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On the Existence of Bounded Solutions to a Class of Nonlinear Initial Value Problems with Delay

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Abstract. We consider a class of nonlinear initial value problems with delay. Using an abstract fixed point theorem, we prove an existence result producing a unique bounded solution.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

In this paper, we study the existence and uniqueness of a bounded solution for the following nonlinear initial value problem with delay:

$$u(t) = \begin{cases} \int_{t-\tau}^{t} g(s, u(s), u'(s))ds, & t \in [0, t_1], \quad t_1 > 0, \\ \phi(t), & t \in [-\tau, 0], \quad \phi \in C^1[-\tau, 0], \end{cases} \tag{1}$$

under assumption

$$\begin{cases} \phi(0) = \int_{-\tau}^{0} g(s, \phi(s), \phi'(s))ds, \\ \phi'(0) = g(0, \phi(0), \phi'(0)) - g(-\tau, \phi(-\tau), \phi'(-\tau)). \end{cases}$$

In this problem, $u(t)$ is the proportion of infectious individuals (in population) at time t, $\tau > 0$ is the length of time for which an individual remains infectious; $u'(t)$ is the speed of infectivity and $g(t, u(t), u'(t))$ is the proportion of new infectious individuals per unit time. For a comprehensive study of integral equations with delay, the reader is referred to Precup [7].

We denote by $\mathcal{X}$ the product space $\mathcal{X} = C^1[-\tau, t_1] \times C[-\tau, t_1]$. Then, we consider the Bielecki metric $d_B : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_2$, where $\mathbb{R}_2$ is the set of all 2×1 matrices, given as

$$d_B((u_1, v_1), (u_2, v_2)) = (\|u_1 - u_2\|_B, \|v_1 - v_2\|_B)^T, \quad \text{for all } (u_1, v_1), (u_2, v_2) \in \mathcal{X},$$

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where

$$\|z\|_B = \max\{|z(t)|e^{-\vartheta(t+\tau)} : t \in [-\tau, t_1]\}, \quad \text{for a chosen } \vartheta > 0 \text{ and any } z \in C[-\tau, t_1].$$

We need the following functional space

$$\mathcal{X}_+ = \{(u, v) \in \mathcal{X} : u(t) \geq 0, t \in [-\tau, t_1]\}.$$

Of course, $\mathcal{X}$ is a complete metric space, $\mathcal{X}_+ \subseteq \mathcal{X}$ is closed in $\mathcal{X}$ and so $\mathcal{X}_+$ is a complete metric space too. Clearly, from (1) we have

$$u'(t) = \begin{cases} g(t, u(t), u'(t)) - g(t-\tau, u(t-\tau), u'(t-\tau)), & t \in [0, t_1], \\ \phi'(t), & t \in [-\tau, 0]. \end{cases}$$

This problem is largely investigated by Bica-Muresan [2], where some existence and uniqueness results of solution for problem (1) are obtained by using classical tools of fixed point theory (see Banach [1] and Perov [6]). In this paper, by using the same approach in Bica-Muresan [2] and a concept of admissibility for mappings (based on an idea of Samet-Vetro-Vetro [9]), we obtain the existence and uniqueness of a bounded solution of problem (1). First we prove an abstract result which is a generalization of Perov's fixed point theorem [6], then we work with a suitable integral operator associated to a large class of nonlinear initial value problems.

2. Mathematical Background and Preliminaries

We fix notation as follows. Let X be a non-empty set. By $\mathbb{R}_+$ we denote the set of all non-negative numbers and by $\mathbb{R}_m$ the set of all $m \times 1$ real matrices. Let $\alpha, \beta \in \mathbb{R}_m$, that is $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)^T$, then by $\alpha \leq \beta$ (resp., $\alpha < \beta$) we mean $\alpha_i \leq \beta_i$ (resp., $\alpha_i < \beta_i$) for each $i \in \{1, 2, \ldots, m\}$. Also, we denote the set of all $m \times m$ matrices with non-negative elements by $M_{m,m}(\mathbb{R}_+)$, the zero $m \times m$ matrix by $\bar{0}$ and the identity $m \times m$ matrix by I. Let $A \in M_{m,m}(\mathbb{R}_+)$, then A is said to be convergent to zero if and only if $A^n \to \bar{0}$ as $n \to \infty$ (see Varga [11]). Also note that $A^0 = I$. From Filip-Petruşel [4], we have:

Theorem 2.1. *Let $A \in M_{m,m}(\mathbb{R}_+)$. The following conditions are equivalent:*

(i) A is convergent to zero;

(ii) the eigenvalues of A are in the open unit disc, that is, $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$;

(iii) the matrix $I - A$ is nonsingular (that is, its determinant is nonzero) and $(I - A)^{-1} = I + A + \cdots + A^n + \cdots$.

Thus, it is easy to give some examples of matrices convergent to zero, from the literature (see Filip-Petruşel [4]). For example, we consider the following:

$$A := \begin{pmatrix} a & a \\ b & b \end{pmatrix}, \text{ where } a, b \in \mathbb{R}_+ \text{ and } a + b < 1;$$

$$B := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \text{ where } a, b, c \in \mathbb{R}_+ \text{ and } \max\{a, c\} < 1.$$

Now, we work in the setting of generalized metric spaces. Precisely, a mapping $d \colon X \times X \to \mathbb{R}_m$ is called a vector-valued metric on X if the following properties are satisfied:

(d_1) $d(x, y) \geq 0$ for all $x, y \in X$; if $d(x, y) = 0$ then $x = y$, and viceversa;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Thus, a non-empty set X equipped with a vector-valued metric d is called a generalized metric space, say (X, d). Notice that the convergence and Cauchyness of a sequence in generalized metric spaces are defined in a similar manner as in usual metric spaces. So Perov [6] proved the following interesting generalization of Banach contraction principle in [1].

Theorem 2.2. *Let (X, d) be a complete generalized metric space and $f: X \to X$ be a mapping for which there exists a matrix $A \in M_{m,m}(R_+)$ such that $d(fx, fy) \leq Ad(x, y)$ for all $x, y \in X$. If A is a matrix convergent to zero, then*

(i) $Fix(f) = \{x^*\}$, where $Fix(f) = \{x \in X : x = fx\}$;

(ii) the sequence of successive approximations $\{x_n\}$ such that $x_n = f^n x_0$ is convergent and admits the limit x^*, for all $x_0 \in X$.

Some interesting contributions to the development of fixed point theory and its applications in this context are obtained by Bica-Muresan [2], Bucur-Guran-Petruşel [3], Filip-Petruşel [4], O'Regan-Shahzad-Agarwal [5], Rus [8], Turinici [10].

3. Fixed Point Theorem

In this section we prove a fixed point theorem useful to obtain the existence and uniqueness of solution of problem (1). The crucial key to establish our generalization of Theorem 2.2 (Perov [6]) is the following notion of admissibility (inspired by Samet-Vetro-Vetro [9]).

Definition 3.1. *Let X be a non-empty set, $\Lambda : X \times X \to M_{m,m}(\mathbb{R}_+)$ and $f : X \to X$ be a mapping. The function f is said to be Λ-admissible if*

$$x, y \in X, \quad \Lambda(x, y) \geq I \implies \Lambda(fx, fy) \geq I,$$

where I is the $m \times m$ identity matrix and the inequality between matrices means entrywise inequality.

Let $\Lambda, A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ such that $(I - A_3 - A_4)^{-1}$ exists. Let $f : X \to X$. The hypotheses are the following:

(i) the matrix $A = (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)$ converges to zero;

(ii) there exists $x_0 \in X$ such that $\Lambda(x_0, fx_0) \geq I$;

(iii) f is Λ-admissible;

(iv) a. for each sequence $\{x_n\} \subseteq X$ such that $\lim_{n\to\infty} x_n = x$ and $\Lambda(x_n, x_{n+1}) \geq I$ for all $n \in \mathbb{N}$, we have $\Lambda(x_n, x) \geq I$ for all $n \in \mathbb{N}$;
or
b. f is continuous.

Now we can have the first theorem producing existence and uniqueness of fixed point for a given mapping f.

Theorem 3.2. *Let (X, d) be a complete generalized metric space and $f: X \to X$ be a mapping such that, for all $x, y \in X$, we have*

$$\Lambda(x, y)d(fx, fy) \leq A_1 d(x, y) + A_2 d(x, fx) + A_3 d(y, fy) + A_4 d(x, fy) + Bd(y, fx) \tag{2}$$

with $\Lambda, A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ satisfying hypotheses (i)-(iv). Then f has a fixed point. Moreover, if for all $x^*, x_* \in Fix(f)$ we have $\Lambda(x^*, x_*) \geq I$ and $A_1 + A_4 + B$ converges to zero then the fixed point is unique.

Proof. Because of hypothesis (ii), we see that there exists $x_0 \in X$ such that $\Lambda(x_0, fx_0) \geq I$. By putting $x_1 = fx_0$ and $x_2 = fx_1$, from (2), we have

$$\begin{aligned} d(x_1, x_2) &= d(fx_0, fx_1) = Id(fx_0, fx_1) \leq \Lambda(x_0, x_1)d(fx_0, fx_1) \\ &\leq A_1 d(x_0, x_1) + A_2 d(x_0, fx_0) + A_3 d(x_1, fx_1) + A_4 d(x_0, fx_1) + Bd(x_1, fx_0) \\ &= A_1 d(x_0, x_1) + A_2 d(x_0, x_1) + A_3 d(x_1, x_2) + A_4 d(x_0, x_2) + Bd(x_1, x_1) \\ &\leq A_1 d(x_0, x_1) + A_2 d(x_0, x_1) + A_3 d(x_1, x_2) + A_4[d(x_0, x_1) + d(x_1, x_2)] + B0. \end{aligned}$$

After routine calculations, we get

$$d(x_1, x_2) \leq (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)d(x_0, x_1) = Ad(x_0, x_1). \tag{3}$$

By putting $x_3 = fx_2$, hypothesis (iii) and (2) imply that

$$\begin{aligned} d(x_2, x_3) &= d(fx_1, fx_2) = Id(fx_1, fx_2) \leq \Lambda(x_1, x_2)d(fx_1, fx_2) \\ &\leq A_1 d(x_1, x_2) + A_2 d(x_1, fx_1) + A_3 d(x_2, fx_2) + A_4 d(x_1, fx_2) + Bd(x_2, fx_1) \\ &= A_1 d(x_1, x_2) + A_2 d(x_1, x_2) + A_3 d(x_2, x_3) + A_4 d(x_1, x_3) + Bd(x_2, x_2) \\ &\leq A_1 d(x_1, x_2) + A_2 d(x_1, x_2) + A_3 d(x_2, x_3) + A_4[d(x_1, x_2) + d(x_2, x_3)] + B0. \end{aligned}$$

This yields

$$d(x_2, x_3) \leq (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4)d(x_1, x_2) = Ad(x_1, x_2). \tag{4}$$

Combining (3) and (4), we deduce that

$$d(x_2, x_3) \leq A^2 d(x_0, x_1).$$

Iterating this process, we construct a sequence $\{x_n\} \subseteq X$ such that $x_n = fx_{n-1}$, $\Lambda(x_{n-1}, x_n) \geq I$ and

$$d(x_n, x_{n+1}) \leq A^n d(x_0, x_1), \quad \text{for all } n \in \mathbb{N}.$$

Next we show that $\{x_n\}$ is a Cauchy sequence. Let n, m be arbitrary natural numbers. By using the triangular inequality (d_3), for all $n, m \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+m}) &\leq \sum_{i=n}^{n+m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{n+m-1} A^i d(x_0, x_1) \leq A^n \left(\sum_{i=0}^{\infty} A^i \right) d(x_0, x_1) \\ &= A^n (I - A)^{-1} d(x_0, x_1) \quad \text{(by condition (iii) of Theorem 2.1).} \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0, \qquad \text{(see hypothesis (i)),}$$

$$\Rightarrow \quad \{x_n\} \text{ is a Cauchy sequence.}$$

From completeness of (X, d), we deduce that there exists $x^* \in X$ such that $x_n \to x^*$. Next, we distinguish two cases.

Case 1: If hypothesis (iv.a) holds then we have $\Lambda(x_n, x^*) \geq I$ for all $n \in \mathbb{N}$. Thus, from (2), we get

$$\begin{aligned} d(fx_n, fx^*) &= Id(fx_n, fx^*) \leq \Lambda(x_n, x^*)d(fx_n, fx^*) \\ &\leq A_1 d(x_n, x^*) + A_2 d(x_n, fx_n) + A_3 d(x^*, fx^*) + A_4 d(x_n, fx^*) + Bd(x^*, fx_n) \\ &= A_1 d(x_n, x^*) + A_2 d(x_n, x_{n+1}) + A_3 d(x^*, fx^*) + A_4 d(x_n, fx^*) + Bd(x^*, x_{n+1}). \end{aligned}$$

By passing to the limit as $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned} & d(x^*, fx^*) \le (A_3 + A_4)d(x^*, fx^*), \\ \Rightarrow \quad & (I - (A_3 + A_4))d(x^*, fx^*) \le 0. \end{aligned}$$

Since the matrix $I - (A_3 + A_4)$ is nonsingular, we deduce that $d(x^*, fx^*) = 0$, and hence $x^* = fx^*$.
Case 2: If hypothesis (iv.b) holds then, for $n \to \infty$, we have $fx_n \to fx^*$, that is $x_{n+1} \to fx^*$and so $fx^* = x^*$.

This concludes the existence part. The uniqueness part is obvious and is obtained by contradiction. Precisely, assume that there exist $x^*, x_* \in Fix(f)$ with $x^* \neq x_*$. Clearly, we have $\Lambda(x^*, x_*) \ge I$ and so (by (2))

$$\begin{aligned} & d(fx^*, fx_*) = Id(fx^*, fx_*) \le \Lambda(x^*, x_*)d(fx^*, fx_*) \\ & \le A_1 d(x^*, x_*) + A_2 d(x^*, fx^*) + A_3 d(x_*, fx_*) + A_4 d(x^*, fx_*) + Bd(x_*, fx^*) \\ & = A_1 d(x^*, x_*) + A_2 d(x^*, x^*) + A_3 d(x_*, x_*) + A_4 d(x^*, x_*) + Bd(x_*, x^*) \\ & = (A_1 + A_4 + B)d(x^*, x_*). \end{aligned}$$

Consequently, by iterating this process, we obtain

$$\begin{aligned} & d(x^*, x_*) \le (A_1 + A_4 + B)^n d(x^*, x_*), \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow \quad & d(x^*, x_*) = 0 \quad (\text{letting } n \to \infty), \\ \Rightarrow \quad & x^* = x_*, \quad \text{a contradiction.} \end{aligned}$$

Thus, the fixed point of f is unique. □

Example 3.3. *The following mappings and matrices satisfy the hypotheses of Theorem 3.2. Let $X = \mathbb{R}^2$ be endowed with the generalized metric d defined by*

$$d(x, y) = (|x_1 - y_1|, |x_2 - y_2|)^T, \quad \textit{for all } x = (x_1, x_2), y = (y_1, y_2) \in X.$$

Let $f \colon X \to X$ be given by

$$fx = \begin{cases} \left(\frac{2x_1}{3} - \frac{x_2}{3} + 1, \frac{x_2}{3} + 1\right), & \textit{for all } x = (x_1, x_2) \in X \textit{ with } x_1 \le 3, \\ \left(x_1 - \frac{x_2}{2} + 1, \frac{x_2}{2} + 1\right), & \textit{for all } x = (x_1, x_2) \in X \textit{ with } x_1 > 3. \end{cases}$$

For the sake of simplicity, we put $fx = f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$, where

$$f_1(x_1, x_2) = \begin{cases} \frac{2x_1}{3} - \frac{x_2}{3} + 1, & \textit{if } x_1 \le 3, \\ x_1 - \frac{x_2}{2} + 1, & \textit{if } x_1 > 3, \end{cases}$$

and

$$f_2(x_1, x_2) = \begin{cases} \frac{x_2}{3} + 1 & \textit{if } x_1 \le 3, \\ \frac{x_2}{2} + 1 & \textit{if } x_1 > 3. \end{cases}$$

Consider $\Lambda : X \times X \to M_{2,2}(\mathbb{R}_+)$ defined by

$$\Lambda(x, y) = \Lambda((x_1, x_2), (y_1, y_2)) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \textit{if } 0 \le x_1, x_2, y_1, y_2 \le 3, \\ \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}, & \textit{if } x_1, x_2, y_1, y_2 > 3, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \textit{otherwise.} \end{cases}$$

We show only that the condition (2) *holds for all* $x, y \in X$, *by distinguishing some cases; we leave to the reader to check the remaining hypotheses of Theorem 3.2. Let* $A_1 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$.

Case 1: If $0 \leq x_1, x_2, y_1, y_2 \leq 3$, *then we have*

$$\Lambda(x,y)d(fx,fy) = \begin{pmatrix} |f_1(x_1,x_2) - f_1(y_1,y_2)| \\ |f_2(x_1,x_2) - f_2(y_1,y_2)| \end{pmatrix} \leq \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} |x_1 - y_1| \\ |x_2 - y_2| \end{pmatrix} = A_1 d(x,y).$$

Case 2: If $x_1, x_2, y_1, y_2 > 3$, *then we have*

$$\Lambda(x,y)d(fx,fy) = \begin{pmatrix} \frac{2}{3}|f_1(x_1,x_2) - f_1(y_1,y_2)| \\ \frac{2}{3}|f_2(x_1,x_2) - f_2(y_1,y_2)| \end{pmatrix} \leq \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} |x_1 - y_1| \\ |x_2 - y_2| \end{pmatrix} = A_1 d(x,y).$$

Case 3: For other choices of x_1, x_2, y_1 *and* y_2, *we have*

$$\Lambda(x,y)d(fx,fy) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} |x_1 - y_1| \\ |x_2 - y_2| \end{pmatrix} = A_1 d(x,y).$$

Thus (2) *holds for all* $x, y \in X$ *with* $A_1 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$ *and* $A_2 = A_3 = A_4 = B = \overline{0}$. *Of course, we have* $A = (I - A_3 - A_4)^{-1}(A_1 + A_2 + A_4) = A_1$, *which is convergent to zero.*

Using two generalized metrics, one can have the following variant of Theorem 3.2.

Theorem 3.4. *Let* (X, d) *be a complete generalized metric space,* ρ *a second generalized metric and* $f : (X, \rho) \to (X, \rho)$ *be a mapping such that, for all* $x, y \in X$, *we have*

$$\Lambda(x,y)\rho(fx,fy) \leq A_1\rho(x,y) + A_2\rho(x,fx) + A_3\rho(y,fy) + A_4\rho(x,fy) + B\rho(y,fx) \tag{5}$$

with $\Lambda, A_1, A_2, A_3, A_4, B \in M_{m,m}(\mathbb{R}_+)$ *satisfying hypotheses (i)-(iii). Further, assume that*

(v) *there exists* $C \in M_{m,m}(\mathbb{R}_+)$ *such that* $d(fx,fy) \leq C\rho(x,y)$, *whenever there exists a sequence* $\{x_i\}_{i=0}^n$ *with* $\Lambda(x_i, x_{i+1}) \geq I$, *where* $x_0 = x$ *and* $x_n = y$;

(vi) $f : (X,d) \to (X,d)$ *is* Λ*-continuous, that is, if* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} d(x_n, x) = 0$ *and* $\Lambda(x_n, x_{n+1}) \geq I$ *for all* $n \in \mathbb{N}$, *then we have* $\lim_{n\to\infty} d(fx_n, fx) = 0$.

Then f *has a fixed point. Moreover, if for all* $x, y \in Fix(f)$ *we have* $\Lambda(x,y) \geq I$ *and* $A_1 + A_4 + B$ *converges to zero then the fixed point is unique.*

Remark 3.5. *The proof of Theorem 3.4 essentially follows step by step the proof of Theorem 3.2, by replacing the generalized metric* d *with* ρ. *The difference between the two proofs is relative to the fact that here we have to establish the Cauchyness of the sequence* $\{x_n\}$ *in respect both of* (X, ρ) *and* (X, d); *because only* (X, d) *is complete by hypothesis. To this aim, by construction of* $\{x_n\}$, *for all* $n, m \in \mathbb{N}$, *we have* $\Lambda(x_i, x_{i+1}) \geq I$ *for each* $i \in \{n, n+1, \cdots, n+m-1\}$. *So, by using hypothesis (v), we get*

$$d(x_{n+1}, x_{n+m+1}) = d(fx_n, fx_{n+m}) \leq C\rho(x_n, x_{n+m}) \leq C[A^n(I - A)^{-1}\rho(x_0, x_1)]$$

(by triangular inequality for ρ *and (iii) of Theorem 2.1). So, passing to the limit as* $n \to \infty$, *we deduce easily that* $\{x_n\}$ *is Cauchy in* (X, d). *Finally, by using hypothesis (vi), we deduce that* f *has a fixed point.*

Example 3.6. *The following mappings and matrices satisfy the hypotheses of Theorem 3.4. Let* $X = \mathbb{R}_+ \setminus \{0\}$ *be endowed with the generalized metrics* ρ *and* d *defined by*

$$\rho(x,y) = (|x-y|, |x-y|)^T, \quad \text{for all } x, y \in X,$$

and

$$d(x,y)=\begin{cases}(|x-y|+1,|x-y|+1)^T, & \text{if } x\in(0,1) \text{ or } y\in(0,1) \text{ or } x,y\in(0,1) \text{ with } x\neq y,\\ (0,0)^T, & \text{if } x=y\in(0,1),\\ (|x-y|,|x-y|)^T, & \text{otherwise.}\end{cases}$$

Let $f\colon X\to X$ *be given by*

$$fx=\begin{cases}x^3, & \text{if } x\in(0,1),\\ \frac{x+20}{5}, & \text{otherwise.}\end{cases}$$

Consider $\Lambda : X\times X\to M_{2,2}(\mathbb{R}_+)$ *defined by*

$$\Lambda(x,y)=\begin{cases}\begin{pmatrix}1&0\\0&1\end{pmatrix}, & \text{if } x,y\geq 1,\\ \begin{pmatrix}0&0\\0&0\end{pmatrix}, & \text{otherwise.}\end{cases}$$

In particular, the condition (5) *holds true with* $A_1=\begin{pmatrix}\frac{1}{5}&0\\0&\frac{1}{5}\end{pmatrix}$ *and* $A_2=A_3=A_4=B=\overline{0}$.

4. Solution of an Initial Value Problem with Delay

In this section we prove a theorem producing the existence of a unique bounded solution of problem (1). We follow the presentation in Bica-Muresan [2] and in Samet-Vetro-Vetro [9]. First, we consider a more general integral operator than the one in Bica-Muresan [2].

Let $f:\mathcal{X}_+\to\mathcal{X}_+$ be the integral operator defined for all $(u,v)\in\mathcal{X}_+$ by

$$f(u,v)(t)=\begin{cases}\left(\int_{t-\tau}^{t} g(s,u(s),v(s))ds,\ h(t,u(t),v(t))\right), & t\in[0,t_1],\\ (\phi(t),\phi'(t)), & t\in[-\tau,0).\end{cases}\tag{6}$$

The hypotheses are the following:

H$_1$: $\zeta:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ is a function such that

(i) there exists $(u_0,v_0)\in\mathcal{X}_+$ such that $\zeta((u_0(t),v_0(t)),f(u_0,v_0)(t))\geq 0$ for all $t\in[-\tau,t_1]$;

(ii) for all $t\in[-\tau,t_1]$, $(u_1,v_1),(u_2,v_2)\in\mathcal{X}_+$, we have

$$\zeta((u_1(t),v_1(t)),(u_2(t),v_2(t)))\geq 0 \implies \zeta(f(u_1,v_1)(t),f(u_2,v_2)(t))\geq 0;$$

(iii) for each sequence $\{(u_n,v_n)\}\subseteq\mathcal{X}_+$ such that $(u_n,v_n)\to(u,v)$ as $n\to\infty$ and $\zeta((u_n,v_n),(u_{n+1},v_{n+1}))\geq 0$ for all $n\in\mathbb{N}$, we have $\zeta((u_n,v_n),(u,v))\geq 0$ for all $n\in\mathbb{N}$.

H$_2$: $g,h\in C([-\tau,t_1]\times\mathbb{R}_+\times\mathbb{R})$ are functions such that

(i) there exist $\underline{g},\overline{g},\underline{\phi},\overline{\phi}\in\mathbb{R}_+$ such that

$$\begin{cases}\underline{g}\leq g(t,u,v)\leq\overline{g}, & t\in[-\tau,t_1],\quad u\in\mathbb{R}_+,\quad v\in\mathbb{R},\\ \underline{\phi}\leq\phi(t)\leq\overline{\phi}, & t\in[-\tau,0];\end{cases}$$

(ii) there exist $a_1, a_2, \rho, \vartheta > 0$ with $a_1\vartheta^{-1}+\rho a_2 < 1$ such that, for all $u, u' \in \mathbb{R}_+, v, v' \in \mathbb{R}$ with $\zeta((u,v),(u',v')) \geq 0$, we have

$$|g(t,u,v) - g(t,u',v')| \leq a_1|u-u'| + a_2|v-v'|, \quad \text{for all } t \in [-\tau, t_1],$$

and

$$|h(t,u,v) - h(t,u',v')| \leq \rho(a_1|u-u'| + a_2|v-v'|), \quad \text{for all } t \in [-\tau, t_1].$$

(iii) we have

$$\begin{cases} \phi(0) = \int_{-\tau}^{0} g(s,\phi(s),\phi'(s))ds, \\ \phi'(0) = g(0,\phi(0),\phi'(0)) - g(-\tau,\phi(-\tau),\phi'(-\tau)) = h(0,\phi(0),\phi'(0)). \end{cases}$$

Now, we can have the theorem producing a unique fixed point of f.

Theorem 4.1. *If hypotheses H_1 and H_2 hold, then the integral operator* (6) *has a unique fixed point in $\mathcal{X}_+$.*

Proof. Note that hypothesis H_2(i) implies that $f(\mathcal{X}_+) \subseteq \mathcal{X}_+$ and so f is well-defined. Since $f(u_1,v_1)(t) = f(u_2,v_2)(t)$ for all $t \in [-\tau, 0]$, then we have

$$d_B(f(u_1,v_1), f(u_2,v_2)) = \Bigg(\max_{t\in[0,t_1]} \left| \int_{t-\tau}^{t} g(s,u_1(s),v_1(s))ds - \int_{t-\tau}^{t} g(s,u_2(s),v_2(s))ds \right| e^{-\vartheta(t+\tau)}, \\ \max_{t\in[0,t_1]} |h(t,u_1(t),v_1(t)) - h(t,u_2(t),v_2(t))| e^{-\vartheta(t+\tau)} \Bigg)^T,$$

for all $(u_1,v_1),(u_2,v_2) \in \mathcal{X}_+$. If $\zeta((u_1(t),v_1(t)),(u_2(t),v_2(t))) \geq 0$ for all $t \in [-\tau, t_1]$, then we have

$$\begin{aligned} &\left| \int_{t-\tau}^{t} g(s,u_1(s),v_1(s))ds - \int_{t-\tau}^{t} g(s,u_2(s),v_2(s))ds \right| \\ &\leq \int_{t-\tau}^{t} \left| g(s,u_1(s),v_1(s)) - g(s,u_2(s),v_2(s)) \right| ds \\ &\leq \int_{t-\tau}^{t} (a_1|u_1(s)-u_2(s)| + a_2|v_1(s)-v_2(s)|)ds \\ &= \int_{t-\tau}^{t} \left(a_1|u_1(s)-u_2(s)|e^{-\vartheta(s+\tau)} + a_2|v_1(s)-v_2(s)|e^{-\vartheta(s+\tau)} \right) e^{\vartheta(s+\tau)} ds \\ &\leq (a_1\|u_1-u_2\|_B + a_2\|v_1-v_2\|_B) \int_{t-\tau}^{t} e^{\vartheta(s+\tau)} ds \\ &= \left(\frac{a_1}{\vartheta}\|u_1-u_2\|_B + \frac{a_2}{\vartheta}\|v_1-v_2\|_B \right) \left(e^{\vartheta(t+\tau)} - e^{\vartheta t} \right). \end{aligned}$$

It follows that

$$\begin{aligned} &\left| \int_{t-\tau}^{t} g(s,u_1(s),v_1(s))ds - \int_{t-\tau}^{t} g(s,u_2(s),v_2(s))ds \right| e^{-\vartheta(t+\tau)} \\ &\leq \left(\frac{a_1}{\vartheta}\|u_1-u_2\|_B + \frac{a_2}{\vartheta}\|v_1-v_2\|_B \right) \left(1 - e^{-\vartheta\tau} \right) \\ &\leq \frac{a_1}{\vartheta}\|u_1-u_2\|_B + \frac{a_2}{\vartheta}\|v_1-v_2\|_B, \quad t \in [0, t_1]. \end{aligned}$$

Therefore

$$\begin{aligned}&\max_{t\in[0,t_1]}\left|\int_{t-\tau}^{t} g(s,u_1(s),v_1(s))ds-\int_{t-\tau}^{t} g(s,u_2(s),v_2(s))ds\right|e^{-\vartheta(t+\tau)}\\&\leq \frac{a_1}{\vartheta}\|u_1-u_2\|_B+\frac{a_2}{\vartheta}\|v_1-v_2\|_B,\end{aligned}$$

for all $(u_1,v_1),(u_2,v_2)\in\mathcal{X}_+$ such that $\zeta((u_1(t),v_1(t)),(u_2(t),v_2(t)))\geq 0$ for all $t\in[-\tau,t_1]$. Similarly, we get

$$\begin{aligned}&|h(t,u_1(t),v_1(t))-h(t,u_2(t),v_2(t))|\\&\leq \rho(a_1|u_1(t)-u_2(t)|+a_2|v_1(t)-v_2(t)|)\\&=\rho\left(a_1|u_1(t)-u_2(t)|e^{-\vartheta(t+\tau)}+a_2|v_1(t)-v_2(t)|e^{-\vartheta(t+\tau)}\right)e^{\vartheta(t+\tau)}\\&\leq \rho\left(a_1\|u_1-u_2\|_B+a_2\|v_1-v_2\|_B\right)e^{\vartheta(t+\tau)},\end{aligned}$$

for all $(u_1,v_1),(u_2,v_2)\in\mathcal{X}_+$ such that $\zeta((u_1(t),v_1(t)),(u_2(t),v_2(t)))\geq 0$ for all $t\in[-\tau,t_1]$. Consequently, we obtain

$$\begin{aligned}&\max_{t\in[0,t_1]}|h(t,u_1(t),v_1(t))-h(t,u_2(t),v_2(t))|e^{-\vartheta(t+\tau)}\\&\leq \rho\left(a_1\|u_1-u_2\|_B+a_2\|v_1-v_2\|_B\right),\end{aligned}$$

for all $(u_1,v_1),(u_2,v_2)\in\mathcal{X}_+$ such that $\zeta((u_1(t),v_1(t)),(u_2(t),v_2(t)))\geq 0$ for all $t\in[-\tau,t_1]$.
Then, for all $(u_1,v_1),(u_2,v_2)\in\mathcal{X}_+$ such that $\zeta((u_1(t),v_1(t)),(u_2(t),v_2(t)))\geq 0$ for all $t\in[-\tau,t_1]$, we have

$$d_B(f(u_1,v_1),f(u_2,v_2))\leq Ad_B((u_1,v_1),(u_2,v_2)),$$

where

$$A=\begin{pmatrix} a_1\vartheta^{-1} & a_2\vartheta^{-1}\\ \rho a_1 & \rho a_2\end{pmatrix}. \tag{7}$$

Consider $\Lambda: X\times X\to M_{2,2}(\mathbb{R}_+)$ defined by

$$\Lambda((u_1,v_1),(u_2,v_2))=\begin{cases}\begin{pmatrix}1&0\\0&1\end{pmatrix}, & \text{if } \zeta((u_1(t),v_1(t)),(u_2(t),v_2(t)))\geq 0 \text{ for all } t\in[-\tau,t_1],\\ \begin{pmatrix}0&0\\0&0\end{pmatrix}, & \text{otherwise.}\end{cases}$$

Finally, for all $(u_1,v_1),(u_2,v_2)\in\mathcal{X}_+$, we have

$$\Lambda((u_1,v_1),(u_2,v_2))d_B(f(u_1,v_1),f(u_2,v_2))\leq Ad_B((u_1,v_1),(u_2,v_2)).$$

Next, the eigenvalues of A are

$$\lambda_1=0 \quad\text{and}\quad \lambda_2=a_1\vartheta^{-1}+\rho a_2. \tag{8}$$

Therefore, from (8) and hypothesis H_2(ii) (i.e., $a_1\vartheta^{-1}+\rho a_2<1$) we infer that λ_1,λ_2 are in the open unit disc and so A is convergent to zero (see conditions (i) and (ii) of Theorem 2.1). Also, the above calculations and the relation between the matrix Λ and function ζ (by using hypotheses H_1) show that the hypotheses of Theorem 3.2 hold with $A_1=A$ given by (7) and $A_2=A_3=A_4=B=\overline{0}$. For instance, by H_1(ii), we have

$$\begin{aligned}&\Lambda((u_1,v_1),(u_2,v_2))\geq I\\ \Rightarrow\ &\zeta((u_1(t),v_1(t)),(u_2(t),v_2(t)))\geq 0\\ \Rightarrow\ &\zeta(f(u_1,v_1)(t),f(u_2,v_2)(t))\geq 0\\ \Rightarrow\ &\Lambda(f(u_1,v_1)(t),f(u_2,v_2)(t))\geq I,\end{aligned}$$

so f is Λ-admissible. Thus, the existence and uniqueness of a fixed point of f in $\mathcal{X}_+$ is an immediate consequence of Theorem 3.2. □

By particularizing the choice of $h \in C([-\tau, t_1] \times \mathbb{R}_+ \times \mathbb{R})$, we can have the theorem producing a unique bounded solution of problem (1). This theorem is more general than the analogous of Bica-Muresan ([2], Theorem 5). Let $h(t, u(t), v(t)) = g(t, u(t), v(t)) - g(t-\tau, u(t-\tau), v(t-\tau))$ for all $t \in [0, t_1]$ and consider the integral operator

$$\widetilde{f}(u,v)(t) = \begin{cases} \left(\int_{t-\tau}^{t} g(s, u(s), v(s))ds,\ g(t, u(t), v(t)) - g(t-\tau, u(t-\tau), v(t-\tau)) \right), & t \in [0, t_1], \\ (\phi(t), \phi'(t)), & t \in [-\tau, 0]. \end{cases}$$

Theorem 4.2. *If hypotheses H_1 and H_2 hold, then problem* (1) *has a unique bounded solution in $\mathcal{X}_+$.*

Proof. The similar reasoning as in the proof of Theorem 4.1 shows that $\widetilde{f}$ has a unique fixed point in $\mathcal{X}_+$, say $\{(u_1^*, v_1^*)\} = Fix(\widetilde{f})$. To avoid repetition, we leave the details and point out just the difference. Precisely, here we obtain the matrix

$$A = \begin{pmatrix} a_1\vartheta^{-1} & a_2\vartheta^{-1} \\ a_1(1+e^{-\vartheta\tau}) & a_2(1+e^{-\vartheta\tau}) \end{pmatrix}$$

with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = a_1\vartheta^{-1} + a_2(1+e^{-\vartheta\tau})$, that is, we have $\rho = (1+e^{-\vartheta\tau})$.

Next, we show that u_1^* is a unique bounded solution of (1). In fact, from hypothesis H_2(i), we get

$$\begin{cases} \underline{g}\tau \le u_1^*(t) = \int_{t-\tau}^{t} g(s, u_1^*(s), v_1^*(s))ds \le \overline{g}\tau, & t \in [0, t_1], \\ \underline{\phi} \le u_1^*(t) \le \overline{\phi}, & t \in [-\tau, 0], \end{cases}$$

and hence the boundedness is proved. It remain to prove that

$$(u_1^*)'(t) = v_1^*(t), \quad t \in [-\tau, t_1],$$

(see Bica-Muresan [2], p. 25). We distinguish the following two cases:
Case 1: If $t \in [0, t_1]$ then, from (6), we have

$$(u_1^*(t), v_1^*(t)) = \widetilde{f}(u_1^*(t), v_1^*(t)), \quad t \in [0, t_1].$$

So

$$\begin{cases} u_1^*(t) = \int_{t-\tau}^{t} g(s, u_1^*(t)(s), v_1^*(s))ds, \\ v_1^*(t) = g(t, u_1^*(t), v_1^*(t)) - g(t-\tau, u_1^*(t-\tau), v_1^*(t-\tau)). \end{cases}$$

It follows easily that

$$(u_1^*)'(t) = g(t, u_1^*(t), v_1^*(t)) - g(t-\tau, u_1^*(t-\tau), v_1^*(t-\tau))$$

and so $(u_1^*)'(t) = v_1^*(t)$ for all $t \in [0, t_1]$.
Case 2: If $t \in [-\tau, 0]$, again from (6), we have

$$\widetilde{f}(u_1^*(t), v_1^*(t)) = (\phi(t), \phi'(t)) = (u_1^*(t), v_1^*(t))$$

and so $u_1^*(t) = \phi(t)$ and $v_1^*(t) = \phi'(t)$. □

Remark 4.3. *Every non-negative constant function ζ reduces Theorem 4.2 to Theorem 5 of Bica-Muresan [2], where (for the sake of exactness) the authors assume $a_2 \in (0, 2^{-1})$. On the other hand, other choices of function ζ are possible. So Theorem 4.1 covers a large class of situations than those of the original version in [2]. For example, by assuming*

$$\zeta((u_1(t), v_1(t)), (u_2(t), v_2(t))) = u_1(t) - u_2(t), \quad \text{for all } t \in [-\tau, t_1], \quad (u_1, v_1), (u_2, v_2) \in \mathcal{X}_+,$$

the ordered approach to the study of initial value problems with delay arises naturally. Technically, this means to consider the product space $\mathcal{X}$ endowed with the partial order $\preceq$ defined by

$$(u_1, v_1), (u_2, v_2) \in \mathcal{X}, \quad (u_1, v_1) \preceq (u_2, v_2) \iff u_1(t) \le u_2(t), \quad t \in [-\tau, t_1].$$

Thus, we have to check the contractive condition in Theorem 4.1 only for couples of points satisfying the partial order. Also, the hypothesis H_1(i) reduces to the existence of an upper solution for problem (1) *and so on.*

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922) 133–181.
[2] A. Bica, S. Muresan, Applications of the Perov's Fixed Point Theorem to Delay Integro-Differential Equations, in: Fixed Point Theory and Applications, Volume 7, Nova Science Publishers, New York, 2007.
[3] A. Bucur, L. Guran, A. Petruşel, Fixed points for multivalued operators on a set endowed with vector-valued metrics and applications, Fixed Point Theory 10 (2009) 19–34.
[4] A. D. Filip, A. Petruşel, Fixed point theorems on spaces endowed with vector-valued metrics, Fixed Point Theory Appl. 2010 (2010), Article ID 281381.
[5] D. O'Regan, N. Shahzad, R. P. Agarwal, Fixed Point Theory for Generalized Contractive Maps on Spaces with Vector-Valued Metrics, in: Fixed Point Theory and Applications, Volume 6, Nova Science Publishers, New York, 2006.
[6] A. I. Perov, On the Cauchy problem for a system of ordinary differential equations, Pviblizhen. Met. Reshen. Differ. Uvavn. 2 (1964) 115–134.
[7] R. Precup, Methods in Nonlinear Integral Equations, Springer Science+Business Media, Dordrecht, 2002.
[8] I. A. Rus, Principles and Applications of the Fixed Point Theory, Dacia, Cluj-Napoca, 1979.
[9] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal. 75 (2012) 2154–2165.
[10] M. Turinici, Finite-dimensional vector contractions and their fixed points, Stud. Univ. Babeş-Bolyai Math. 35 (1990) 30–42.
[11] R. S. Varga, Matrix Iterative Analysis, Springer Series in Computational Mathematics, Volume 27, Springer, Berlin, 2000.