Filomat 31:11 (2017), 3599–3610
https://doi.org/10.2298/FIL1711599A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Characterization of the Essential Approximation Pseudospectrum on a Banach Space

Aymen Ammar[a], Aref Jeribi[a], Kamel Mahfoudhi[a]

[a]*Department of Mathematics, University of Sfax,Faculty of Sciences of Sfax, Route de soukra Km 3.5, B.P. 1171, 3000, Sfax Tunisia*

Abstract. One impetus for writing this paper is the issue of approximation pseudospectrum introduced by M. P. H. Wolff in the journal of approximation theory (2001). The latter study motivates us to investigate the essential approximation pseudospectrum of closed, densely defined linear operators on a Banach space. We begin by defining it and then we focus on the characterization, the stability and some properties of these pseudospectra.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

This research paper is centralized on the study of the approximation pseudospectrum. We survey the historical development of this subject. In 1967, J. M. Varah [18] introduced the first idea of pseudospectra. In 1986, J. H. Wilkinson [19] came with the modern interpretation of pseudospectrum where he defined it for an arbitrary matrix norm induced by a vector norm. Throughout the 1990s, L. N. Trefethen [13–15, 17] not only initiated the study of pseudospectrum for matrices and operators, but also he talked of approximate eigenvalues and pseudospectrum and used this notion to study interesting problems in mathematical physics. By the same token, several authors worked on this field. For example, we may refer to E. B. Davies [5], A. Harrabi [6] and M. P. H. Wolff [20] who has introduced the term approximation pseudospectrum for linear operators.

Pseudospectra are interesting objects by themselves since they carry more information than spectra, especially about transient instead of just asymptotic behaviour of dynamical systems. Also, they have better convergence and approximation properties than spectra. The definition of pseudospectra of a closed densely defined linear operator T, for every $\varepsilon > 0$, is given by:

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\right\}.$$

or by

$$\Sigma_\varepsilon(T) := \sigma(T) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \|(\lambda - T)^{-1}\| \geq \frac{1}{\varepsilon}\right\}.$$

2010 *Mathematics Subject Classification*. Primary 47A53 - 47A55 ; Secondary 54B35 -47A13
Keywords. Approximation pseudospectrum, essential approximation pseudospectrum.
Received: 15 March 2017; Accepted: 20 April 2017
Communicated by Vladimir Rakočević
Email addresses: ammar_aymen84@yahoo.fr (Aymen Ammar), Aref.Jeribi@fss.rnu.tn (Aref Jeribi), kamelmahfoudhi72@yahoo.com (Kamel Mahfoudhi)

By convention, we write $\|(\lambda - T)^{-1}\| = \infty$ if $\lambda \in \sigma(T)$, (spectrum of T). For $\varepsilon > 0$, it can be shown that $\sigma_\varepsilon(T)$ is a larger set and is never empty. The pseudospectra of T are a family of strictly nested closed sets, which grow to fill the whole complex plane as $\varepsilon \to \infty$ (see [6, 14, 15]). From these definitions, it follows that the pseudospectra associated with various ε are nested sets. Then for all $0 < \varepsilon_1 < \varepsilon_2$, we have

$$\sigma(T) \subset \sigma_{\varepsilon_1}(T) \subset \sigma_{\varepsilon_2}(T) \text{ and } \sigma(T) \subset \Sigma_{\varepsilon_1}(T) \subset \Sigma_{\varepsilon_2}(T),$$

and that the intersections of all the pseudospectra are the spectra,

$$\bigcap_{\varepsilon>0} \sigma_\varepsilon(T) = \sigma(T) = \bigcap_{\varepsilon>0} \Sigma_\varepsilon(T).$$

In [1–4], A. Ammar and A. Jeribi defined the notion of Weyl pseudospectra of a closed densely defined linear operator on a Banach space X by:

$$\begin{aligned} \sigma_{w,\varepsilon}(T) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma_\varepsilon(T+K), \\ &:= \bigcup_{\|D\|<\varepsilon} \sigma_w(T+D) \qquad \text{(see [2, Theorem 2.3])}, \end{aligned}$$

where

$$\sigma_w(T) := \bigcap_{K\in\mathcal{K}(X)} \sigma(T+K),$$

and $\mathcal{K}(X)$ is the subspace of compact operators from X into X.

In [20], M. P. H. Wolff has given a motivation to study the essential approximation pseudospectrum. In this paper, the notion of essential approximation pseudospectrum can be extended by devoting our studies to the essential approximation spectrum. For $\varepsilon > 0$ and $T \in \mathcal{C}(X)$, we define

$$\begin{aligned} \sigma_{eap,\varepsilon}(T) &= \bigcap_{K\in\mathcal{K}(X)} \sigma_{ap,\varepsilon}(T+K), \qquad (1) \\ \Sigma_{eap,\varepsilon}(T) &= \bigcap_{K\in\mathcal{K}(X)} \Sigma_{ap,\varepsilon}(T+K), \end{aligned}$$

where

$$\begin{aligned} \sigma_{ap,\varepsilon}(T) &:= \sigma_{ap}(T) \cup \left\{\lambda \in \mathbb{C} : \inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| < \varepsilon\right\}, \\ \Sigma_{ap,\varepsilon}(T) &:= \sigma_{ap}(T) \cup \left\{\lambda \in \mathbb{C} : \inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| \le \varepsilon\right\}, \end{aligned}$$

and

$$\sigma_{ap}(T) := \left\{\lambda \in \mathbb{C} : \inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| = 0\right\}.$$

In the following, we measure the sensitivity of the set $\sigma_{ap}(T)$ with respect to additive perturbations of T by an operator $D \in \mathcal{L}(X)$ of a norm less than ε. So we define the approximation pseudospectrum of T by

$$\sigma_{ap,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{ap}(T+D), \qquad \text{(see Theorem 3.3)} \qquad (2)$$

and we characterize the essential approximation pseudospectrum by: $\lambda \notin \sigma_{eap,\varepsilon}(T)$ if, and only if, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have $\lambda - T - D \in \Phi_+(X)$ and $i(\lambda - T - D) \le 0$.

The essential approximation pseudospectrum $\sigma_{eap,\varepsilon}(.)$ nicely blends these properties of the essential and the approximation pseudospectrum, and accordingly we are interested by the following essential approximation spectrum

$$\sigma_{eap}(T) := \bigcap_{K\in\mathcal{K}(X)} \sigma_{ap}(T+K). \tag{3}$$

We have already mentioned that (1) inherits an ε-version from (3). We will also show that there is an essential version of (2), that is

$$\sigma_{eap,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{eap}(T+D) \qquad \text{(see Theorem 4.5).}$$

Throughout the paper, we denote by $\mathcal{L}(X)$ (resp. $\mathcal{C}(X)$) the set of all bounded (resp. closed, densely defined) linear operators from X into X. For $T \in \mathcal{C}(X)$, we denote by $\rho(T)$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively, the resolvent set, the null space and the range of T. The nullity of T, $\alpha(T)$, is defined as the dimension of $\mathcal{N}(T)$ and the deficiency of T, $\beta(T)$, is defined as the codimension of $\mathcal{R}(T)$ in X.
The set of upper semi-Fredholm operators from X into X is defined by

$$\Phi_+(X) := \{T \in \mathcal{C}(X) : \ \alpha(T) < \infty, \ \ \mathcal{R}(T) \ \text{ is closed in } X\},$$

the set of all lower semi-Fredholm linear operators is defined by

$$\Phi_-(X) := \{T \in \mathcal{C}(X) : \ \beta(T) < \infty, \ \ \mathcal{R}(T) \ \text{ is closed in } X\}.$$

The set of all semi-Fredholm linear operators is defined by

$$\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X), \ \text{ and}$$

the class $\Phi(X)$ of all Fredholm linear operators is defined by

$$\Phi(X) := \Phi_+(X) \cap \Phi_-(X).$$

The set of bounded Fredholm linear operators from X into X is defined by

$$\Phi^b(X) := \Phi(X) \cap \mathcal{L}(X).$$

The set of bounded upper (resp. lower) semi-Fredholm linear operators from X into X is defined by

$$\Phi^b_+(X) := \Phi_+(X) \cap \mathcal{L}(X) \quad (\text{resp. } \Phi^b_-(X) := \Phi_-(X) \cap \mathcal{L}(X)).$$

The index of a semi-Fredholm linear operator T is defined by $i(T) = \alpha(T) - \beta(T)$. Clearly, $i(T)$ is an integer or $\pm\infty$. If $T \in \Phi(X)$, then $i(T) < \infty$. If $T \in \Phi_+(X)\backslash\Phi(X)$ then $i(T) = -\infty$ and if $T \in \Phi_-(X)\backslash\Phi(X)$ then $i(T) = +\infty$. An operator $F \in \mathcal{L}(X)$ is called an upper semi-Fredholm perturbation, if $T+F \in \Phi_+(X)$ whenever, $T \in \Phi_+(X)$. The set of upper semi-Fredholm perturbations is denoted by $\mathcal{F}_+(X)$. If we replace $\Phi_+(X)$ by $\Phi^b_+(X)$, we obtain the set $\mathcal{F}^b_+(X)$.

Theorem 1.1. *Let X a Banach space.*

(i) [8, Lemma 2.1] *Let $T \in \mathcal{C}(X)$ and $K \in \mathcal{L}(X)$. Then*

- (i_1) *If $T \in \Phi_+(X)$ and $K \in \mathcal{F}_+(X)$, then $T + K \in \Phi_+(X)$ and $i(T + K) = i(T)$.*
- (i_2) *If $T \in \Phi^b_+(X)$ and $K \in \mathcal{F}^b_+(X)$, then $T + K \in \Phi^b_+(X)$ and $i(T + K) = i(T)$.*

(ii) [1, Theorem 6.3.1] *If the set $\Phi^b(X)$ is not empty, then*

(ii_1) *$F \in \mathcal{F}_+^b(X)$ and $T \in \mathcal{L}(X)$ imply that $TF \in \mathcal{F}_+^b(X)$ and $FT \in \mathcal{F}_+^b(X)$.*

(iii)[8, Theorem 3.9] *Let $T \in \Phi_+(X)$. Then the following statements are equivalent:*

(iii_1) *$i(T) \leq 0$.*

(iii_2) *T can be expressed in the form $T = S + K$ where $K \in \mathcal{K}(X)$ and $S \in \mathcal{C}(X)$ is an operator with closed range and $\alpha(S) = 0$.*

Definition 1.2. *Let X be a Banach space. A linear operator B from X to X is called T-compact if $\mathcal{D}(T) \subset \mathcal{D}(B)$ and whenever a sequence (x_n) of elements of $\mathcal{D}(T)$ satisfies*

$$\|x_n\| + \|Tx_n\| \leq c, \quad n = 1, 2, \ldots,$$

then (Bx_n) has a subsequence convergent in X.

The paper is organized as follows. Section 2 contains preliminary and auxiliary properties that we will need in order to prove the main results of the other sections. The main aim of section 3 is to characterize the essential approximation pseudospectrum of closed, densely defined linear operators on a Banach space. Then we give different definitions of approximation pseudospectrum and we establish relations between approximation pseudospectrum and the union of the spectra approximation point of all perturbed operators with perturbations that have norms strictly less than ε. Finally, we will prove the invariance of the essential approximation pseudospectrum and establish some results of perturbation on the context of closed, densely defined linear operators on a Banach space.

2. Preliminaries

The goal of this section consists in establishing some preliminary results which will be needed in the sequel.

The following Lemma is developed by P. H. Wolff in [20, Lemma 2.1].

Lemma 2.1. *If $T \in \mathcal{C}(X)$ and $\varepsilon > 0$, then $\Sigma_{ap,\varepsilon}(T)$ is closed.*

Remark 2.2. (i) *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$, then the set $\Sigma_{ap,\varepsilon}(T)$ is obtained from the set $\sigma_{ap,\varepsilon}(T)$ by taking a non-strict inequality instead of a strict inequality. This set makes the approximation pseudospectrum an open set.*

(ii) *It follows from the set $\Sigma_{ap,\varepsilon}(T)$ and the set $\sigma_{ap,\varepsilon}(T)$ that the boundary of $\Sigma_{ap,\varepsilon}(T)$, $\partial\Sigma_{ap,\varepsilon}(T)$ satisfies*

$$\partial\Sigma_{ap,\varepsilon}(T) = \left\{ \lambda \in \mathbb{C} \ : \inf_{x \in \mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| = \varepsilon \right\},$$

and $\partial\Sigma_{ap,\varepsilon}(T)$ depends continuously on ε.

(iii) *The set $\sigma_{ap,\varepsilon}(T)$ is an open set for all $\varepsilon > 0$, in contrast with the set $\Sigma_{ap,\varepsilon}(T)$ which is closed. For $\varepsilon > 0$, it can be shown that $\Sigma_{ap,\varepsilon}(T)$ is a larger set and is never empty and $\Sigma_{ap,\varepsilon}(T)$ is a nested closed family with $\lim_{\varepsilon \to 0^+} \Sigma_{ap,\varepsilon}(T) = \sigma_{ap}(T)$.*

(iv) *$\sigma_{ap,\varepsilon}(T)$ and $\Sigma_{ap,\varepsilon}(T)$ are related as follows $\Sigma_{ap,\varepsilon}(T) = \sigma_{ap,\varepsilon}(T) \cup \left\{ \lambda \in \mathbb{C} \ : \inf_{x \in \mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| = \varepsilon \right\}$.*

Now, we present the following simple and useful result:

Proposition 2.3. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$.*

(i) *$\sigma_{ap,\varepsilon}(T) \subset \sigma_\varepsilon(T)$.*

(*ii*) $\sigma_{ap}(T) = \bigcap_{\varepsilon>0} \sigma_{ap,\varepsilon}(T)$.

(*iii*) *If* $\varepsilon_1 < \varepsilon_2$, *then* $\sigma_{ap}(T) \subset \sigma_{ap,\varepsilon_1}(T) \subset \sigma_{ap,\varepsilon_2}(T)$.

(*iv*) *If* $T \in \mathcal{L}(X)$ *and* $\lambda \in \sigma_{ap,\varepsilon}(T)$, *then* $|\lambda| < \varepsilon + \|T\|$.

(*v*) *If* $\alpha \in \mathbb{C}$ *and* $\varepsilon > 0$, *then* $\sigma_{ap,\varepsilon}(T + \alpha) = \alpha + \sigma_{ap,\varepsilon}(T)$.

(*vi*) *If* $\alpha \in \mathbb{C}\backslash\{0\}$ *and* $\varepsilon > 0$, *then* $\sigma_{ap,|\alpha|\varepsilon}(\alpha T) = \alpha\sigma_{ap,\varepsilon}(T)$.

Proof. (*i*) If $\lambda \notin \sigma_\varepsilon(T)$, then $\|(\lambda - T)^{-1}\| \leq \frac{1}{\varepsilon}$. Moreover,

$$\begin{aligned}
\frac{1}{\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\|)} &= \sup_{x\in\mathcal{D}(T),\ \|x\|=1} \frac{\|x\|}{\|(\lambda - T)x\|} \\
&= \sup_{0\neq x\in\mathcal{D}(T)} \frac{\|x\|}{\|(\lambda - T)x\|} \\
&= \sup_{y\in X\backslash\{0\}} \frac{\|(\lambda - T)^{-1}y\|}{\|y\|} = \|(\lambda - T)^{-1}\| \leq \frac{1}{\varepsilon},
\end{aligned}$$

hence, $\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\|) > \varepsilon$. So $\lambda \notin \sigma_{ap,\varepsilon}(T)$.

(*ii*) It is clear that $\sigma_{ap}(T) \subset \sigma_{ap,\varepsilon}(T)$, then $\sigma_{ap}(T) \subset \bigcap_{\varepsilon>0} \sigma_{ap,\varepsilon}(T)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \sigma_{ap,\varepsilon}(T)$, then for all $\varepsilon > 0$, we have $\lambda \in \sigma_{ap,\varepsilon}(T)$. There are two possible cases:

1^{st} *case* : If $\lambda \in \sigma_{ap}(T)$, we get the desired result.

2^{nd} *case* : If $\lambda \in \left\{\lambda \in \mathbb{C} : \inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| < \varepsilon\right\}$, taking limits as $\varepsilon \to 0^+$, we get for all $x \in \mathcal{D}(T)$ such that $\|x\| = 1$, $\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| = 0$. We infer that $\lambda \in \sigma_{ap}(T)$.

(*iii*) Let $\lambda \in \sigma_{ap,\varepsilon_1}(T)$, then $\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| < \varepsilon_1 < \varepsilon_2$. Hence $\lambda \in \sigma_{ap,\varepsilon_2}(T)$.

(*iv*) Let $\lambda \in \sigma_{ap,\varepsilon}(T)$, then $\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| < \varepsilon$, and also $\Big|\|\lambda| - \|Tx\|\Big| < \|(\lambda - T)x\|$. Hence $|\lambda| < \varepsilon + \|T\|$.

(*v*) Let $\lambda \in \sigma_{ap,\varepsilon}(T+\alpha)$, then $\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|((\lambda-\alpha)-T)x\| < \varepsilon$. Hence $\lambda-\alpha \in \sigma_{ap,\varepsilon}(T)$. This yields to $\lambda \in \alpha+\sigma_{ap,\varepsilon}(T)$. For the second inclusion it is the same reasoning.

(*vi*) Let $\lambda \in \sigma_{ap,|\alpha|\varepsilon}(\alpha T)$, then

$$\begin{aligned}
\inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - \alpha T)x\| &= \inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|\alpha(\frac{\lambda}{\alpha} - T)x\| \qquad \alpha \neq 0, \\
&= |\alpha| \inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\frac{\lambda}{\alpha} - T)x\| \\
&< |\alpha|\varepsilon.
\end{aligned}$$

Hence $\dfrac{\lambda}{\alpha} \in \sigma_{ap,\varepsilon}(T)$. So $\sigma_{ap,|\alpha|\varepsilon}(\alpha T) \subseteq \alpha\sigma_{ap,\varepsilon}(T)$. However, the reverse inclusion is similar. □

Remark 2.4. *P. H. Wolff shows that for all* $\varepsilon > 0$ *that* $\Sigma_{ap,\varepsilon}(T) \neq \Sigma_\varepsilon(T)$, (*see Example 1,* [20]).

Proposition 2.5. *Let* $T \in \mathcal{C}(X)$ *and* $\varepsilon > 0$.

(*i*) $\Sigma_{ap,\varepsilon}(T) \subset \Sigma_\varepsilon(T)$.

(*ii*) $\bigcap_{\varepsilon>0} \Sigma_{ap,\varepsilon}(T) = \sigma_{ap}(T)$.

(iii) If $\varepsilon_1 < \varepsilon_2$*, then* $\sigma_{ap}(T) \subset \Sigma_{ap,\varepsilon_1}(T) \subset \Sigma_{ap,\varepsilon_2}(T)$.

Proof. The proof of (*i*), (*ii*) and (*iii*) may be achieved in the same way as the proof of (*i*), (*ii*) and (*iii*) for Proposition 2.3. □

3. A Characterization of Approximation Pseudospectrum

In this section, we turn to the problem when the closure of $\sigma_{ap,\varepsilon}(T)$ is equal to $\Sigma_{ap,\varepsilon}(T)$ holds. Our first result is the following.

Theorem 3.1. *Let* $T \in \mathcal{C}(X)$ *and* $\varepsilon > 0$*. Then* $\lim\limits_{\varepsilon \to \varepsilon_0} \Sigma_{ap,\varepsilon}(T) = \Sigma_{ap,\varepsilon_0}(T)$.

Proof. The approximation pseudospectrum is a family increase in function ε. Then for $0 < \varepsilon_0 < \varepsilon$, we have $\sigma_{ap}(T) \subseteq \Sigma_{ap,\varepsilon_0}(T) \subseteq \Sigma_{ap,\varepsilon}(T)$. Hence

$$\lim_{\varepsilon \to \varepsilon_0} \Sigma_{ap,\varepsilon}(T) = \bigcap_{\varepsilon > \varepsilon_0} \Sigma_{ap,\varepsilon}(T).$$

Proposition 2.3-(ii) justifies the equality $\bigcap\limits_{\varepsilon > \varepsilon_0} \Sigma_{ap,\varepsilon}(T) = \Sigma_{ap,\varepsilon_0}(T)$. □

We define the following hypothesis for T :

$$(\mathcal{H}):\left\{ \text{There is no open set in } \rho_{ap}(T) := \mathbb{C}\backslash\sigma_{ap}(T) \text{ on which the } \lambda \longmapsto \inf_{x \in \mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| \text{ is constant.} \right\}$$

Theorem 3.2. *Let* $T \in \mathcal{C}(X)$ *and* $\varepsilon > 0$*. If* $(\mathcal{H})$ *holds, then* $\overline{\sigma_{ap,\varepsilon}(T)} = \Sigma_{ap,\varepsilon}(T)$.

Proof. Since $\sigma_{ap,\varepsilon}(T) \subset \Sigma_{ap,\varepsilon}(T)$ and $\Sigma_{ap,\varepsilon}(T)$ is closed, then $\overline{\sigma_{ap,\varepsilon}(T)} \subset \Sigma_{ap,\varepsilon}(T)$. In order to prove the inverse inclusion, we take $\lambda \in \Sigma_{ap,\varepsilon}(T)$. We notice the existence of two cases:

<u>1^{st}*case*</u> : If $\lambda \in \sigma_{ap,\varepsilon}(T)$, then $\lambda \in \overline{\sigma_{ap,\varepsilon}(T)}$.

<u>2^{nd}*case*</u> : If $\lambda \in \Sigma_{ap,\varepsilon}(T) \setminus \sigma_{ap,\varepsilon}(T)$, then $\inf\limits_{x \in \mathcal{D}(T),\ \|x\|=1,} \|(\lambda - T)x\| = \varepsilon$. By using Hypothesis $(\mathcal{H})$, there exists a sequence $\lambda_n \in \rho_{ap}(T)$ such that $\lambda_n \to \lambda$, and

$$\inf_{x \in \mathcal{D}(T),\ \|x\|=1} \|(\lambda_n - T)x\| < \inf_{x \in \mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| = \varepsilon \ .$$

We deduce that $\lambda_n \in \sigma_{ap,\varepsilon}(T)$ and also that $\lambda_n \to \lambda$, which implies that $\lambda \in \overline{\sigma_{ap,\varepsilon}(T)}$, then, $\Sigma_{ap,\varepsilon}(T) \subset \overline{\sigma_{ap,\varepsilon}(T)}$. □

Theorem 3.3. *Let* $T \in \mathcal{C}(X)$ *and* $\varepsilon > 0$*. Then the following properties are equivalent:*

(i) $\lambda \in \sigma_{ap,\varepsilon}(T)$.

(ii)There exists a bounded operator $D \in \mathcal{L}(X)$ *such that* $\|D\| < \varepsilon$ *and* $\lambda \in \sigma_{ap}(T + D)$.

Proof. (*i*) ⇒ (*ii*) Let $\lambda \in \sigma_{ap,\varepsilon}(T)$. There are two possible cases:
<u>1^{st}*case*</u> : If $\lambda \in \sigma_{ap}(T)$, then it is sufficient to take $D = 0$.

<u>2^{nd}*case*</u> : If $\lambda \notin \sigma_{ap}(T)$, then there exists $x_0 \in X$ such that $\|x_0\| = 1$ and $\|(\lambda - T)x_0\| < \varepsilon$. By using the Hahn Banach Theorem, (see [11]) there exists $x' \in X'$ (dual of X) such that $\|x'\| = 1$ and $x'(x_0) = \|x_0\|$. Consider the operator D defined by the formula $D : X \to X,\ \ x \longmapsto Dx := x'(x)(\lambda - T)x_0$. Then D is a linear operator everywhere defined on X. It is bounded, since $\|Dx\| = \|x'(x)(\lambda - T)x_0\| \leq \|x'\|\|x\|\|(\lambda - T)x_0\|$, for $x \neq 0$. Therefore,

$$\frac{\|Dx\|}{\|x\|} \leq \|(\lambda - T)x_0\|.$$

Hence $\|D\| < \varepsilon$. We claim that $\inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T - D)x\| = 0$. Let $x_0 \in X$, then

$$\inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T - D)x\| \leq \|(\lambda - T - D)x_0\| \leq \|(\lambda - T)x_0 - x'(x_0)(\lambda - T)x_0\| = 0.$$

$(ii) \Rightarrow (i)$ We assume that there exists a bounded operator $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $\lambda \in \sigma_{ap}(T + D)$, which means that $\inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T - D)x\| = 0$. In order to prove that $\inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| < \varepsilon$, we can write,

$$\|(\lambda - T)x_0\| = \|(\lambda - T - D + D)x_0\| \leq \|(\lambda - T - D)x_0\| + \|Dx_0\|.$$

Then, $\inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| < \varepsilon$. □

Remark 3.4. *We can derive from Theorem 3.3 the following result. Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then*

$$\sigma_{ap,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{ap}(T + D).$$

Theorem 3.5. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then*

$$\sigma_{ap,\varepsilon}(T) = \bigcup_{D\in\Theta_\varepsilon(X)} \sigma_{ap}(T + D),$$

where

$$\Theta_\varepsilon(X) := \Big\{D \in \mathcal{L}(X) \ : \ \|D\| < \varepsilon \text{ and } \dim\big(\mathcal{R}(D)\big) \leq 1\Big\}.$$

Proof. Let $\lambda \in \sigma_{ap,\varepsilon}(T)$. We will discuss the two following cases:

1^{st}case : If $\lambda \in \sigma_{ap}(T)$, then it is sufficient to take $D = 0$.

2^{nd}case : If $\lambda \notin \sigma_{ap}(T)$, then there exists $x_0 \in X$ such that $\|x_0\| = 1$ and $\|(\lambda - T)x_0\| < \varepsilon$. Putting $\|x_0\| = \|(\lambda - T)^{-1}(\lambda - T)x_0\|$ implies that $\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}$. Then we can find $y_0 \in X$ such that $\|y_0\| = 1$ and $\|(\lambda - T)^{-1}y_0\| > \frac{1}{\varepsilon}$. Hence $\|(\lambda - T)^{-1}y_0\| = \frac{1}{\delta}$, where $\delta < \varepsilon$. By using the Hahn Banach Theorem, there exists $x' \in X'$ such that $\|x'\| = 1$ and

$$x'\left((\lambda - T)^{-1}y_0\right) = \|(\lambda - T)^{-1}y_0\| = \frac{1}{\delta}.$$

Now, we can define the rank-one operator by, $D : X \to X,\ \ x \longmapsto Dx := \delta x'(x)y_0$. Clearly, D is a linear operator everywhere defined on X. It is bounded, since

$$\|Dx\| = \|\delta x'(x)y_0\| \leq \delta\|x'\|\|y_0\|\|x\|.$$

Then $\|D\| \leq \delta < \varepsilon$. Furthermore,

$$D\left((\lambda - T)^{-1}y_0\right) = \delta x'\left((\lambda - T)^{-1}y_0\right)y_0 = \delta\frac{1}{\delta}y_0 = y_0.$$

Putting $x = (\lambda - T)^{-1}y_0$, we will discuss these two cases:

1^{st}case : If $x = x_0$, we obtain

$$\begin{aligned}\inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T - D)x\| &\leq \|(\lambda - T - D)x_0\| \\ &\leq \|(\lambda - T)x_0 - Dx_0\| = \|y_0 - y_0\| = 0.\end{aligned}$$

2^{nd}case : If $x \neq x_0$. First, let $x = 0$, then $(\lambda - T)^{-1}y_0 = 0$, which is a contradiction with $\|(\lambda - T)^{-1}y_0\| = \frac{1}{\delta}$. Second, let $x \neq 0$, then $Dx = y_0 = (\lambda - T)x$. Hence, $\inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T - D)x\| = 0$.

We deduce that $\lambda \in \sigma_{ap}(T + D)$ and $D \in \Theta_\varepsilon(X)$. The second inclusion is clear. □

Theorem 3.6. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Let $E \in \mathcal{L}(X)$ such that $\|E\| < \varepsilon$. Then*

$$\sigma_{ap,\varepsilon-\|E\|}(T) \subseteq \sigma_{ap,\varepsilon}(T+E) \subseteq \sigma_{ap,\varepsilon+\|E\|}(T).$$

Proof. Let $\lambda \in \sigma_{ap,\varepsilon-\|E\|}(T)$. Then by Theorem 3.3 there exists a bounded operator $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon - \|E\|$ such that

$$\lambda \in \sigma_{ap}(T+D) = \sigma_{ap}\Big((T+E)+(D-E)\Big).$$

The fact that $\|D-E\| \leq \|D\|+\|E\| < \varepsilon$ allows us to deduce that $\lambda \in \sigma_{ap,\varepsilon}(T+E)$. Using a similar reasoning to the first inclusion, we deduce that $\lambda \in \sigma_{ap,\varepsilon+\|E\|}(T)$. $\square$

Theorem 3.7. *Let $T \in \mathcal{C}(X)$ and $V \in \mathcal{L}(X)$ be invertible. Let $B = V^{-1}TV$. Then*

$$\sigma_{ap}(B) = \sigma_{ap}(T),$$

and for $\varepsilon > 0$ and $k = \|V^{-1}\|\|V\|$, we have

$$\sigma_{ap,\frac{\varepsilon}{k}}(T) \subseteq \sigma_{ap,\varepsilon}(B) \subseteq \sigma_{ap,k\varepsilon}(T), \text{ and} \tag{4}$$

$$\Sigma_{ap,\frac{\varepsilon}{k}}(T) \subseteq \Sigma_{ap,\varepsilon}(B) \subseteq \Sigma_{ap,k\varepsilon}(T). \tag{5}$$

Proof. we can write

$$\|(\lambda-B)x\| = \|V^{-1}(\lambda-T)Vx\| \leq k\|(\lambda-T)x\|, \tag{6}$$

$$\|(\lambda-T)x\| = \|V(\lambda-B)V^{-1}x\| \leq k\|(\lambda-B)x\|. \tag{7}$$

Let $\lambda \in \sigma_{ap}(T)$, which implies that $\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda-T)x\| = 0$. By using Relation (6), it follows that $\inf\limits_{x\in\mathcal{D}(B),\ \|x\|=1} \|(\lambda-B)x\| = 0$. Hence $\lambda \in \sigma_{ap}(B)$. The converse is similar: it is sufficient to use Relation (7).

For the second result, If $\lambda \in \sigma_{ap,\frac{\varepsilon}{k}}(T)$, then using Relation (6), we obtain $\lambda \in \sigma_{ap,\varepsilon}(B)$. if $\lambda \in \sigma_{ap,\varepsilon}(B)$, then using Relation (7), we obtain $\lambda \in \sigma_{ap,k\varepsilon}(B)$. The Second formula in (4) holds and the proof is similar to Relation (5). $\square$

The closure of $\sigma_{ap,\varepsilon}(T)$ is always contained in $\Sigma_{ap,\varepsilon}(T)$, but equality holds if, and only if, T does not have constant infimum norm on any open set. The present part addresses the question on whether or not a similar equality holds in the case of non-strict inequalities:

$$\Sigma_{ap,\varepsilon}(T) \overset{?}{=} \bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D).$$

Theorem 3.8. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. We have*

$$\bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D) \subset \Sigma_{ap,\varepsilon}(T).$$

Proof. Let $\lambda \notin \Sigma_{ap,\varepsilon}(T)$, then $\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda-T)x\| > \varepsilon$. In order to prove that $\lambda \notin \bigcup\limits_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D)$, which means that, $\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda-T-D)x\| > 0$ for all $D \in \mathcal{L}(X)$ such that $\|D\| \leq \varepsilon$, we have $\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda-T-D)x\| \geq \inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \Big|\|(\lambda-T)x\| - \|Dx\|\Big| \geq \inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \Big|\|(\lambda-T)x\| - \varepsilon\|x\|\Big| > 0.$ $\square$

We first consider the following example:

Example 3.9. *Let* $l_1(\mathbb{N}) = \left\{(x_j)_{j\geq 1} : x_j \in \mathbb{C} \text{ and } \sum_{j=1}^{+\infty} |x_j| < \infty\right\}$ *be equipped with the following norm* $\|x\| := \sum_{j=1}^{\infty} |x_j|$ *and we define the operator T by:*

$$\begin{cases} T : l_1(\mathbb{N}) \to l_1(\mathbb{N}), \\ \quad x \longmapsto Tx, \\ \text{where } Tx = \left((1+2\varepsilon)x_1 - \sum_{j=2}^{\infty} x_j, -\varepsilon_2 x_2, \dots, -\varepsilon_n x_n, \dots\right), \end{cases}$$

$x = (x_1, x_2, \dots, x_n, \dots) \in l_1(\mathbb{N})$ *and* ε_n*, where* $n = 2, 3, \dots$ *is a sequence of positive numbers monotonically decreasing to* 0*. It was proved by E. Shargorodsky in* [12]*, that* $\inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(2\varepsilon I - T)x\| = \varepsilon$*, and for all* $D \in \mathcal{L}(X)$, $\|D\| \leq \varepsilon$*, we have* $2\varepsilon \in \rho(T+D)$*. It follows that* $2\varepsilon \in \Sigma_{ap,\varepsilon}(T)$ *and* $2\varepsilon \notin \sigma_{ap}(T+D)$ *for all* $\|D\| \leq \varepsilon$*. Then* $2\varepsilon \notin \bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D)$*. Hence* $\bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D) \subsetneq \Sigma_{ap,\varepsilon}(T)$*.*

Theorem 3.10. *Let* $T \in \mathcal{C}(X)$ *and* $\varepsilon > 0$*. If* $(\mathcal{H})$ *holds, then*

$$\Sigma_{ap,\varepsilon}(T) = \bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D).$$

Proof. It follows from Theorem 3.8 and Theorem 3.3 that

$$\overline{\sigma_{ap,\varepsilon}(T)} \subseteq \overline{\bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D)} \subseteq \Sigma_{ap,\varepsilon}(T)$$

We assume that $(\mathcal{H})$ holds, then $\overline{\sigma_{ap,\varepsilon}(T)} = \Sigma_{ap,\varepsilon}(T)$, hence

$$\overline{\bigcup_{\|D\|\leq\varepsilon} \sigma_{ap}(T+D)} = \Sigma_{ap,\varepsilon}(T).$$

It follows from Theorem 3.8 and Theorem 3.3 that Theorem 3.10 is an equality if, and only if, the level set $\left\{\lambda \in \mathbb{C} : \inf_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T)x\| = \varepsilon\right\}$ is a subset of $\bigcup_{\|D\|=\varepsilon} \sigma_{ap}(T+D)$. □

4. Essential Approximation Pseudospectrum

In this section, we have the following useful stability result for the essential approximation pseudospectrum.

Definition 4.1. *Let* $T \in \mathcal{C}(X)$*. We define the essential approximation spectrum of the operator T by*:

$$\sigma_{eap}(T) = \bigcap_{K\in\mathcal{K}(X)} \sigma_{ap}(T+K).$$

The following result gives the essential approximation spectrum of the operator T in terms of upper semi-Fredholm linear operators.

Proposition 4.2. [8, Proposition 3.1] *Let* $T \in \mathcal{C}(X)$*. Then*

$$\lambda \notin \sigma_{eap}(T) \text{ if, and only if, } \lambda - T \in \Phi_+(X) \text{ and } i(\lambda - T) \leq 0.$$

In what follows, we will bring a new definition of the essential approximation pseudospectrum.

Definition 4.3. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. We define the essential approximation pseudospectrum of the operator T by*

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma_{ap,\varepsilon}(T + K).$$

Proposition 4.4. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then*

(i) $\bigcap_{\varepsilon>0} \sigma_{eap,\varepsilon}(T) = \sigma_{eap}(T)$.

(ii) If $\varepsilon_1 < \varepsilon_2$, *then* $\sigma_{eap}(T) \subset \sigma_{eap,\varepsilon_1}(T) \subset \sigma_{eap,\varepsilon_2}(T)$.

(iii) $\sigma_{eap,\varepsilon}(T + F) = \sigma_{eap,\varepsilon}(T)$ *for all* $F \in \mathcal{K}(X)$.

Proof. (i) $\sigma_{eap}(T) \subset \sigma_{eap,\varepsilon}(T)$. Indeed, Let $\lambda \notin \sigma_{eap,\varepsilon}(T)$. Then there exists $K \in \mathcal{K}(X)$, such that $\inf\limits_{x\in\mathcal{D}(X),\ \|x\|=1} \|(\lambda - T - K)x\| > \varepsilon > 0$. Hence $\lambda \notin \sigma_{eap}(T)$, so $\sigma_{eap}(T) \subset \bigcap_{\varepsilon>0} \sigma_{eap,\varepsilon}(T)$. Conversely, let $\lambda \in \bigcap\limits_{\varepsilon>0} \sigma_{eap,\varepsilon}(T)$. Hence for all $\varepsilon > 0$, we have $\lambda \in \sigma_{eap,\varepsilon}(T)$. Then for every $K \in \mathcal{K}(X)$ we obtain $\lambda \in \sigma_{ap,\varepsilon}(T + K)$. This implies that $\inf\limits_{x\in\mathcal{D}(X),\ \|x\|=1} \|(\lambda - T - K)x\| < \varepsilon$. Taking limits as $\varepsilon \to 0^+$, we infer that $\lambda \in \sigma_{eap}(T)$.

(ii) Let $\lambda \in \sigma_{eap,\varepsilon_1}(T)$, then there exists $K \in \mathcal{K}(X)$, such that $\inf\limits_{x\in\mathcal{D}(X),\ \|x\|=1} \|(\lambda - T - K)x\| < \varepsilon_1 < \varepsilon_2$. So $\lambda \in \sigma_{eap,\varepsilon_2}(T)$.

(iii) It follows immediately from Definition 4.3 that $\sigma_{eap,\varepsilon}(T + F) = \sigma_{eap,\varepsilon}(T)$ for all $F \in \mathcal{K}(X)$. □

In what follows, Theorem 4.5 gives a characterization of the essential approximation pseudospectrum by means of semi-Fredholm operators.

Theorem 4.5. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then the following properties are equivalent:*

(i) $\lambda \notin \sigma_{eap,\varepsilon}(T)$.

(ii) For all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have $\lambda - T - D \in \Phi_+(X)$ and $i(\lambda - T - D) \leq 0$.

Proof. $(i) \Rightarrow (ii)$ Let $\lambda \notin \sigma_{eap,\varepsilon}(T)$. It follows that there exists a compact operator K on X such that $\lambda \notin \sigma_{ap,\varepsilon}(T + K)$. By using Theorem 3.3, we notice that $\lambda \notin \sigma_{ap}(T + D + K)$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. So,

$$\lambda - T - D - K \in \Phi_+(X) \ \text{ and } \ i(\lambda - T - D - K) \leq 0,$$

for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Using Theorem 1.1, we get, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$,

$$\lambda - T - D \in \Phi_+(X) \ \text{ and } \ i(\lambda - T - D) \leq 0.$$

$(ii) \Rightarrow (i)$ We assume that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have

$$\lambda - T - D \in \Phi_+(X) \ \text{ and } \ i(\lambda - T - D) \leq 0.$$

Based on Lemma 1.1, $\lambda - T - D$ can be expressed in the form $\lambda - T - D = S + K$, where $K \in \mathcal{K}(X)$ and $S \in \mathcal{C}(X)$ is an operator with closed range and $\alpha(S) = 0$. So

$$\lambda - T - D - K = S \ \text{ and } \ \alpha(\lambda - T - D - K) = 0.$$

By using [11, Theorem 3.12] there exists a constant $c > 0$ such that

$$\|(\lambda - T - D - K)x\| \geq c\|x\|, \ \text{ for all } \ x \in \mathcal{D}(T).$$

This proves that $\inf\limits_{x\in\mathcal{D}(T),\ \|x\|=1} \|(\lambda - T - D - K)x\| \geq c > 0$. Thus $\lambda \notin \sigma_{ap}(T + D + K)$, and therefore $\lambda \notin \sigma_{eap,\varepsilon}(T)$. □

Remark 4.6. *It follows immediately from Theorem 4.5 that $\lambda \notin \sigma_{eap,\varepsilon}(T)$ if, and only if, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we obtain*

$$\lambda - T - D \in \Phi_+(X) \text{ and } i(\lambda - T - D) \leq 0.$$

This is equivalent to

$$\sigma_{eap,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{eap}(T + D).$$

From Definition 4.3 and Proposition 4.4, we get the following corollary:

Corollary 4.7. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then*

$$\sigma_{eap}(T) = \lim_{\varepsilon\to 0} \overline{\bigcap_{K\in\mathcal{K}(X)} \sigma_{ap,\varepsilon}(T + K)} = \bigcap_{\varepsilon>0} \left(\bigcup_{\|D\|<\varepsilon} \sigma_{eap}(T + D) \right).$$

Theorem 4.8. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then*

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{F\in\mathcal{F}^+(X)} \sigma_{ap,\varepsilon}(T + F).$$

Proof. Let $\lambda \notin \bigcap_{F\in\mathcal{F}^+(X)} \sigma_{ap,\varepsilon}(T + F)$, then there exists $F \in \mathcal{F}^+(X)$ such that $\lambda \notin \sigma_{ap,\varepsilon}(T + F)$. From Theorem 3.3, we see that $\lambda \notin \sigma_{ap}(T + F + D)$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Therefore,

$$\lambda - T - F - D \in \Phi_+(X) \text{ and } i(\lambda - T - F - D) \leq 0.$$

Using Theorem 1.1, we conclude that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$,

$$\lambda - T - D \in \Phi_+(X) \text{ and } i(\lambda - T - D) \leq 0.$$

Finally, Theorem 4.5 shows that $\lambda \notin \sigma_{eap,\varepsilon}(T)$. For the second inclusion, it is clear that

$$\bigcap_{F\in\mathcal{F}^+(X)} \sigma_{ap,\varepsilon}(T + F) \subset \bigcap_{F\in\mathcal{K}(X)} \sigma_{ap,\varepsilon}(T + F) := \sigma_{eap,\varepsilon}(T),$$

because $\mathcal{K}(X) \subset \mathcal{F}^+(X)$. □

Remark 4.9. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$.*

(*i*) *Using Theorem 4.8, we infer that $\sigma_{eap,\varepsilon}(T + F) = \sigma_{eap,\varepsilon}(T)$ for all $F \in \mathcal{F}^+(X)$.*

(*ii*) *Let $\Im(X)$ be a subset of $\mathcal{L}(X)$. If $\mathcal{K}(X) \subset \Im(X) \subset \mathcal{F}^+(X)$, then $\sigma_{eap,\varepsilon}(T) = \bigcap_{M\in\Im(X)} \sigma_{ap,\varepsilon}(T + M)$ and $\sigma_{eap,\varepsilon}(T + J) = \sigma_{eap,\varepsilon}(T)$ for all $J \in \Im(X)$.*

Lemma 4.10. *Let $\varepsilon > 0$, T and B be two elements of $\mathcal{C}(X)$. Assume that for a bounded operator D such that $\|D\| < \varepsilon$, the operator B is $(T + D)$-compact, then $\sigma_{eap,\varepsilon}(T) = \sigma_{eap,\varepsilon}(T + B)$.*

Proof. Let $\lambda \notin \sigma_{eap,\varepsilon}(T)$, then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have $\lambda - T - D \in \Phi_+(X)$ and $i(\lambda - T - D) \leq 0$. Since B is $(T + D)$-compact and applying [10, Theorem 3.3], we get

$$\lambda - T - B - D \in \Phi_+(X) \text{ and } i(\lambda - T - B - D) \leq 0.$$

Therefore, $\lambda \notin \sigma_{eap,\varepsilon}(T + D)$. We conclude that $\sigma_{eap,\varepsilon}(T + B) \subset \sigma_{eap,\varepsilon}(T)$. Conversely, let $\lambda \notin \sigma_{eap,\varepsilon}(T + B)$. Then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have $\lambda - T - B - D \in \Phi_+(X)$ and $i(\lambda - T - B - D) \leq 0$. On the other hand, B is $(T + D)$-compact. Using [10, Theorem 2.12], we deduce that B is $(T + B + D)$-compact, then

$$\lambda - T + D \in \Phi_+(X) \text{ and } i(\lambda - T - D) \leq 0.$$

Therefore $\lambda \notin \sigma_{eap,\varepsilon}(T)$. This proves that $\sigma_{eap,\varepsilon}(T) \subset \sigma_{eap,\varepsilon}(T + B)$. □

References

[1] A. Jeribi, *Spectral theory and applications of linear operators and block operator matrices, Springer-Verlag, New-York,* (2015).
[2] A. Ammar and A. Jeribi, *A characterization of the essential pseudospectra on a Banach space, Arab. J. Math.* 2 (2013), 139–145 .
[3] A. Ammar and A. Jeribi, *A characterization of the essential pseudospectra and application to a transport equation, Extra Math.* 28 (2013), 95–112 .
[4] A. Ammar, B. Boukettaya and A. Jeribi, *A note on the essential pseudospectra and application,Linear and Multilinear Algebra* 64, no. 8 (2016), 1474–1483 .
[5] E. B. Davies, *Linear Operators and their spectra,Cambridge Studies in Advanced Mathematics, 106. Camb. Univ. Press, Cambridge* (2007).
[6] A. Harrabi, *Pseudospectre d'une suite d'opérateurs bornés, RAIRO Modél. Math. Anal. Numér.* 32 (1998), 671–680 .
[7] R. Hagger, M. Lindner and M. Seidel, *Essential pseudospectra and essential norms of band-dominated operators, J. Math. Anal. Appl.* 437 (2016), no. 1, 255–291.
[8] A. Jeribi and N. Moalla, *A characterization of some subsets of Schechter's essential spectrum and application to singular transport equation, J. Math. Anal. Appl.* 358 (2009), 434–444 .
[9] T. Kato, *Perturbation theory for linear operators,Die Grundlehren der math. Wissenschaften, Band 132, Springer-Verlag, New-York,* (1966).
[10] M. Schechter, *Spectra of partial differential operators,* North-Holland, Amsterdam, New-York (1986).
[11] M. Schechter, *Principles of functional analysis, Second edition. Graduate Studies in Mathematics, American Mathematical Society, Providence, RI,* 36, (2002).
[12] E. Shargorodsky, *On the definition of pseudospectra,Bull. London Math. Soc.* 41 (2009), 524–534 .
[13] L. N. Trefethen, *Approximation theory and numerical linear algebra, Algorithms for Approximation II, JC Mason and MG Cox, eds., Chapman and Hall, London* (1990) , 336–360.
[14] L. N. Trefethen, *Pseudospectra of matrices. numerical analysis,* (1991). *In Pitman Research notes in Mathematics Series, Longman Science and Technology, Harlow* 260 (1992), 234–266 .
[15] L. N. Trefethen, *Pseudospectra of linear operators, SAIM rev.* 39 (3) (1997), 383–406.
[16] L. N. Trefethen and M. Embree, *Spectra and pseudospectra: The behavior of nonnormal matrices and operators,Prin. Univ. Press, Princeton and Oxford,* (2005).
[17] S. C. Reddy and L. N. Trefethen, *Lax-stability of fully discrete spectral methods via stability regions and pseudo-eigenvalues, Comput. Meth. Appl. Mech. and Engrg.* 80 , no. 1 (1990),147–164 .
[18] J. M. Varah, *The computation of bounds for the invariant subspaces of a general matrix operator,Ph.D. thesis, stanford university ProQuest LLC, Ann Arbor,* (1967).
[19] J. H. Wilkinson, *Sensitivity of eigenvalues II, Util. Math.* 30 (1986), 243–286.
[20] M. P. H. Wolff, *Discrete approximation of unbounded operators and approximation of their spectra, J. Approx. Theo.* 113 (2001), 229–244.