Filomat 31:11 (2017), 3573–3583
https://doi.org/10.2298/FIL1711573F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Three-Step Iterative Algorithm for a Pair of Total Asymptotically Nonexpansive Mappings in Uniformly Convex Metric Spaces

Hafiz Fukhar-ud-din[a], Safeer Hussain Khan[b]

[a]*Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia and Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan*
[b]*Department of Mathematics, Statistics and Physics, Qatar University, Doha 2713, State of Qatar.*

Abstract. We introduce and study a three-step iterative algorithm for a pair of total asymptotically nonexpansive mappings in a uniformly convex metric space. The proposed algorithm includes Mann and Ishikawa iterative algorithms, the iterative algorithm of Khan and Takahashi [13] and the three-step iterative algorithm of Xu and Noor [26] as special cases. Our results are new and generalize several recent results in Hilbert spaces, uniformly convex Banach spaces and $CAT(0)$ spaces, simultaneously.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

Let C be a nonempty subset of a metric space X and $\{x_n\}$ a bounded sequence in X. Define a functional $r(., \{x_n\}) : X \to \mathbb{R}^+$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n) \text{ for all } x \in X.$$

Set

$$r(\{x_n\}) = \inf_{x\in C} r(x, \{x_n\}) \text{ and } A(\{x_n\}) = \{y \in C : r(y, \{x_n\}) = r(\{x_n\})\}$$

as asymptotic center and asymptotic radius of $\{x_n\}$ with respect to C, respectively.

A sequence $\{x_n\}$ $\triangle-$converges to x if x is the unique asymptotic center for every subsequence $\{u_n\}$ of $\{x_n\}$ and we write $\triangle - \lim_{n\to\infty} x_n = x$.

It has been shown in [18] that $\triangle-$convergence coincides with weak convergence in a Banach space with Opial's property.

2010 *Mathematics Subject Classification*. Primary 47H09, 47H10; Secondary 49M05
Keywords. Convex metric space, an iterative algorithm, total asymptotically nonexpansive mapping, common fixed point, $\triangle$-convergence and strong convergence.
Received: 12 April 2017; Accepted: 20 May 2017
Communicated by Vladimir Rakočević
The author Hafiz Fukhar-ud-din would like to acknowledge the support provided by the Deanship of Scientific Research(DSR) at King Fahd University of Petroleum & Minerals (KFUPM) for funding this work through project No. IN141047.
Corresponding author: Safeer Hussain Khan
Email addresses: hfdin@kfupm.edu.sa (Hafiz Fukhar-ud-din), safeerhussain5@yahoo.com, safeer@qu.edu.qa (Safeer Hussain Khan)

In 2006, Alber et al. [1] introduced the concept of total asymptotically nonexpansive mappings as the generalization of a few classes of mappings including
(1) nonexpansive mappings
(2) asymptotically nonexpansive mappings
(3) asymptotically nonexpansive mappings in the intermediate sense
(4) generalized asymptotically nonexpansive mappings.

A mapping $T : C \to C$ is a total asymptotically nonexpansive if there exist non-negative real sequences $\{k_n\}$ and $\{\varphi_n\}$ with $k_n \to 0$ and $\varphi_n \to 0$ and a strictly increasing continuous function $\xi : [0, \infty) \to [0, \infty)$ with $\xi(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + k_n \xi(d(x, y)) + \varphi_n$$

for all $x, y \in C$, $n \in \mathbb{N}$.

When $y \in F(T) = \{x \in C : Tx = x\}$, the mapping T is called total asymptotically quasi- nonexpansive.

Convergence theorems for the mappings in (1)-(4) in the framework of uniformly convex Banach spaces and CAT(0) spaces via different iterative algorithms have been obtained by a number of researchers (e.g., [2, 3, 12, 16, 23, 26] and the references therein).

Khan and Takahashi [13] used the two-step iterative algorithm

$$\begin{aligned} & x_1 \in C, \\ & y_n = \beta_n T_2^n x_n + (1 - \beta_n) x_n, \\ & x_{n+1} = \alpha_n T_1^n y_n + (1 - \alpha_n) x_n, \quad n \in \mathbb{N}, \end{aligned} \tag{1}$$

to approximate common fixed points of a pair of asymptotically nonexpansive mappings T_1, T_2 on a nonempty closed and convex subset C of a uniformly convex Banach space and $\alpha_n, \beta_n \in [b, c]$ for some $b, c \in (0, 1)$ (see also [8, 14]).

Noor [19], in 2000, introduced a three-step iterative algorithm and used it for approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [9] applied a three-step iterative algorithms for finding the approximate solution of the elastoviscoplasticity problem, eigenvalue problem and liquid crystal theory. In 2002, Xu and Noor [26] introduced the following algorithm:

$$\begin{aligned} & x_1 \in C, \\ & z_n = \gamma_n T^n x_n + (1 - \gamma_n) x_n, \\ & y_n = \beta_n T^n z_n + (1 - \beta_n) x_n, \\ & x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \in \mathbb{N}. \end{aligned} \tag{2}$$

They used it to approximate fixed points of asymptotically nonexpansive mappings in a uniformly convex Banach space assuming the conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ as under:
(C1) : $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$
(C2) : $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

The iterative algorithms (1)-(2) involve convex combinations formed through the built-in linear structure of the Banach space and also contain modified Mann and Ishikawa iterative algorithms as special cases. Most of the problems in various disciplines of science are nonlinear in nature, whereas fixed point theory explored in the setting of Banach spaces majorly depends on the linear structure of the underlying space. One of the nonlinear framework for fixed point theory is a metric space embedded with a convex structure $W : X^2 \times I \to X$ satisfying

$$d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha) d(u, y) \tag{3}$$

for all $x, y, u \in X$ and $\alpha \in I = [0, 1]$ (see [24]). A metric space X together with a convex structure W is known as a convex metric space. For the sake of simplicity, we denote a convex metric space also by X. A nonempty subset C of X is convex if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in I$. Some examples of convex metric spaces are normed spaces and their convex subsets, Hadamard manifolds and $CAT(0)$ spaces.

Kohlenbach [17] enriched the concept of convex metric space as "hyperbolic space" by including three additional conditions in the definition of a convex metric space, see for its use, for example, [4].

A convex metric space X is uniformly convex [22] if for any $\varepsilon > 0, r > 0, x, y, z \in X$,there exists $\alpha > 0$ such that $d(z, x) \leq r, d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$ imply that $d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leq r(1-\alpha) < r$.

The class of uniformly convex metric space is nonlinear in nature and is a general theoretical setting with rich geometrical properties for metric fixed point theory.

To know the importance of different iterative algorithms for the approximation of fixed points of total asymptotically nonexpansive mappings in uniformly convex Banach spaces, $CAT(0)$ spaces and hyperbolic spaces, we refer the interested reader to [7, 11, 15, 21, 25].

We define a unified iterative algorithm containing the algorithms (1)-(2) as special cases, in a general nonlinear setup, namely, uniformly convex metric space. Our algorithm reads as under:

$$\begin{aligned} & x_1 \in C, \\ & z_n = W\left(T_1^n x_n, x_n, \gamma_n\right), \\ & y_n = W\left(T_2^n z_n, x_n, \beta_n\right), \\ & x_{n+1} = W\left(T_1^n y_n, x_n, \alpha_n\right), \quad n \in \mathbb{N}. \end{aligned} \tag{4}$$

In this paper, we approximate common fixed points of two mappings which are uniformly continuous and total asymptotically nonexpansive via the algorithm (4) in a uniformly convex metric space. Throughout the paper, we denote $F(T_1) \cap F(T_2)$ by F.

For the development of our main section, some key results are needed as listed below.

Lemma 1.1. *[5] Let C be a nonempty, closed and convex subset of a complete uniformly convex metric space X. Then every bounded sequence $\{x_n\}$ in C has a unique asymptotic center in C.*

Lemma 1.2. *[5] Let C be a nonempty closed and convex subset of a uniformly convex complete metric space. Let $\{x_n\}$ be a bounded sequence in C such that $A_C(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in C such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$ (a real number), then $\lim_{m\to\infty} y_m = y$.*

Lemma 1.3. *([20]) If $\{r_n\}, \{s_n\}$ and $\{t_n\}$ are non-negative real sequences satisfying $r_{n+1} \leq (1 + s_n)r_n + t_n$ for all $n \in \mathbb{N}, \sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.*

Lemma 1.4. *[6] Let X be a uniformly convex metric space with continuous convex structure W. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{u_n\}$ and $\{v_n\}$ are sequences in X such that $\limsup_{n\longrightarrow\infty} d(u_n, x) \leq r$, $\limsup_{n\longrightarrow\infty} d(v_n, x) \leq r$ and $\lim_{n\longrightarrow\infty} d(W(u_n, v_n, a_n), x) = r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(u_n, v_n) = 0$.*

2. Convergence Analysis

We first prove the following helpful lemmas.

Lemma 2.1. *Let C be a nonempty closed and convex subset of a convex metric space X. Let $T_i : C \to C$ be total asymptotically nonexpansive mappings with sequences $\{k_{in}\}$ and $\{\varphi_{in}\}$, $n \in \mathbb{N}$, $i = 1, 2$ and ξ_i satisfy the following conditions:*

(C3) : $\sum\limits_{n=1}^{\infty} k_{in} < \infty$ *and* $\sum\limits_{n=1}^{\infty} \varphi_{in} < \infty$;

(C4) : *there exist constants $M_i > 0$, $M_i^* > 0$ such that $\xi_i(t) \leq M_i^* \lambda$ for all $\lambda \geq M$.*

Let $\{x_n\}$ be the sequence as defined in (4). If $F \neq \phi$, then $\lim\limits_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.

Proof. Using (C4) and strictly increasing function ξ_i, we obtain

$$\xi_i(\lambda) \leq \xi_i(M_i) + \lambda M_i^* \text{ for } i = 1, 2. \tag{5}$$

We apply (3) to (4) and $p \in F$ and obtain

$$\begin{aligned} d(z_n, p) &= d\left(W\left(T_1^n x_n, x_n, \gamma_n\right), p\right) \\ &\leq \gamma_n d\left(T_1^n x_n, p\right) + (1-\gamma_n) d(x_n, p) \\ &\leq \gamma_n \left[d(x_n, p) + k_{1n}\xi_1(d(x_n, p)) + \varphi_{1n}\right] + (1-\gamma_n) d(x_n, p) \\ &\leq \gamma_n \left[d(x_n, p) + k_{1n}\xi_1(M_1) + k_{1n}M_1^* d(x_n, p) + \varphi_{1n}\right] + (1-\gamma_n) d(x_n, p) \\ &\leq \left(1 + k_{1n}M_1^*\right) d(x_n, p) + k_{1n}\xi_1(M_1) + \varphi_{1n} \end{aligned} \tag{6}$$

and

$$\begin{aligned} d(y_n, p) &= d\left(W\left(T_2^n z_n, x_n, \beta_n\right), p\right) \\ &\leq \beta_n d\left(T_2^n z_n, p\right) + (1-\beta_n) d(x_n, p) + (1-\beta_n) d(x_n, p) \\ &\leq \beta_n \left[d(z_n, p) + k_{2n}\xi_2(d(z_n, p)) + \varphi_{2n}\right] \\ &\leq \beta_n \left[d(z_n, p) + k_{2n}\xi_2(M_2) + k_{2n}M_2^* d(z_n, p) + \varphi_{2n}\right] + (1-\beta_n) d(x_n, p) \\ &\leq \left(1 + k_{2n}M_2^*\right) d(z_n, p) + k_{2n}\xi_2(M_2) + \varphi_{2n}. \end{aligned} \tag{7}$$

Inserting (6) into (7), we get

$$\begin{aligned} d(y_n, p) &\leq \left(1 + k_{2n}M_2^*\right)\begin{bmatrix} \left(1 + k_{1n}M_1^*\right) d(x_n, p) \\ +k_{1n}\xi_1(M_1) + \varphi_{1n} \end{bmatrix} + k_{2n}\xi_2(M_2) + \varphi_{2n} \\ &= \left[1 + k_{1n}M_1^* + k_{2n}M_2^* + k_{1n}M_1^* k_{2n}M_2^*\right] d(x_n, p) \\ &\quad + k_{1n}\left(1 + k_{2n}M_2^*\right)\xi_1(M_1) + \varphi_{1n}\left(1 + k_{2n}M_2^*\right) + k_{2n}\xi_2(M_2) + \varphi_{2n}. \end{aligned} \tag{8}$$

Also

$$\begin{aligned} d(x_{n+1}, p) &= d\left(W\left(T_1^n y_n, x_n, \alpha_n\right), p\right) \\ &\leq \alpha_n d\left(T_1^n y_n, p\right) + (1-\alpha_n) d(x_n, p) \\ &\leq \alpha_n \left\{d(y_n, p) + k_{1n}\ \xi_1(d(y_n, p)) + \varphi_{1n}\right\} + (1-\alpha_n) d(x_n, p) \\ &\leq \alpha_n \left\{\left(1 + k_{1n}M_1^*\right) d(y_n, p) + k_{1n}\xi_1(M_1) + \varphi_{1n}\right\} + (1-\alpha_n) d(x_n, p) \\ &\leq \left(1 + k_{1n}M_1^*\right) d(y_n, p) + k_{1n}\xi_1(M_1) + \varphi_{1n} + (1-\alpha_n) d(x_n, p). \end{aligned} \tag{9}$$

Inserting (8) into (9), we get

$$\begin{aligned} d(x_{n+1}, p) &\leq \left(1 + k_{1n}M_1^*\right)\begin{bmatrix} \left[1 + k_{1n}M_1^* + k_{2n}M_2^* + k_{1n}M_1^* k_{2n}M_2^*\right] d(x_n, p) \\ +k_{1n}\left(1 + k_{2n}M_2^*\right)\xi_1(M_1) + \varphi_{1n}\left(1 + k_{2n}M_2^*\right) \\ +k_{2n}\xi_2(M_2) + \varphi_{2n} \end{bmatrix} + k_{1n}\xi_1(M_1) + \varphi_{1n} \\ &= \begin{bmatrix} 1 + k_{1n}M_1^* + k_{1n}M_1^*\left(1 + k_{1n}M_1^*\right) \\ +k_{2n}M_2^*\left(1 + k_{1n}M_1^*\right) + k_{1n}M_1^* k_{2n}M_2^*\left(1 + k_{1n}M_1^*\right) \end{bmatrix} d(x_n, p) \\ &\quad + k_{1n}\left(1 + k_{2n}M_2^*\right)\xi_1(M_1)\left(1 + k_{1n}M_1^*\right) + \varphi_{1n}\left(1 + k_{2n}M_2^*\right)\left(1 + k_{1n}M_1^*\right) \\ &\quad + k_{2n}\xi_2(M_2)\left(1 + k_{1n}M_1^*\right) + \varphi_{2n}\left(1 + k_{1n}M_1^*\right) + k_{1n}\xi_1(M_1) + \varphi_{1n}. \end{aligned} \tag{10}$$

Since $\{k_{in}\}$ and $\{\varphi_{in}\}$ are bounded, we set $\max\{k_{in}, \varphi_{in}\} \leq \delta$ for some $\delta > 0$.

Put

$$s_n = k_{1n}M_1^* + k_{1n}M_1^*\left(1+\delta M_1^*\right) + k_{2n}M_2^*\left(1+\delta M_1^*\right) + k_{1n}M_1^*\delta M_2^*\left(1+\delta M_1^*\right)$$

and

$$\begin{aligned} t_n &= k_{1n}\left(1+\delta M_2^*\right)\xi_1\left(M_1\right)\left(1+\delta M_1^*\right) + \varphi_{1n}\left(1+\delta M_2^*\right)\left(1+\delta M_1^*\right) \\ &\quad + k_{2n}\xi_2\left(M_2\right)\left(1+\delta M_1^*\right) + \varphi_{2n}\left(1+\delta M_1^*\right) + k_{1n}\xi_1\left(M_1\right) + \varphi_{1n}. \end{aligned}$$

In the light of (C3), $\sum\limits_{n=1}^{\infty} s_n < \infty$ *and* $\sum\limits_{n=1}^{\infty} t_n < \infty$.

Therefore we apply Lemma 1.3 to (10) and get that $\lim\limits_{n\to\infty} d\left(x_n, p\right)$ *exists for each* $p \in F$. □

Lemma 2.2. *Let C be a nonempty closed and convex subset of a uniformly convex metric space X with continuous convex structure W. Let $T_i : C \to C$ be total asymptotically nonexpansive mappings with $F \neq \phi$ and satisfy* (C3) − (C4). *Then for the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ given in* (4)*, we have the following assertions: (i) If* (C1) *is satisfied, then* $\lim_{n\to\infty} d\left(T_1^n y_n, x_n\right) = 0$*; (ii) If* (C2) *is satisfied and* $\liminf_{n\to\infty} \alpha_n > 0$*, then* $\lim_{n\to\infty} d\left(T_2^n z_n, x_n\right) = 0$.

Proof. Suppose (i) holds. Then there exist $a, b \in (0,1)$ such that $0 < a \le \alpha_n \le b < 1$.

By Lemma 2.1, $\lim\limits_{n\to\infty} d\left(x_n, p\right)$ exists for $p \in F$. Let $\lim\limits_{n\to\infty} d\left(x_n, p\right) = c$ for some $c \ge 0$.We discuss the case when $c > 0$. Now, $\lim\limits_{n\to\infty} d\left(x_n, p\right) = c$ can be expressed as

$$\lim_{n\to\infty} d\left(W\left(T_1^n y_n, x_n, \alpha_n\right), p\right) = c. \tag{11}$$

As in (9), we have that

$$\begin{aligned} d\left(T_1^n y_n, p\right) &\le d\left(y_n, p\right) + k_{1n}\,\xi_1\left(d\left(y_n, p\right)\right) + \varphi_{1n} \\ &\le \left(1 + k_{1n}M_1^*\right) d(y_n, p) + k_{1n}\xi_1\left(M_1\right) + \varphi_{1n} \\ &\le \left(1 + k_{1n}M_1^*\right)\left[\begin{array}{c} \left(1 + k_{1n}M_1^* + k_{2n}M_2^* + k_{1n}M_1^*k_{2n}M_2^*\right) d(x_n, p) \\ + k_{1n}\left(1 + k_{2n}M_2^*\right)\xi_1\left(M_1\right) + \varphi_{1n}\left(1 + k_{2n}M_2^*\right) \\ + k_{2n}\xi_2\left(M_2\right) + \varphi_{2n} \end{array}\right] + k_{1n}\xi_1\left(M_1\right) + \varphi_{1n}. \end{aligned}$$

Taking $\limsup_{n\to\infty}$ on both sides in the above inequality and with the help of (C3), we get that

$$\limsup_{n\to\infty} d\left(T_1^n y_n, p\right) \le c. \tag{12}$$

In the light of (11) and (12), we utilize Lemma 1.4 with values $x = p, r = c, a_n = \alpha_n, u_n = T_1^n y_n, v_n = x_n$ and get that

$$\lim_{n\to\infty} d\left(T_1^n y_n, x_n\right) = 0. \tag{13}$$

Suppose (ii) holds. Then there exists $\alpha, a, b \in (0,1)$ such that $0 < a \le \beta_n \le b < 1$ and $\alpha \le \alpha_n$ for all $n \in \mathbb{N}$. From (8), we have that

$$\limsup_{n\to\infty} d\left(y_n, p\right) \le c. \tag{14}$$

Re-arranging the terms in the inequality (9) and then using the given fact that $0 < \alpha \le \alpha_n$ for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d\left(x_n, p\right) &\le \left(1 + k_{1n}M_1^*\right) d(y_n, p) + k_{1n}\xi_1\left(M_1\right) + \varphi_{1n} \\ &\quad + \frac{d\left(x_n, p\right) - d\left(x_{n+1}, p\right)}{\alpha}. \end{aligned} \tag{15}$$

By $\liminf_{n\to\infty}$ on both sides of (15), we get

$$c \leq \liminf_{n\to\infty} d\left(y_n, p\right). \tag{16}$$

A combined effect of (14) and (16) yields that

$$\lim_{n\to\infty} d\left(y_n, p\right) = c.$$

That is,

$$\lim_{n\to\infty} d\left(W\left(T_2^n z_n, x_n, \beta_n\right), p\right) = c. \tag{17}$$

Since

$$\begin{aligned} d\left(T_2^n z_n, p\right) &\leq d(z_n, p) + k_{2n}\xi_2\left(d(z_n, p)\right) + \varphi_{2n} \\ &\leq \left(1 + k_{2n}M_2^*\right) d(z_n, p) + k_{2n}\xi_2\left(M_2\right) + \varphi_{2n} \\ &\leq \left(1 + k_{2n}M_2^*\right)\left[\left(1 + k_{1n}M_1^*\right) d(x_n, p) + k_{1n}\xi_1\left(M_1\right) + \varphi_{1n}\right] + k_{2n}\xi_2\left(M_2\right) + \varphi_{2n}, \end{aligned}$$

$\limsup_{n\to\infty}$ on both sides in the above inequality and using (C3), we get that

$$\limsup_{n\to\infty} d\left(T_2^n z_n, p\right) \leq c. \tag{18}$$

Again utilizing Lemma 1.4 with values $x = p, r = c, a_n = \beta_n, u_n = T_2^n z_n, v_n = x_n$ together with (17) and (18), we have that

$$\lim_{n\to\infty} d\left(T_2^n z_n, x_n\right) = 0.$$

□

Lemma 2.3. *Let C be a nonempty closed and convex subset of a uniformly convex metric space X. Let $T_1, T_2 : C \to C$ be uniformly continuous and total asymptotically nonexpansive mappings with $F \neq \phi$ and $\{x_n\}$ be given in* (4)*. If the conditions* (C1) − (C4) *are satisfied, then we have*

$$\lim_{n\to\infty} d\left(T_1 x_n, x_n\right) = 0 = \lim_{n\to\infty} d\left(T_2 x_n, x_n\right).$$

Proof. It has been obtained in Lemma 2.2 that

$$\lim_{n\to\infty} d\left(T_1^n y_n, x_n\right) = 0 = \lim_{n\to\infty} d\left(T_2^n z_n, x_n\right).$$

Since

$$\begin{aligned} d\left(x_n, y_n\right) &= d\left(x_n, W\left(T_2^n z_n, x_n, \beta_n\right)\right) \\ &\leq \beta_n d\left(x_n, T_2^n z_n\right) \to 0, \end{aligned}$$

therefore

$$\begin{aligned} d\left(T_1^n x_n, x_n\right) &\leq d\left(T_1^n x_n, T_1^n y_n\right) + d\left(T_1^n y_n, x_n\right) \\ &\leq d\left(x_n, y_n\right) + k_{1n}\ \xi_1\left(d\left(x_n, y_n\right)\right) + \varphi_{1n} + d\left(T_1^n y_n, x_n\right) \\ &\leq \left(1 + k_{1n}M_1^*\right) d\left(x_n, y_n\right) + k_{1n}\xi_1\left(M_1\right) + \varphi_{1n} + d\left(T_1^n y_n, x_n\right) \to 0. \end{aligned}$$

Similarly

$$\begin{aligned} d\left(x_n, z_n\right) &= d\left(x_n, W\left(T_1^n x_n, x_n, \gamma_n\right)\right) \\ &\leq \gamma_n d\left(x_n, T_1^n x_n\right) \to 0 \end{aligned}$$

gives that

$$\begin{aligned} d\left(T_2^n x_n, x_n\right) &\leq d\left(T_2^n x_n, T_2^n z_n\right) + d\left(T_2^n z_n, x_n\right) \\ &\leq d\left(x_n, z_n\right) + k_{2n}\ \xi_2\left(d\left(x_n, z_n\right)\right) + \varphi_{2n} + d\left(T_2^n z_n, x_n\right) \\ &\leq \left(1 + k_{2n} M_2^*\right) d\left(x_n, z_n\right) + k_{2n}\ \xi_2\left(M_2\right) + \varphi_{2n} + d\left(T_2^n z_n, x_n\right) \to 0. \end{aligned} \tag{19}$$

Also observe that

$$\begin{aligned} d\left(x_n, x_{n+1}\right) &= d\left(x_n, W\left(T_1^n y_n, x_n, \alpha_n\right)\right) \\ &\leq \alpha_n d\left(x_n, T_1^n y_n\right) \to 0. \end{aligned} \tag{20}$$

The inequality

$$d\left(x_n, T_1 x_n\right) \leq d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, T_1^{n+1} x_{n+1}\right) + d\left(T_1^{n+1} x_{n+1}, T_1^{n+1} x_n\right) + d\left(T_1^{n+1} x_n, T x_n\right),$$

uniformly continuity of T_1 and (19)-(20) provide that

$$\lim_{n\to\infty} d\left(x_n, T_1 x_n\right) = 0.$$

Similarly

$$\lim_{n\to\infty} d\left(x_n, T_2 x_n\right) = 0.$$

That is

$$\lim_{n\to\infty} d\left(x_n, T_1 x_n\right) = 0 = \lim_{n\to\infty} d\left(x_n, T_2 x_n\right).$$

□

Remark 2.4. *Lemma 2.3 extends Lemma 2.2 of Xu and Noor [26] for two mappings without any additional condition on control parameters and Lemma 2.2 of Khan and Takahashi [13] for three- step iterative algorithm in the setting of convex metric spaces.*

Now we obtain our $\triangle-$convergence theorem.

Theorem 2.5. *Let C be a nonempty closed and convex subset of a complete uniformly convex metric space X. Let $T_1, T_2 : C \to C$ be uniformly continuous and total asymptotically nonexpansive mappings with $F \neq \phi$ and $\{x_n\}$ be given in* (4). *If the conditions* (C1) − (C4) *are satisfied, then* $\triangle - \lim_{n\to\infty} x_n = x \in F$.

Proof. For each $p \in F$, $\lim_{n\to\infty}(x_n, p)$ exists (Lemma 2.1) and hence $\{x_n\}$ is bounded. Therefore $\{x_n\}$ has a unique asymptotic centre, that is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By definition of asymptotic centre, $u \in C$.We claim that u is a fixed point of T_1.

By Lemma 2.3, we have

$$\lim_{n\to\infty} d\left(u_n, T_1 u_n\right) = 0. \tag{21}$$

In the presence of strictly increasing function ξ_1 and $(C2)$, we calculate, for any $m, n \in \mathbb{N}$,that

$$\begin{aligned} d\left(T_1^m u, u_n\right) &\leq d\left(T_1^m u, T_1^m u_n\right) + d\left(T_1^m u_n, T_1^{m-1} u_n\right) + \ldots + d\left(T_1^2 u_n, T_1 u_n\right) + d\left(T_1 u_n, u_n\right) \\ &\leq d\left(u, u_n\right) + k_{1m}\xi_1\left(d\left(u, u_n\right)\right) + \varphi_{1m} + \sum_{j=0}^{m-1} d(T_1^j u_n, T_1^{j+1} u_n) \\ &\leq \left(1 + k_{1m} M_j^*\right) d(u, u_n) + k_{1m}\xi_1\left(M_1\right) + \varphi_{1m} + \sum_{j=0}^{m-1} d(T_1^j u_n, T_1^{j+1} u_n). \end{aligned}$$

In the light of uniform continuity of T_1^j $(j = 2, 3, 4, ..., m-1)$ and (21), the above inequality reduces to

$$r\left(T_1^m u, \{u_n\}\right) \leq \left(1 + k_{1m} M_j^*\right) r\left(u, \{u_n\}\right).$$

That is,

$$\limsup_{m\to\infty} r\left(T_1^m u, \{u_n\}\right) \leq r\left(u, \{u_n\}\right).$$

By the definition of $A(\{u_n\})$,we have that

$$r\left(u, \{u_n\}\right) \leq \liminf_{m\to\infty} r\left(T_1^m u, \{u_n\}\right).$$

Therefore

$$\lim_{m\to\infty} r\left(T_1^m u, \{u_n\}\right) = r\left(u, \{u_n\}\right).$$

By Lemma 2, $\lim_{m\to\infty} T_1^m u = u$. Since T_1 is uniformly continuous, therefore $T_1 u = T_1\left(\lim_{m\to\infty} T_1^m u\right) = \lim_{m\to\infty} T_1^{m+1} u = u$. That is, $u \in F(T_1)$.By similar calculations, we have that $u \in F(T_2)$. That is, $u \in F$. Therefore, by Lemma 5 $\lim_{n\to\infty} d(x_n, u)$ exists. Suppose $x \neq u$. Then by the uniqueness of asymptotic centres, we have

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

which is not true. Hence $x = u$.Therefore, $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\triangle - \lim_{n\to\infty} x_n = x \in F$. □

Recall that (i) $T : C \to C$ is completely continuous if $d(x_n, Tx_n) \to 0$ imply that $\{Tx_n\}$ has a convergent subsequence (ii) $T_1, T_2 : C \to C$ with $F \neq \phi$, satisfy condition (AV) if

$$\frac{d\left(x, T_1 x\right) + d\left(x, T_2 x\right)}{2} \geq f(d(x, F)) \text{ for } x \in C,$$

where $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$.

Next, we prove strong convergence theorem via (4).

Theorem 3.7. *Under the hypotheses of Theorem*2.5, *assume that one of the following conditions is satisfied:*
(a) Every subsequence of $\{x_n\}$ is convergent
(b) T_1^m *(or* T_2^m*) is completely continuous for some* $m \geq 1$
(c) T_1 and T_2 satisfy condition(AV).
Then $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *in* (4) *converge strongly to the same point of* F.

Proof. Lemma 2.3 provides that

$$\lim_{n\to\infty} d(x_n, T_1x_n) = 0 = \lim_{n\to\infty} d(x_n, T_2x_n).$$

Assume that (a) holds. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_i} \to z \in C$. Then (20) assures that $z \in F$. As $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$ (Lemma 2.1), so $x_n \to z$.
Let (b) be given. Suppose T_1^m is completely continuous for some $m \geq 1$.

Note that

$$d\left(T_1^m x_n, x_n\right) \leq d\left(T_1^m x_n, T_1^{m-1}x_n\right) + d\left(T_1^{m-1}x_n, T_1^{m-2}x_n\right) + ... + d\left(T_1^2 x_n, T_1x_n\right) + d(T_1x_n, x_n) \to 0$$

because T_1^j $(j = 2, 3, 4, ..., m-1)$ are uniformly continuous. Since $\{x_n\}$ is bounded and T_1^m is completely continuous, so $\{T_1^m x_n\}$ has a convergent subsequence $\{T_1^m x_{n_j}\}$ (say). Suppose $T_1^m x_{n_j} \to z \in C$.

Then

$$d\left(x_{n_j}, z\right) \leq d\left(x_{n_j}, T_1^m x_{n_j}\right) + d\left(T_1^m x_{n_j}, z\right) \to 0.$$

Hence $x_{n_j} \to z$. The rest of the proof is similar to the case (a).

Finally assume (c). Then by condition (AV), we obtain

$$\frac{d(x_n, T_1x_n) + d(x_n, T_2x_n)}{2} \geq f(d(x_n, F))$$

By $\liminf_{z\in F}$ on both sides in (10), we obtain that

$$d(x_{n+1}, F) \leq (1 + s_n)\, d(x_n, F) + t_n$$

where $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$.

By Lemma 2.1, $\lim_{n\to\infty} d(x_n, F)$ exists. We claim that $\lim_{n\to\infty} d(x_n, F) = 0$. Assume that $c = \lim_{n\to\infty} d(x_n, F) > 0$. Then we can choose a natural number n_0 such that $0 < \frac{c}{2} < d(x_{n+1}, F)$ for all $n \geq n_0$. Using condition (AV)

$$0 < f\left(\frac{c}{2}\right) < f(d(x_{n+1}, F)) \leq \frac{d(x_n, T_1x_n) + d(x_n, T_2x_n)}{2} \to 0$$

as $n \to \infty$. That is $f\left(\frac{c}{2}\right) = 0$ for $c > 0$ which is against the definition of f in condition (AV). Therefore, $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we claim that $\{x_n\}$ is a Cauchy sequence. Since $\sum_{n=1}^{\infty} s_n < \infty$, so $\sum_{n=1}^{\infty} s_n = s$(say) and hence $\prod_{n=1}^{\infty}(1 + s_n) = s$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} t_n < \infty$, there exists $n_0 \geq 1$ such that $d(x_{n_0}, F) < \frac{\varepsilon}{4s+4}$ and $\sum_{n=n_0}^{\infty} t_n < \frac{\varepsilon}{4s}$.

Let $m > n \geq n_0$ and $p \in F$.Then by (21), we have

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_m, F) + d(x_n, F) \\
&\leq \prod_{i=n_0}^{m-1}(1 + s_i)\, d(x_{n_0}, F) + \prod_{i=n_0}^{m-1}(1 + s_i) \sum_{n=n_0}^{m-1} t_i + \prod_{i=n_0}^{n-1}(1 + s_i)\, d(x_{n_0}, F) + \prod_{i=n_0}^{n-1}(1 + s_i) \sum_{n=n_0}^{n-1} t_i \\
&\leq \prod_{i=n_0}^{\infty}(1 + s_i)\, d(x_{n_0}, F) + \prod_{i=n_0}^{\infty}(1 + s_i) \sum_{n=n_0}^{\infty} t_i + \prod_{i=n_0}^{\infty}(1 + s_i)\, d(x_{n_0}, F) + \prod_{i=n_0}^{\infty}(1 + s_i) \sum_{n=n_0}^{\infty} t_i \\
&< 2\left[(1 + s)\frac{\varepsilon}{4s + 4} + s\frac{\varepsilon}{4s}\right] = \varepsilon.
\end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in C. Let $\lim_{n\to\infty} x_n = q$. Then $d(q, F) = d(\lim_{n\to\infty} x_n, F) = \lim_{n\to\infty} d(x_n, F) = 0$.Since F is closed, we obtain $q \in F$. Hence $\{x_n\}$ converges strongly to a point of F.

As

$$\lim_{n\to\infty} d(x_n, z_n) = 0 = \lim_{n\to\infty} d(x_n, y_n),$$

therefore the limits of the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ coincide. □

Remark 2.6. *Observe that*
(i) CAT (0) *spaces, uniformly convex Banach spaces and Hilbert spaces are uniformly convex metric spaces, therefore our results also hold in CAT* (0) *spaces, uniformly convex Banach spaces and Hilbert spaces, simultaneously.*
(ii) Nonexpansive mapping, asymptotically nonexpansive mappings, asymptotically nonexpansive mappings in the intermediate sense and generalized asymptotically nonexpansive mappings all are total asymptotically nonexpansive, therefore our theorems generalize the corresponding ones in [2, 3, 13, 16, 26] etc.
(iii) The iterative algorithm(4) *unifies the algorithms*(1) − (2) *and as a result all the theorems in [13, 26] follow as corollaries of our corresponding theorems.*

Remark 2.7. *The essentials of hypotheses in our results are natural in view of the following observations: Take* $X = \mathbb{R}, C = [0,2], T_1, T_2 : C \to C$ *are*

$$T_1(x) = \begin{cases} 1 \; if \;\; x \in [0,1] \\ \sqrt{2-x} \; if \, x \in [1,2] \end{cases}$$

and

$$T_2(x) = \begin{cases} 1 \; if \;\; x \in [0,1] \\ \frac{1}{\sqrt{3}}\sqrt{4-x^2} \; if \, x \in [1,2], \end{cases}$$

$\alpha_n = \frac{4n+5}{5n}, \beta_n = \frac{2n+3}{3n}$.*Then both* T_1 *and* T_2 *are total asymptotically nonexpansive and uniformly continuous mappings (see [16]),* $F(T_1) \cap F(T_2) = \{1\}$, $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

References

[1] Ya.I. Alber, C.E. Chidume, H. Zegeye, *Approximating fixed point of total asymptotically nonexpansive mappings,* Fixed Point Theory Appl., (2006) Art. ID 10673.

[2] S.S.Chang , L.Wang, H.W. Joesph Lee, C.K. Chan and L.Yang, *Demiclosed principle and* $\triangle-$*convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces,* Appl. Math. Comput. 219 (5)(2012), 2611–2617. doi:10.1016/j.amc.2012.08.095.

[3] Y. J. Cho, H. Y. Zhou, G. Guo, *Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings,* Comput. Math. Appl. **47** (2004), 707-717.

[4] H. Fukhar-ud-din, *Fixed point iterations in hyperbolic spaces,* J. Nonlin. Convex Anal. 17(6) (2016), 1159-1169.

[5] H. Fukhar-ud-din, *Existence and approximation of fixed points in convex metric spaces*, Carpathian J. Math. 30 (2014), 175-185.

[6] H. Fukhar-ud-din, *One step iterative scheme for a pair of nonexpansive mappings in a convex metric space,* Hacet. J. Math. Stat., Vol. 44 (2015), 1023 – 1031.

[7] H. Fukhar-ud-din, A. R. Khan and N. Hussain, *Approximating common fixed points of total asymptotically nonexpansive mappings in CAT(0) spaces,* J. Nonlinear Sci. Appl., 10 (2017), 771–779.

[8] H. Fukhar-ud-din and S.H.Khan, *Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications*, J. Math. Anal. Appl. 328 (2007), 821-829.

[9] R. Glowinski and P. Le Tallec, *Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics,* SIAM, Philadelphia, 1989.

[10] E. Karapinar, H. Salahifard and S.M. Vaezpour, *Demiclosedness principle for total asymptotically nonexpansive mappings in CAT(0) spaces,* J. Applied Math., Volume 2014, Article ID 738150, 10 pages.

[11] M. A. A. Khan, H. Fukhar-ud-din and A. Kalsoom, *Existence and higher arity iteration for total asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces*, Fixed Point Theory Appl. (2016), 2016:3.

[12] A. R. Khan, M. A. Khamsi and H. Fukhar-ud-din, *Strong convergence of a general iteration scheme in* CAT(0)*-spaces,* Nonlinear Anal. 74 (2011), 783-79

[13] S. H. Khan and W. Takahashi, *Approximating common fixed points of two asymptotically nonexpansive mappings*, Sci. Math. Japon, **53** (2001), 143-148.

[14] S. H. Khan and I. Yildirim, *Common fixed points of two generalized asymptotically quasi-nonexpansive mappings*, Annals of the Alexandru Ioan Cuza University - Mathematics. ISSN (Online) 1221-8421, DOI: 10.1515/aicu-2015-0023, March 2015.

[15] G. E. Kim, *Strong convergence to a fixed point of a total asymptotically nonexpansive mapping*, Fixed Point Theory Appl., 2013, 2013:302.

[16] G. E. Kim, *Strong convergence for asymptotically nonexpansive mappings in the intermediate sense*, Fixed Point Theory Appl., 2014, 2014: 162.

[17] U. Kohlenbach, *Some logical metatheorems with applications in functional analysis,* Trans. Am. Math. Soc., 357(1) (2005), 89–128.

[18] T. Kuczumow, *An almost convergence and its applications,* Ann. Univ. Mariae Curie-Sklodowska Sect. A 32 (1978), 79–88.

[19] M.A.Noor, *New approximation schemes for general variational inequalities,* J. Math. Anal. Appl. 251 (2000), 217–229.
[20] L. Qihou, *Iterative sequences for asymptotically quasi-nonexpansive mappings with error member,* J. Math. Anal. Appl. 259(2001), 18-24.
[21] G. S. Saluja and M. Postolache, *Three-step iterations for total asymptotically nonexpansive mappings in CAT(0) spaces*, Filomat 31(5) (2017), 1317–133.
[22] T. Shimizu and W. Takahashi, *Fixed points of multivalued mappings in certain convex metric spaces*. Topol. Methods Nonlinear Anal., 8 (1996), 197–203.
[23] S. Suantai, *Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings,* J. Math. Anal. Appl. **311** (2005), 506-517.
[24] W. Takahashi, *A convexity in metric spaces and nonexpansive mappings*, Kodai Math. Sem. Rep., 22(1970),142-149.
[25] B. S. Thakur, D. Thakur and M. Postolache, *Modified Picard-Mann hybrid iteration process for total asymptotically nonexpansive mappings*, Fixed Point Theory Appl. (2015), 2015:140.
[26] B. L. Xu and M. A. Noor, *Fixed- points iteration for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **267** (2002), 444-453.