly, (x^k) converges to $\mathbf{x}$ in (X, τ_T), so by (i), $d_{s,(a_n)}(x^k, \mathbf{x}) \to 0$. On the other hand,

$$d_{s,(a_n)}(x^k, \mathbf{x}) \geq \sup_{j\in\mathbb{N}^*} a_{n_j} d(x^k_{n_j}, x) \geq \varepsilon_0 d(x, y),$$

so letting k tend to ∞, we obtain $0 \geq \varepsilon_0 d(x, y) > 0$, a contradiction. Thus $(a_n) \in c_0$.
Now, suppose that (X, d) is unbounded. Then there exists a sequence (x_k) such that $d(x_k, x_0) > \frac{1}{a_k}$ for any $k \in \mathbb{N}$. Set $\mathbf{x} := (x_0, x_0, ...)$ and $x^k := (x^k_i)_{i\in\mathbb{N}^*}$, where

$$x^k_i := \begin{cases} x_0 & \text{if} \quad i \leq k, \\ x_k & \text{if} \quad i > k. \end{cases}$$

Then (x^k) converges to $\mathbf{x}$ in (X, τ_T), so by (i), $d_{s,(a_n)}(x^k, \mathbf{x}) \to 0$. However, if $k \geq 1$, then

$$d_{s,(a_n)}(x^k, \mathbf{x}) = \sup_{j\geq k+1} a_j d(x_j, x_0) \geq a_{k+1} d(x_{k+1}, x_0) > 1,$$

which yields a contradiction.

(*ii*) ⇒ (*i*): By the last part of Proposition 2.1, it suffices to show that the convergence in (X, τ_T) implies the convergence with respect to $d_{s,(a_n)}$. Assume that $x^k \xrightarrow{\tau_T} x$, where $x^k = (x^k_i)_{i\in\mathbb{N}^*}$ and $x = (x_i)_{i\in\mathbb{N}^*}$. That means $\lim_{k\to\infty} d(x^k_i, x_i) = 0$ for any $i \in \mathbb{N}^*$. Fix $\varepsilon > 0$. Since $a_n \to 0$, there is $p \in \mathbb{N}^*$ such that for $i > p$, $a_i < \frac{\varepsilon}{\operatorname{diam} X}$. Then $a_i d(x^k_i, x_i) < \varepsilon$ for $i > p$ and $k \in \mathbb{N}^*$. Since $\lim_{k\to\infty} d(x^k_i, x_i) = 0$ for $i = 0, 1, ..., p$, there is $j \in \mathbb{N}^*$ such that for $k \geq j$ and $i = 0, ..., p$, $d(x^k_i, x_i) < \frac{\varepsilon}{a_i}$. Then for $k \geq j$, $d_{s,(a_n)}(x^k, x) \leq \varepsilon$. Thus we get that $d_{s,(a_n)}(x^k, x) \to 0$. □

Using a similar argument as in the proofs of Propositions 2.1 and 2.2, it is possible to prove the following two results for metrics $d_{p,(a_n)}$.

Proposition 2.3. *Let (X, d) be a metric space such that X is not a singleton, (a_n) be a sequence of nonnegative reals and $p \in [1, \infty)$. The following statements are equivalent:*

(i) $d_{p,(a_n)}$ is a metric on $\ell_\infty(X)$;

(ii) $a_n > 0$ for any $n \in \mathbb{N}^*$ and $(a_n) \in l_1$.

Moreover, if (a_n) is as in (ii), then the convergence with respect to $d_{p,(a_n)}$ implies the convergence in the Tychonoff product topology.

Proposition 2.4. *Let (X, d) be a metric space such that X is not a singleton and (a_n) be a sequence of positive reals such that $(a_n) \in l_1$. The following statements are equivalent:*

(i) $\tau_T = \tau_{d_{p,(a_n)}}$;

(ii) (X, d) is bounded.

In what follows, when writing $d_{s,(a_n)}$ (or $d_{p,(a_n)}$) we automatically assume that (a_n) is chosen so that $d_{s,(a_n)}$ (or $d_{p,(a_n)}$) is a metric.
A natural question arises whether these metrics are complete if d is so. Clearly, if $a_n = 1$ for all $n \in \mathbb{N}^*$, then $d_{s,(a_n)}$ is exactly the metric d_s considered by Secelean, so it is complete. Also, if $\inf\{a_n : n \in \mathbb{N}^*\} > 0$, then the metrics $d_{s,(a_n)}$ and d_s are Lipschitz equivalent, hence $d_{s,(a_n)}$ is also complete.
The following example shows that the answer can be negative if $a_n \to 0$.

Example 2.5. Let $(X, d) := (\mathbb{R}, |\cdot|)$ and for every $k \in \mathbb{N}^*$, let $x^k := (0, 1, .., k, 0, 0, ...)$. Then:
- $d_{s,(a_n)}(x^k, x^{k+1}) = (k + 1)a_{k+1}$, so if $\sum(k + 1)a_{k+1} < \infty$, then (x^k) is Cauchy in $d_{s,(a_n)}$;
- if $p \geq 1$, then $d_{p,(a_n)}(x^k, x^{k+1}) = (k + 1)a^{1/p}_{k+1}$, so if $\sum(k + 1)a^{1/p}_{k+1} < \infty$, then (x^k) is Cauchy in $d_{p,(a_n)}$.
On the other hand, (x^k) cannot be convergent since, by Propositions 2.1 and 2.3, convergence in any of metrics $d_{s,(a_n)}, d_{p,(a_n)}$ implies the convergence of each coordinate.

Corollary 2.6. *Assume that* $a_n \to 0$.
(1) If (X,d) *is bounded and complete, then* $\ell_\infty(X)$ *is complete with respect to any of metrics* $d_{s,(a_n)}, d_{p,(a_n)}$.
(2) If (X,d) *is complete and* $(x^k) = ((x_i^k)_{i\in\mathbb{N}^*})$ *is a Cauchy sequence in* $\ell_\infty(X)$ *(with respect to any of metrics* $d_{s,(a_n)}, d_{p,(a_n)}$*) such that the set* $\{x_i^k : i,k \in \mathbb{N}^*\}$ *is bounded in X, then* (x^k) *is convergent to* $x = (x_i)$, *where* $x_i = \lim_{k\to\infty} x_i^k$, $i \in \mathbb{N}^*$.
(3) If $x^k = (x_i^k)_{i\in\mathbb{N}^*}$, $k \in \mathbb{N}^*$, *and* $x = (x_i)$ *are elements of* $\ell_\infty(X)$ *such that the set* $\{x_i^k : i,k \in \mathbb{N}^*\}$ *is bounded, then* $x^k \xrightarrow{d'} x$ *iff* $x^k \to x$ *with respect to the Tychonoff topology on* $\ell_\infty(X)$, *where* d' *is any of metrics* $d_{s,(a_n)}, d_{p,(a_n)}$.
(4) If (X,d) *is compact, then* $\ell_\infty(X)$ *is compact with respect to any of metrics* $d_{s,(a_n)}, d_{p,(a_n)}$.

Proof. (1). If $(x^k) = ((x_i^k)_{i\in\mathbb{N}^*})$ is a Cauchy sequence in $\ell_\infty(X)$, then each $(x_i^k)_{k\in\mathbb{N}^*}$ is Cauchy in (X,d), hence convergent to some $x_i \in X$. Then by Propositions 2.2 and 2.4, $x^k \to (x_i)$ with respect to any of metrics $d_{s,(a_n)}, d_{p,(a_n)}$.
(2) follows from (1) used for the subspace $\ell_\infty(Y) \subset \ell_\infty(X)$, where $Y := \overline{\{x_i^k : i,k \in \mathbb{N}^*\}}$.
(3) follows from Propositions 2.2 and 2.4 and (1), used for $Y := \overline{\{x_i^k : i,k \in \mathbb{N}^*\}}$.
(4) is a direct consequence of Propositions 2.2 and 2.4. □

Remark 2.7. It is worth to remark that the definitions of metrics $d_{s,(a_n)}$ and $d_{p,(a_n)}$ base on the same ideas as definitions of weighted L_p-sum of spaces considered in functional analysis (see for example [2]). However, our setting is strictly metric.

2.2. Particular versions of metrics $d_{s,(a_n)}$ *and* $d_{p,(a_n)}$*: metrics* $d_{s,q}$ *and* $d_{p,q}$

From now on we will assume that (a_n) is a geometric sequence (q^n) for $q \in (0,1]$. As we will show, the obtained results in such a case imply corresponding results for the general case of (a_n).
For $q \in (0,1]$, denote $d_{s,q} := d_{s,(q^n)}$, that is,

$$d_{s,q}(x,y) := \sup\{q^n d(x_n,y_n) : n \in \mathbb{N}^*\} \quad \text{for any } x = (x_n), y = (y_n) \in \ell_\infty(X).$$

By Proposition 2.1, $d_{s,q}$ is a metric. Observe that in this notation, the supremum metric d_s is exactly the metric $d_{s,1}$.
If additionally $q < 1$ and $p \in [1,\infty)$, denote $d_{p,q} := d_{p,(q^n)}$, that is,

$$d_{p,q}(x,y) := \left(\sum_{n=0}^{\infty} q^n d^p(x_n,y_n)\right)^{1/p} \quad \text{for any } x = (x_n), y = (y_n) \in \ell_\infty(X).$$

By Proposition 2.3, $d_{p,q}$ is a metric.
The following result shows some connections between $d_{s,q}$ and $d_{p,q}$.

Proposition 2.8. *In the above frame, assume that* $q < 1$ *and* $p \geq 1$. *Then the following statements hold:*

(i) $d_{s,q} \leq d_{p,q^p}$;

(ii) if $q \leq q' \leq 1$, *then* $d_{s,q} \leq d_{s,q'}$;

(iii) if $q^{1/p} < q' \leq 1$, *then* $d_{p,q} \leq \left(1 - \frac{q}{(q')^p}\right)^{-1/p} d_{s,q'}$;

(iv) for every $x,y \in \ell_\infty(X)$, $\lim_{p'\to\infty} d_{p',q}(x,y) = d_{s,1}(x,y)$.

Proof. Let $x = (x_n), y = (y_n) \in \ell_\infty(X)$.
We prove (i). Since $q^n \to 0$ and $(d(x_n,y_n))$ is bounded, we have for some $k_0 \in \mathbb{N}^*$:

$$d_{s,q}(x,y) = \sup_{n\in\mathbb{N}^*} q^n d(x_n,y_n) = q^{k_0} d(x_{k_0},y_{k_0}) = \left(q^{pk_0} d^p(x_{k_0},y_{k_0})\right)^{1/p} \leq \left(\sum_{n=0}^{\infty} (q^p)^n d^p(x_n,y_n)\right)^{1/p} = d_{p,q^p}(x,y).$$

(ii) follows from the fact that for any $n \in \mathbb{N}^*$, $q^n d(x_n, y_n) \le (q')^n d(x_n, y_n)$.
(iii) follows from

$$d_{p,q}(x,y) = \left(\sum_{n=0}^{\infty} q^n d^p(x_n, y_n)\right)^{1/p} = \left(\sum_{n=0}^{\infty} \frac{q^n}{(q')^{pn}}\big((q')^n d(x_n, y_n)\big)^p\right)^{1/p}$$

$$\le \left(\sum_{n=0}^{\infty}\left(\frac{q}{(q')^p}\right)^n\right)^{1/p} d_{s,q'}(x,y) = \left(1 - \frac{q}{(q')^p}\right)^{-1/p} d_{s,q'}(x,y).$$

We prove (iv). Let $\varepsilon > 0$. Then there exists $k_0 \in \mathbb{N}^*$ such that:

$$d_{s,1}(x,y) \le d(x_{k_0}, y_{k_0}) + \varepsilon = \frac{1}{q^{k_0/p}}\left(q^{k_0} d^p(x_{k_0}, y_{k_0})\right)^{1/p} + \varepsilon \le \frac{1}{q^{k_0/p}} d_{p,q}(x,y) + \varepsilon.$$

Hence, $q^{k_0/p} d_{s,1}(x,y) \le d_{p,q}(x,y) + \varepsilon$ and therefore: $d_{s,1}(x,y) \le \liminf_{p\to\infty} d_{p,q}(x,y) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $d_{s,1}(x,y) \le \liminf_{p\to\infty} d_{p,q}(x,y)$. On the other hand, (iii) (with $q' = 1$) implies that $\limsup_{p\to\infty} d_{p,q}(x,y) \le d_{s,1}(x,y)$. Thus we arrive to the desired equality. □

By the previous section, if X is bounded, then all metrics $d_{s,q}$ and $d_{p,q}$ are equivalent (and generate the Tychonoff topology on $\ell_\infty(X) = \prod_{k=0}^{\infty} X$). In general, this is not the case. For example, $d_{s,q}$ and d_{p,q^p} need not be equivalent (recall point (i) of the above proposition), as the next example shows:

Example 2.9. Let $q \in (0,1)$, $(\mathbb{R}, |\cdot|)$ be the Euclidean space and $p \ge 1$. For $k \in \mathbb{N}$, let $x^k = (x_i^k)_{i\in\mathbb{N}^*}$ be defined by

$$x_i^k := \begin{cases} \frac{1}{(k+1)^{1/p} q^i} & \text{if} \quad i \le k, \\ 0 & \text{if} \quad i > k. \end{cases}$$

Then $x^k \to (0)$, the zero sequence, with respect to $d_{s,q}$, but does not converge with respect to d_{p,q^p}. Indeed, for every $k \in \mathbb{N}^*$,

$$d_{s,q}(x^k, (0)) = \sup\{q^n d(x_n^k, 0) : n \in \mathbb{N}^*\} = \frac{1}{(k+1)^{1/p}},$$

but

$$d_{p,q^p}(x^k, (0)) = \left(\sum_{n=0}^{\infty} (q^p)^n d^p(x_n^k, 0)\right)^{1/p} = \left(\sum_{n=0}^{k} \frac{1}{(k+1)}\right)^{1/p} = 1. \tag{5}$$

3. Main Results

3.1. *Sequences of generalized iterates and a selfmap of $\ell_\infty(X)$*

For any mapping $f : \ell_\infty(X) \to X$, define $\tilde{f} : \ell_\infty(X) \to \ell_\infty(X)$ as follows:

$$\tilde{f}((x_n)) = (f((x_n)), x_0, x_1, ...) \text{ for any } (x_n) \in \ell_\infty(X).$$

Now if $x := (x_n) \in \ell_\infty(X)$, then we set $\tilde{x}^0 := x$ and

$$x^1 := f(\tilde{x}^0) \quad \text{and} \quad \tilde{x}^1 := \tilde{f}(\tilde{x}^0).$$

Assume that for some $k \in \mathbb{N}$, we defined $x^i \in X$, and $\tilde{x}^i \in \ell_\infty(X)$ for $i \in \{1, ..., k\}$. Then set

$$x^{k+1} := f(\tilde{x}^k) \quad \text{and} \quad \tilde{x}^{k+1} := \tilde{f}(\tilde{x}^k).$$

In this way we defined sequences $(x^k) \subset X$ and $(\tilde{x}^k) \subset \ell_\infty(X)$. Observe that for every $k \in \mathbb{N}$,

$$\tilde{x}^k = (f(\tilde{x}^{k-1}), ..., f(\tilde{x}^0), x_0, x_1, ...) = (x^k, ..., x^1, x_0, x_1, ...). \tag{6}$$

Clearly, $(\tilde{x}^k)$ is the sequence of iterates of $x = (x_n) \in \ell_\infty(X)$ of mapping $\tilde{f}$. We will say that (x^k) is *the sequence of generalized iterates of function f at x.*
Recall that $x_* \in X$ is a generalized fixed point of f, if

$$f(x_*, x_*, ...) = x_*.$$

Definition 3.1. A generalized fixed point $x_* \in X$ of map $f : \ell_\infty(X) \to X$ is called a *generalized contractive fixed point* (GCFP), if for every $x \in \ell_\infty(X)$, the sequence (x^k) of generalized iterates converges to x_*.

The above definition is a counterpart of the notion of a contractive fixed point of a selfmap of a metric space introduced by Leader and Hoyle [3]:
if $g : Y \to Y$, then a fixed point $y_* \in Y$ of g is called *a contractive fixed point* (CFP), if for every $y \in Y$, the sequence of iterates $(g^k(y))$ converges to y_*.
We will show that the existence of a GCFP of f is strongly related to the existence of a CFP of $\tilde{f}$. We start with the lemma which follows directly from (6):

Lemma 3.2. *In the above frame let $x = (x_n) \in \ell_\infty(X)$.*

(i) If $\tilde{x}^k \to \mathbf{x}$ for some $\mathbf{x} \in \ell_\infty(X)$ with respect to the Tychonoff topology, then $\mathbf{x} = (x, x, x, ...)$ for some $x \in X$, and $x^k \to x$.

(ii) If $x^k \to x$ for some $x \in X$, then $\tilde{x}^k \to \mathbf{x}$ with respect to the Tychonoff topology, where $\mathbf{x} = (x, x, x, ...)$.

We are ready to state the theorem:

Theorem 3.3. *In the above frame,*

(i) f has a GCFP iff $\tilde{f}$ has a CFP with respect to the Tychonoff topology on $\ell_\infty(X)$;

(ii) if $\mathbf{x}_*$ is a CFP of $\tilde{f}$ with respect to the Tychonoff topology, then $\mathbf{x}_* = (x_*, x_*, ...)$, where x_* is a GCFP of f.

Proof. Let $\mathbf{x}_*$ be a CFP of $\tilde{f}$. Then by Lemma 3.2(i), $\mathbf{x}_* = (x_*, x_*, ...)$ for some $x_* \in X$ and $x^k \to x_*$ for every $x \in \ell_\infty(X)$ (as $\tilde{x}^k \to \mathbf{x}_*$ by hypothesis). Also,

$$(f(x_*, x_*, ...), x_*, x_*, ...) = \tilde{f}(x_*, x_*, ...) = (x_*, x_*, ...),$$

so x_* is a generalized fixed point of f, and in view of the above observations, it is a GCFP. Conversely, if x_* is a GCFP of f, then by Lemma 3.2(ii), $\tilde{x}^k \to \mathbf{x}_*$ for any $x \in \ell_\infty(X)$, where $\mathbf{x}_* = (x_*, x_*, x_*, ...)$. As $\mathbf{x}_*$ is obviously a fixed point of $\tilde{f}$, it is a CFP. □

Remark 3.4. It is worth to observe that the convergence of the sequence $(\tilde{x}^k)$ of iterates of $\tilde{f}$ with respect to the Tychonoff topology is equivalent to the convergence with respect to any of metrics $d_{p,q}$ and $d_{s,q}$ if $q < 1$. Indeed, this follows from Corollary 2.6 and (6).

3.2. A fixed point theorem

If $f : \ell_\infty(X) \to X$, then let $L_{s,q}(f)$ be the Lipschitz constant of f with respect to $d_{s,q}$ on $\ell_\infty(X)$, and let $L_{p,q}(f)$ be the Lipschitz constant of f with respect to $d_{p,q}$ on $\ell_\infty(X)$. Similarly, by $\tilde{L}_{s,q}(\tilde{f})$ and $\tilde{L}_{p,q}(\tilde{f})$ we denote the Lipschitz constants of corresponding map $\tilde{f}$.

Remark 3.5. In this framework, Secelean's Theorem 1.4 says that if $L_{s,1}(f) < 1$, then f admits a unique generalized fixed point, and for every $(x_n) \in \ell_\infty(X)$, the sequence (y_k) defined by (4) converges to this fixed point.
Our main result says what happens if we assume contractive conditions with respect to $d_{p,q}$ or $d_{s,q}$ with $q < 1$.

The next lemma shows the relationships between Lipschitz constants of f and $\tilde{f}$ with respect to the considered metrics.

Lemma 3.6. *In the above frame, if $f : \ell_\infty(X) \to X$, then*

(i) $\tilde{L}_{s,q}(\tilde{f}) \le \max\{q, L_{s,q}(f)\}$, *where* $q \le 1$;

(ii) $\tilde{L}_{p,q}(\tilde{f}) \le \left((L_{p,q}(f))^p + q\right)^{1/p}$, *where* $q < 1$ *and* $p \ge 1$.

Proof. Let $x = (x_n), y = (y_n) \in \ell_\infty(X)$. We have:

$$d_{s,q}(\tilde{f}(x), \tilde{f}(y)) = \sup\{d(f(x), f(y)), qd(x_0, y_0), ..., q^n d(x_{n-1}, y_{n-1}), ...\}$$

$$\le \sup\left(\{L_{s,q}(f)d_{s,q}(x, y)\} \cup \{q^n d(x_{n-1}, y_{n-1}) : n \in \mathbb{N}\}\right)$$

$$\le \max\{L_{s,q}(f)d_{s,q}(x, y), qd_{s,q}(x, y)\} = \max\{q, L_{s,q}(f)\}d_{s,q}(x, y),$$

so we get (i). If $p \ge 1$ and $q < 1$, then

$$d_{p,q}(\tilde{f}(x), \tilde{f}(y)) = \left(d^p(f(x), f(y)) + \sum_{n=1}^{\infty} q^n d^p(x_{n-1}, y_{n-1})\right)^{1/p}$$

$$\le \left((L_{p,q}(f))^p d^p_{p,q}(x, y) + q \sum_{n=1}^{\infty} q^{n-1} d^p(x_{n-1}, y_{n-1})\right)^{1/p} \le \left((L_{p,q}(f))^p d^p_{p,q}(x, y) + qd^p_{p,q}(x, y)\right)^{1/p}$$

$$= \left((L_{p,q}(f))^p + q\right)^{1/p} d_{p,q}(x, y),$$

so we get (ii). □

We are ready to state the main result of the paper.

Theorem 3.7. *Assume that (X, d) is a complete metric space, and $f : \ell_\infty(X) \to X$ satisfies one of the following conditions:*

(*Q*) $L_{s,q}(f) < 1$ *for some* $q \in (0, 1)$;

(*P*) $L_{p,q}(f) < (1 - q)^{1/p}$ *for some* $q \in (0, 1)$ *and* $p \in [1, \infty)$.

Then f has a GCFP.
Moreover, if $x_* \in X$ is a GCFP of f and $x \in \ell_\infty(X)$, then

(i) if $L_{s,q}(f) < 1$ *for some* $q < 1$, *it holds*

$$d(x^k, x_*) \le L_{s,q}(f)\frac{\max\{L_{s,q}(f), q\}^{k-1}}{1 - \max\{L_{s,q}(f), q\}} d_{s,q}(\tilde{x}^1, \tilde{x}^0); \tag{7}$$

(ii) if $L_{p,q}(f) < (1 - q)^{1/p}$, *it holds*

$$d(x^k, x_*) \le L_{p,q}(f)\frac{((L_{p,q}(f))^p + q)^{\frac{k-1}{p}}}{1 - ((L_{p,q}(f))^p + q)^{\frac{1}{p}}} d_{p,q}(\tilde{x}^1, \tilde{x}^0). \tag{8}$$

Proof. We first deal with the case $L_{p,q}(f) < (1-q)^{1/p}$. By Lemma 3.6, the Lipschitz constant of $\tilde{f}$ satisfies

$$\tilde{L}_{p,q}(\tilde{f}) \le (L_{p,q}(f)^p + q)^{1/p} < ((1-q)+q)^{1/p} = 1,$$

so $\tilde{f}$ is a Banach contraction with respect to $d_{p,q}$ on $\ell_\infty(X)$. Now take any $x = (x_n) \in \ell_\infty(X)$. Then for every $m > k$, we have

$$d_{p,q}(\tilde{x}^k, \tilde{x}^m) \le d_{p,q}(\tilde{x}^k, \tilde{x}^{k+1}) + \dots + d_{p,q}(\tilde{x}^{m-1}, \tilde{x}^m) \le d_{p,q}\left(\tilde{f}^k(\tilde{x}^0), \tilde{f}^k(\tilde{x}^1)\right) + \dots + d_{p,q}\left(\tilde{f}^{m-1}(\tilde{x}^0), \tilde{f}^{m-1}(\tilde{x}^1)\right)$$

$$\le (\tilde{L}_{p,q}(\tilde{f}))^k (d_{p,q}(\tilde{x}^0, \tilde{x}^1) + \dots + (\tilde{L}_{p,q})^{m-k-1} d_{p,q}(\tilde{x}^0, \tilde{x}^1)) \le \frac{(\tilde{L}_{p,q}(\tilde{f}))^k}{1 - \tilde{L}_{p,q}(\tilde{f})} d_{p,q}(\tilde{x}^0, \tilde{x}^1),$$

which means that $(\tilde{x}^k)$ is a Cauchy sequence with respect to $d_{p,q}$. Moreover

$$d(x^{k+1}, x^{m+1}) = d(f(\tilde{x}^k), f(\tilde{x}^m)) \le L_{p,q}(f) d_{p,q}(\tilde{x}^k, \tilde{x}^m) \le L_{p,q}(f) \frac{(\tilde{L}_{p,q}(\tilde{f}))^k}{1 - \tilde{L}_{p,q}(\tilde{f})} d_{p,q}(\tilde{x}^0, \tilde{x}^1),$$

which means that (x^k) is a Cauchy sequence in X. Hence the set $\{x^k : k \in \mathbb{N}\} \cup \{x_n : n \in \mathbb{N}^*\}$ is bounded and by Corollary 2.6(2), $\tilde{x}^k \xrightarrow{d_{p,q}} \mathbf{x}$ for some $\mathbf{x} \in \ell_\infty(X)$. Since $\tilde{f}$ is continuous with respect to $d_{p,q}$, the point $\mathbf{x}$ is a fixed point of $\tilde{f}$, which must be unique as $\tilde{L}_{p,q}(\tilde{f}) < 1$. Hence $\mathbf{x}$ is a CFP of $\tilde{f}$ (with respect to Tychonoff topology – see Remark 3.4), and by Theorem 3.3, $\mathbf{x} = (x_*, x_*, \dots)$, where x_* is a CGFP of f. Moreover, by the above computations, for every $x \in X$ and $m > k$, we have

$$d(x^{k+1}, x^{m+1}) \le L_{p,q}(f) \frac{(\tilde{L}_{p,q}(\tilde{f}))^k}{1 - \tilde{L}_{p,q}(\tilde{f})} d_{p,q}(\tilde{x}^0, \tilde{x}^1) \le L_{p,q}(f) \frac{((L_{p,q}(f))^p + q)^{\frac{k}{p}}}{1 - ((L_{p,q}(f))^p + q)^{\frac{1}{p}}} d_{p,q}(\tilde{x}^1, \tilde{x}^0). \tag{9}$$

Letting $m \to \infty$, we get

$$d(x^k, x_*) \le L_{p,q}(f) \frac{((L_{p,q}(f))^p + q)^{\frac{k-1}{p}}}{1 - ((L_{p,q}(f))^p + q)^{\frac{1}{p}}} d_{p,q}(\tilde{x}^1, \tilde{x}^0)$$

for all $k \in \mathbb{N}$.
To get the assertion for assumption (Q) we could follow the same lines. However, as we will see in a moment, conditions (Q) and (P) are equivalent. □

Remark 3.8. As was announced, a bit surprisingly, conditions (P) and (Q) are equivalent. In fact, each of them is also equivalent to a particular version of (P). More precisely, for every $f : \ell_\infty(X) \to X$, the following conditions are equivalent:

(i) f satisfies (Q), that is, for some $q \in (0,1)$, $L_{s,q}(f) < 1$;

(ii) f satisfies (P), that is, for some $q \in (0,1)$ and $p \in [1,\infty)$, $L_{p,q}(f) < (1-q)^{1/p}$;

(iii) for every $q \in (0,1)$ there exists $p \in [1,\infty)$ such that $L_{p,q}(f) < (1-q)^{1/p}$.

We first prove *(i)* ⇒ *(iii)*. Assume that $L_{s,q}(f) < 1$ for some $q \in (0,1)$, and choose any $q_0 \in (0,1)$. Observe that

$$\lim_{p\to\infty} (1-q_0)^{1/p} = 1,$$

so we can take $p \in [1,\infty)$ so that $L_{s,q}(f) < (1-q_0)^{1/p}$ and also $q^p \le q_0$. Then let $q' \in [q,1)$ be such that $(q')^p = q_0$. By Proposition 2.8(i),(ii) we have for all $x, y \in \ell_\infty(X)$,

$$d(f(x), f(y)) \le L_{s,q}(f) d_{s,q}(x,y) \le L_{s,q}(f) d_{s,q'}(x,y) \le L_{s,q}(f) d_{p,(q')^p}(x,y) = L_{s,q}(f) d_{p,q_0}(x,y).$$

Hence $L_{p,q_0}(f) \le L_{s,q}(f) < (1-q_0)^{1/p}$. Thus we get (*iii*).
Implication (*iii*) ⇒ (*ii*) is obvious.
Finally, we prove (*ii*) ⇒ (*i*). Assume that $L_{p,q}(f) < (1-q)^{1/p}$ for some $q \in (0,1)$ and $p \in [1,\infty)$. By Proposition 2.8(iii) for $q' > q^{1/p}$:

$$L_{s,q'}(f) \le \frac{L_{p,q}(f)}{(1-\frac{q}{(q')^p})^{1/p}}.$$

Taking the limit with $q' \to 1$, we get

$$\lim_{q' \to 1} L_{s,q'}(f) \le \frac{L_{p,q}(f)}{(1-q)^{1/p}} < \frac{(1-q)^{1/p}}{(1-q)^{1/p}} = 1,$$

which means that $L_{s,q'}(f) < 1$ for some $q' < 1$ and we get (i).

Remark 3.9. In view of (iii) from Remark 3.8, we see that for every $q_0 \in (0,1)$, condition (P) is equivalent to

$$(P_{q_0}) \quad L_{p,q_0}(f) < (1-q_0)^{1/p} \quad \text{for some } p \in [1,\infty).$$

Later we will see that we cannot restrict to arbitrary q_0 in (*Q*), and also we cannot restrict to arbitrary $p_0 \in [1,\infty)$ in (P).

Remark 3.10. Since (P) and (Q) are equivalent, formally it is enough to consider just one of them ((Q) seems to be more natural). On the other hand, the theory works properly for both types of metrics. In particular, we get natural estimations (7) and (8).

Remark 3.11. By Proposition 2.8(ii) we see that for any $q \in (0,1)$, $L_{s,1}(f) \le L_{s,q}(f)$. Hence if (Q) (or, equivalently, (P)) is satisfied, then also the assumptions of Theorem 1.4 are satisfied. (In fact, at the end of [8], Secelean considered the metric $d_{1,\frac{1}{2}}$ and observed these relationships.). It turns out that the converse is not true, as the next example shows.

Example 3.12. Let $X := [0,1]$ and $f((x_n)) := \frac{1}{2}\sup\{x_n : n \in \mathbb{N}^*\}$. Then clearly $L_{s,1}(f) = \frac{1}{2} < 1$, so the assumptions of Theorem 1.4 are satisfied and $x_* = 0$ is a generalized fixed point of f. However, if $x = (x_n) \in \ell_\infty([0,1])$ is such that for some $i \in \mathbb{N}^*$, $x_i := \delta > 0$, then for any $k \in \mathbb{N}$, $x^k \ge \frac{1}{2}\delta$. In particular, the sequence of generalized iterations (x^k) does not converge to $x_* = 0$ and f has no GCFP.

Remark 3.13. Theorem 3.7 can be formulated in a more general way. Namely, assume that (X,d) is complete and a sequence (a_n) of positive reals satisfies $M := \sup_{n\in\mathbb{N}} \frac{a_n}{a_{n-1}} < 1$, and let $f : \ell_\infty(X) \to X$ be such that one of the following conditions holds:

(i) $a_0 L_{s,(a_n)}(f) < 1$;

(ii) $L_{p,(a_n)}(f) < \left(\frac{1-M}{a_0}\right)^{1/p}$ for some $p \in [1,\infty)$,

where $L_{s,(a_n)}(f)$ and $L_{p,(a_n)}(f)$ are Lipschitz constants of f with respect to metrics $d_{s,(a_n)}$ and $d_{p,(a_n)}$, respectively. Then f has a GCFP.
However, this assertion follows directly from Theorem 3.7. Indeed, for any $x = (x_n), y = (y_n) \in \ell_\infty(X)$ we have:

$$d_{p,(a_n)}(x,y) = \left(\sum_{n=0}^{\infty} a_n d^p(x_n,y_n)\right)^{1/p} \le \left(\sum_{n=0}^{\infty} a_0 M^n d^p(x_n,y_n)\right)^{1/p} = a_0^{1/p} d_{p,M}(x,y)$$

and therefore $L_{p,M}(f) \le a_0^{1/p} L_{p,(a_n)}(f)$, so (ii) implies (P). Similarly, we can see that (i) implies (Q).

In the last section we are going to use Theorem 3.7 to prove Theorem 1.1. However, now we will show another connection between mappings on finite Cartesian products and mappings defined on spaces of sequences:

Theorem 3.14. *Assume that (X, d) is a complete metric space and let $f : \ell_\infty(X) \to X$ satisfy (Q) (or, equivalently, (P)) for $q \in (0, 1)$. Choose any $x \in X$ and for any $n \in \mathbb{N}$, define $f_n : X^n \to X$ as follows:*

$$\forall_{(x_0,...,x_{n-1})\in X^n}\ f_n(x_0, ..., x_{n-1}) := f(x_0, ..., x_{n-1}, x, x, ...). \tag{10}$$

Then for any $n \in \mathbb{N}$, $Lip(f_n) \le L_{s,q}(f)$ (w.r.t. maximum metric d_m on X^n) and the sequence (x_*^n) of generalized fixed points of f_ns′ (whose existence follows from Theorem 1.1) converges to x_*, a generalized fixed point of f. More precisely, for every $n \in \mathbb{N}$,

$$d(x_*^n, x_*) \le q^n \frac{L_{s,q}(f)}{1 - L_{s,q}(f)} d(x_*, x). \tag{11}$$

Proof. Assume that $L_{s,q}(f) < 1$ for some $q \in (0, 1)$. For every $n \in \mathbb{N}$, we have

$$d\left(f_n(x_0, ..., x_{n-1}), f_n(y_0, ..., y_{n-1})\right) = d\left(f(x_0, ..., x_{n-1}, x, x, ...), f(y_0, ..., y_{n-1}, x, x, ...)\right)$$

$$\le L_{s,q}(f) d_{s,q}((x_0, ..., x_{n-1}, x, ...), (y_0, ..., y_{n-1}, x, ...))$$

$$= L_{s,q}(f) \max\{q^k d(x_k, y_k) : k = 0, ..., n-1\} \le L_{s,q}(f) d_m((x_0, ..., x_{n-1}), (y_0, ..., y_{n-1})).$$

Hence $Lip(f_n) \le L_{s,q}(f) < 1$ and the assumptions of Theorem 1.1 are fulfilled. Thus f_n has a fixed point $x_*^n \in X$. Then we have

$$d(x_*^n, x_*) = d(f(x_*^n, ..., x_*^n, x, ...), f(x_*, ..., x_*, x_*, ...)) \le L_{s,q}(f) d_{s,q}((x_*^n, ..., x_*^n, x, ...), (x_*, ..., x_*, x_*, ...))$$

$$= L_{s,q}(f) \max\{d(x_*^n, x_*), q^n d(x_*, x)\} \le L_{s,q}(f)(d(x_*^n, x_*) + q^n d(x_*, x)).$$

Hence

$$d(x_*^n, x_*) \le q^n \frac{L_{s,q}(f)}{1 - L_{s,q}(f)} d(x_*, x).$$

□

Finally, we give an example which shows that the thesis of the above theorem need not hold under the assumption $L_{s,1}(f) < 1$:

Example 3.15. Consider function f from Example 3.12. Take any $x > 0$ and for every $n \in \mathbb{N}$, let $f_n : [0, 1]^n \to [0, 1]$ be defined by (10), i.e.,

$$f_n(x_0, .., x_{n-1}) := f(x_0, ..., x_{n-1}, x, x, ...).$$

Then, clearly, $f_n(x_0, ..., x_{n-1}) \ge \frac{1}{2}x$ for every $(x_0, ..., x_{n-1})$. In fact, $x_*^n = \frac{1}{2}x$, so (x_*^n) does not converge to $x_* = 0$.

4. An Example

To illustrate the considered machinery, we will calculate Lipschitz constants $L_{p,q}(f)$ and $L_{s,q}(f)$ in the case of mappings $f : \ell_\infty(\mathbb{R}) \to \mathbb{R}$ of the form

$$f(x) = \sum_{n\in\mathbb{N}^*} b_n x_n, \quad \text{for } x = (x_n) \in \ell_\infty(\mathbb{R}) \tag{12}$$

for some sequence (b_n) of reals with $\sum_{n\in\mathbb{N}^*} |b_n| < \infty$. We will use these calculations in a discussion connected with Remark 3.8.

Proposition 4.1. *If $f : \ell_\infty(\mathbb{R}) \to \mathbb{R}$ is defined by (12), then*

$$L_{s,q}(f) = \sum_{n=0}^{\infty} \frac{|b_n|}{q^n}$$

and if $q < 1$, then

$$L_{p,q}(f) = \begin{cases} \left(\sum_{n=0}^{\infty} \frac{|b_n|^{p/(p-1)}}{(q^n)^{1/(p-1)}}\right)^{(p-1)/p} & \text{if} \quad p > 1, \\ \sup\left\{\frac{|b_n|}{q^n} : n \in \mathbb{N}^*\right\} & \text{if} \quad p = 1. \end{cases}$$

Proof. Let $q < 1$. Set $I_1 := \sup_{n\in\mathbb{N}^*} \frac{|b_n|}{q^n}$. Then for every $x = (x_n), y = (y_n) \in \ell_\infty(\mathbb{R})$, we have

$$d(f(x), f(y)) = \left|\sum_{n=0}^{\infty} b_n x_n - b_n y_n\right| \leq \sum_{n=0}^{\infty} |b_n| \left|x_n - y_n\right|$$

$$= \sum_{n=0}^{\infty} \frac{|b_n|}{q^n} q^n \left|x_n - y_n\right| \leq I_1 \sum_{n=0}^{\infty} q^n \left|x_n - y_n\right| = I_1 d_{1,q}(x, y).$$

Now assume that $I_1 < \infty$, and let $\varepsilon > 0$. Then there is n_0 such that $\frac{|b_{n_0}|}{q^{n_0}} \geq I_1 - \varepsilon$. If $y_n = 0$ for all $n \in \mathbb{N}^*$ and $x_n = 0$ for $n \neq n_0$ and $x_{n_0} = 1$, then

$$|f(x) - f(y)| = |b_{n_0} x_{n_0}| = \frac{|b_{n_0}|}{q^{n_0}} q^{n_0} |x_{n_0}| = \frac{|b_{n_0}|}{q^{n_0}} d_{1,q}(x, y) \geq (I_1 - \varepsilon) d_{1,q}(x, y).$$

Hence $L_{1,q} = I_1$. In a similar way we can show that $L_{1,q} = I_1$ when $I_1 = \infty$.
Now assume $p > 1$. Then for every $x = (x_n), y = (y_n) \in \ell_\infty(\mathbb{R})$, we have by the Hölder inequality:

$$|f(x) - f(y)| = \left|\sum_{n=0}^{\infty} b_n(x_n - y_n)\right| \leq \sum_{n=0}^{\infty} \frac{|b_n|}{(q^n)^{1/p}} \cdot (q^n)^{1/p} \left|x_n - y_n\right|$$

$$\leq \left(\sum_{n=0}^{\infty} \frac{|b_n|^{p/(p-1)}}{(q^n)^{1/(p-1)}}\right)^{(p-1)/p} \left(\sum_{n=0}^{\infty} q^n |x_n - y_n|^p\right)^{1/p} = \left(\sum_{n=0}^{\infty} \frac{|b_n|^{p/(p-1)}}{(q^n)^{1/(p-1)}}\right)^{(p-1)/p} d_{p,q}((x_n), (y_n)).$$

Observe that the first inequality is the equality if $b_n(x_n - y_n) \geq 0$ for all $n \in \mathbb{N}^*$. Moreover, from the Hölder inequality we know that the second inequality is the equality iff the sequences $\left(\left(\frac{|b_n|}{(q^n)^{1/p}}\right)^{p/(p-1)}\right)$, $\left(\left((q^n)^{1/p}|x_n - y_n|\right)^p\right)$ are linearly dependent. For every $n \in \mathbb{N}^*$, let $y_n := 0$, and define

$$x_n := \begin{cases} \operatorname{sgn}(b_n)\left(\frac{|b_n|}{q^n}\right)^{1/(p-1)} & \text{if} \quad n \leq N, \\ 0 & \text{if} \quad n > N, \end{cases} \tag{13}$$

where $\operatorname{sgn}(\cdot)$ denotes the sign function. Then by previous observations, replacing (b_n) by (b'_n) defined by $b'_n := b_n$ for $n \leq N$ and $b'_n := 0$ for $n > N$, we have

$$|f(x) - f(y)| = \left(\sum_{n=0}^{N} \frac{|b_n|^{p/(p-1)}}{(q^n)^{1/(p-1)}}\right)^{(p-1)/p} d_{p,q}(x, y).$$

Since N was taken arbitrarily, we get $L_{p,q}(f) = \left(\sum_{n=0}^{\infty} \frac{|b_n|^{p/(p-1)}}{(q^n)^{1/(p-1)}}\right)^{(p-1)/p}$.
Finally, for any $q \leq 1$ and every $x = (x_n), y = (y_n) \in \ell_\infty(\mathbb{R})$, we have

$$|f(x) - f(y)| \leq \sum_{n=0}^{\infty} |b_n||x_n - y_n| = \sum_{n=0}^{\infty} \frac{|b_n|}{q^n} q^n |x_n - y_n| \leq \left(\sum_{n=0}^{\infty} \frac{|b_n|}{q^n}\right) d_{s,q}(x, y).$$

Now let $y_n := 0$ for all $n \in \mathbb{N}^*$, fix any $N \in \mathbb{N}$, and define $x_n := 0$ for $n > N$ and for $n = 0, ..., N$, set $x_n := \frac{\operatorname{sgn}(b_n)}{q^n}$. Then

$$|f(x) - f(y)| = \left|\sum_{n=0}^{N} b_n x_n\right| = \left|\sum_{n=0}^{N} \frac{b_n}{q^n} q^n \frac{\operatorname{sgn}(b_n)}{q^n}\right| = \sum_{n=0}^{N} \frac{|b_n|}{q^n} \cdot 1 = \left(\sum_{n=0}^{N} \frac{|b_n|}{q^n}\right) d_{s,q}(x, y).$$

Since N was arbitrary, we have $L_{s,q}(f) = \sum_{n\in\mathbb{N}^*} \frac{|b_n|}{q^n}$. $\square$

Remark 4.2. It is very likely that the above result can be obtained from functional analysis machinery, since f is a linear map which is a sum of linear maps. However, we presented here the proof for the sake of completeness.

Example 4.3. We will consider functions $f : \ell_\infty(\mathbb{R}) \to \mathbb{R}$ of the form (12) with different sequences (b_n).
(1) Let $b_0 = 0$ and $b_n = b^n$, where $b \in (0, \frac{1}{2}]$ is fixed. By Proposition 4.1,

$$L_{s,q}(f) = \sum_{n=1}^{\infty} \left(\frac{b}{q}\right)^n = \begin{cases} \frac{b}{q-b} & \text{if } q > b, \\ \infty & \text{if } q \le b. \end{cases}$$

Now if $b < 1/2$, then $L_{s,q}(f) < 1$ iff $q > 2b$. This shows that in the formulation of condition (Q) we cannot restrict to some particular value q_0 (compare Remark 3.9).
If $b = 1/2$, then $\sum_{n=1}^{\infty} b_n = 1$, so every $x \in \mathbb{R}$ is a generalized fixed point of f. Also $\lim_{q\to 1} L_{s,q}(f) = 1$, which shows that in Theorem 3.7 we cannot assume that $L_{s,q}(f) > 1$.
(2) Let $b_n = 0$ for $n \ne 1$ and $b_1 = b$, where $b \in (0, 1]$ is fixed. By Proposition 4.1, for every $p \in [1, \infty)$,

$$L_{p,q}(f) = \frac{b}{q^{1/p}}.$$

Now if $b < 1$, then $L_{p,q}(f) < (1 - q)^{1/p}$ iff $b < (q - q^2)^{1/p}$. In particular, we can choose $q \in (0, 1)$ and $p \in [1, \infty)$ such that $L_{p,q}(f) < (q - q^2)^{1/p}$. However, if we fix $p_0 \in [1, \infty)$, then

$$\sup\left\{(q - q^2)^{1/p} : p \in [1, p_0], q \in (0, 1)\right\} = \frac{1}{4^{1/p_0}} < 1.$$

This shows that in the formulation of condition (P) we cannot restrict to some particular value of p_0.
If $b = 1$, then every $x \in \mathbb{R}$ is a generalized fixed point of f. Also, for every $q \in (0, 1)$, $\lim_{p\to\infty} \frac{L_{p,q}(f)}{(1-q)^{1/p}} = \lim_{p\to\infty} \frac{1}{(q-q^2)^{1/p}} = 1$. This shows that in Theorem 3.7 we cannot assume that $L_{p,q}(f) > (1 - q)^{1/p}$.

5. Applications

At first we show that Theorem 3.7 implies Theorem 1.1 and, in particular, the classical Banach fixed point theorem. Recall that by X^m we denote the Cartesian product of m copies of X and we endow X^m with the maximum metric

$$d_m((x_0, ..., x_{m-1}), (y_0, ..., y_{m-1})) := \max\{d(x_0, y_0), ..., d(x_{m-1}, y_{m-1})\}.$$

Proof. (of Theorem 1.1) Choose $q \in (0, 1)$ such that $Lip(g) < q^{m-1}$. Define $f : \ell_\infty(X) \to X$ by $f(x_0, x_1, x_2, ...) := g(x_0, ..., x_{m-1})$. For every $x = (x_n), y = (y_n) \in \ell_\infty(X)$, we have

$$\begin{aligned} d(f(x), f(y)) &= d(g(x_0, ..., x_{m-1}), g(y_0, ..., y_{m-1})) \le Lip(g) \max\{d(x_0, y_0), ..., d(x_{m-1}, y_{m-1})\} \\ &\le \frac{Lip(g)}{q^{m-1}} \max\{q^0 d(x_0, y_0), ..., q^{m-1} d(x_{m-1}, y_{m-1})\} \le \frac{Lip(g)}{q^{m-1}} d_{s,q}(x, y). \end{aligned}$$

Hence $L_{s,q}(f) \le \frac{Lip(g)}{q^{m-1}} < 1$, so mapping f satisfies the assumptions of Theorem 3.7. It remains to observe that if $x_0, ..., x_{m-1} \in X$, then sequence (x_k) defined by (1), is the sequence of generalized iterates of f at $x := (x_{m-1}, ..., x_0, x_0, x_0, ...)$. $\square$

The second application of our result deals with a recursive procedure which "looks back" at all previously defined elements.

Example 5.1. Fix $(b_n) \subset \mathbb{R}$, $c \in \mathbb{R}$, and consider the sequence (x^k) defined by the following linear recursion:

$$\begin{cases} x^1 := c, \\ x^k := c + b_0 x^{k-1} + b_1 x^{k-2} + \ldots + b_{k-2} x^1, \quad k \geq 2. \end{cases}$$

Then (x^k) is the sequence of iterates of $y = (0, 0, \ldots)$, of the map $f(x) := \sum_{n=0}^{\infty} b_n x_n + c$. Thus if the assumptions of Theorem 3.7 are satisfied (the Lipschitz constants can be calculated as in Proposition 4.1), then $x^k \to x_*$, where x_* is the GCFP of f, that is

$$x_* = f(x_*, x_*, \ldots) = \sum_{n=0}^{\infty} b_n x_* + c = \left(\sum_{n=0}^{\infty} b_n \right) x_* + c,$$

which gives $x_* = \frac{c}{1 - \left(\sum_{n=0}^{\infty} b_n\right)}$.

For example, assume that $b_n := \frac{1}{3 \cdot 2^n}$, $n \in \mathbb{N}^*$ and $c = 1$. Setting $q = \frac{4}{5}$, we have that $L_{s,q}(f) = \frac{8}{9}$, so f fulfills the assumptions of Theorem 3.7. Thus $x^k \to x_* = \frac{c}{1 - \sum_{n \in \mathbb{N}^*} b_n} = 3$. Moreover, by the second part of Theorem 3.7, for every $k \in \mathbb{N}$,

$$|x^k - 3| \leq L_{s,q}(f) \frac{\max\{L_{s,q}(f), q\}^{k-1}}{1 - \max\{L_{s,q}(f), q\}} d_{s,q}(\tilde{x}^1, (0)) = 9 \left(\frac{8}{9} \right)^k$$

since $d_{s,q}(\tilde{x}^1, (0)) = d_{s,q}((c, 0, 0, \ldots), (0, 0, \ldots)) = c = 1$.

Remark 5.2. As we have already mentioned, Secelean [8] used his theorem to study the Hutchinson–Barnsley theory of fractals for maps defined on $\ell_\infty(X)$. In our paper [5] we use results of this article to obtain an appropriate version of the Hutchinson–Barnsley theory in such setting.

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