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On the Equivalence between Perov Fixed Point Theorem and Banach Contraction Principle

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Abstract. There are many results in the fixed point theory that were presented as generalizations of Banach theorem and other well-known fixed point theorems, but later proved equivalent to these results. In this article we prove that Perov's existence result follows from Banach theorem by using renormization of normal cone and obtained metric. The observed estimations of approximate point given by Perov, could not be obtained from consequences of Banach theorem on metric spaces.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

Well-known Banach fixed point theorem, also known as Banach contraction principle, was a foundation for a development of metric fixed point theory and found applications in various areas. There were many generalizations of this result in the last years. We can observe two main directions in this area of research, including different contraction conditions or introducing analogous concept on different spaces such as partial, cone-metric, b-metric spaces, etc. Russian mathematician A. I. Perov [22] defined generalized cone metric space by defining a metric with values in $\mathbb{R}^n$. Then, this concept of metric space allowed him to define a new class of mappings, known as Perov contractions, which satisfy contractive condition similar to Banach's, but with a matrix $A \in \mathbb{R}^{n\times n}$ with nonnegative entries instead of constant q. This result found main application in the area of differential equations ([23, 26, 29]).

In [6] was presented extension of Perov theorem on a cone metric space, normal or solid. The concept of cone metric space (vector value metric space, K-metric space) has a long history (see [15, 27, 33]) and first fixed point theorems in cone metric spaces were obtained by Schröder [30, 31] in 1956. Cone metric space may be considered as a generalization of metric space and it is focus of the research in metric fixed point theory last few decades (see, e.g., [1, 2, 4], [9], [13], [16], [18], [28], [32] for more details). Concept of cone metric includes generalized metric in the sense of Perov, and contractive condition defined in [6] introduces a bounded linear operator instead of a matrix. Other requirements for this operator varies based on the ki

This paper is focused on a relation between Banach and Perov theorem along with its generalizations on a complete normal cone metric space. Some equivalents between metric and normal cone metric spaces are presented and used to obtain different proof approach for several Perov type results.

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2. Preliminaries

Some basic definitions and facts which are applied in subsequent sections are collected in this section. Since some correlations will be made, we give basic overview on generalized metric space in the sense of Perov, cone metric spaces and b-metric spaces.

Let X be a nonempty set and $n \in \mathbb{N}$.

Definition 2.1. *([22]) A mapping $d : X \times X \mapsto \mathbb{R}^n$ is called a vector-valued metric on X if the following statements are satisfied for all $x, y, z \in X$.*

(d_1) $d(x,y) \geq 0_n$ and $d(x,y) = 0_n \Leftrightarrow x = y$, where $0_n = (0,\dots,0) \in \mathbb{R}^n$;

(d_2) $d(x,y) = d(y,x)$;

(d_3) $d(x,y) \leq d(x,z) + d(z,y)$.

If $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in \mathbb{R}^n$, then $x \leq y$ means that $x_i \leq y_i$, $i = \overline{1,n}$.
Throughout this paper we denote by $\mathcal{M}_{n,n}$ the set of all $n \times n$ matrices, by $\mathcal{M}_{n,n}(\mathbb{R}^+)$ the set of all $n \times n$ matrices with nonnegative entries. We write Θ_n for the zero $n \times n$ matrix and I_n for the identity $n \times n$ matrix and further on we identify row and column vector in $\mathbb{R}^n$.
A matrix $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ is said to be convergent to zero if $A^m \to \Theta_n$, as $m \to \infty$.

Theorem 2.1. *(Perov [22, 23]) Let (X,d) be a complete generalized metric space, $f : X \mapsto X$ and $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ a matrix convergent to zero, such that*

$$d(f(x), f(y)) \leq A(d(x,y)), \quad x, y \in X.$$

Then:

(i) *f has a unique fixed point $x^* \in X$;*

(ii) *the sequence of successive approximations $x_n = f(x_{n-1}), n \in \mathbb{N}$, converges to x^* for any $x_0 \in X$;*

(iii) *$d(x_n, x^*) \leq A^n(I_n - A)^{-1}(d(x_0,x_1))$, $n \in \mathbb{N}$;*

(iv) *if $g : X \mapsto X$ satisfies the condition $d(f(x), g(x)) \leq c$ for all $x \in X$ and some $c \in \mathbb{R}^n$, then by considering the sequence $y_n = g^n(x_0), n \in \mathbb{N}$, one has*

$$d(y_n, x^*) \leq (I_n - A)^{-1}(c) + A^n(I_n - A)^{-1}(d(x_0,x_1)), \ n \in \mathbb{N}.$$

This result was extended on a setting of cone metric spaces.

Definition 2.2. *Let E be a real Banach space with a zero vector θ. A subset P of E is called a cone if:*

(i) *P is closed, nonempty and $P \neq \{\theta\}$;*

(ii) *$a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;*

(iii) *$P \cap (-P) = \{\theta\}$.*

Given a cone $P \subseteq E$, the partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ denotes $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ is the interior of P.

The cone P in a real Banach space E is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq K \|y\|. \tag{2.1}$$

The least positive number satisfying (2.1) is called the normal constant of P. The cone P is called solid if $\operatorname{int} P \neq \emptyset$.

Definition 2.3. *[15] Let X be a nonempty set, and let P be a cone on a real ordered Banach space E. Suppose that the mapping $d : X \times X \mapsto E$ satisfies:*

(d_1) $\theta \le d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d_3) $d(x, y) \le d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is a cone metric space.

It is known that the class of cone metric spaces is bigger than the class of metric spaces. A lot of fixed point results, such as Banach contraction principle, are proved in the frame of cone metric spaces ([1, 2, 4], [13],[18–20]).

Suppose that E is a Banach space, P is a solid cone in E, whenever it is not normal, and $\le$ is the partial order on E with respect to P.

Definition 2.4. *The sequence $\{x_n\} \subseteq X$ is convergent in X if there exists some $x \in X$ such that*

$$(\forall\, c \gg \theta)(\exists\, n_0 \in \mathbb{N})\, n \ge n_0 \implies d(x_n, x) \ll c.$$

We say that a sequence $\{x_n\} \subseteq X$ converges to $x \in X$ and denote that with $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$, $n \to \infty$. Point x is called a limit of the sequence $\{x_n\}$.

Definition 2.5. *The sequence $\{x_n\} \subseteq X$ is a Cauchy sequence if*

$$(\forall\, c \gg \theta)(\exists\, n_0 \in \mathbb{N})\, n, m \ge n_0 \implies d(x_n, x_m) \ll c.$$

Every convergent sequence is a Cauchy sequence, but reverse do not hold. If any Cauchy sequence in a cone metric space (X, d) is convergent, then X is a complete cone metric space.

As proved in [15], if P is a normal cone, even in the case $\operatorname{int} P = \emptyset$, then $\{x_n\} \subseteq X$ converges to $x \in X$ if and only if $d(x_n, x) \to \theta$, $n \to \infty$. Similarly, $\{x_n\} \subseteq X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$, $n, m \to \infty$. Also, if $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$, then $d(x_n, y_n) \to d(x, y)$, $n \to \infty$. Let us emphasise that this equivalences do not hold if P is a non-normal cone.

Perov generalized metric space is obviously a kind of a normal cone metric space. Defined partial ordering determines a normal cone $P = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \ge 0, i = \overline{1, n}\}$ on $\mathbb{R}^n$, with the normal constant $K = 1$. Evidently, $A(P) \subseteq P$ if and only if $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$.

One of the results in [6] is a new generalization of Banach contraction principle in the sense of Perov.

Theorem 2.2. *Let (X, d) be a complete cone metric space with a solid cone P, $d : X \times X \mapsto E$, $f : X \mapsto X$, $A \in \mathcal{B}(E)$, with $r(A) < 1$ and $A(P) \subseteq P$, such that*

$$d(f(x), f(y)) \le A\left(d(x, y)\right), \quad x, y \in X. \tag{2.2}$$

Then:

(i) f has a unique fixed point $z \in X$;

(ii) For any $x_0 \in X$ the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$, converges to z and

$$d(x_n, z) \le A^n(I - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N};$$

(iii) Suppose that $g : X \mapsto X$ satisfies the condition $d(f(x), g(x)) \le c$ for all $x \in X$ and some $c \in P$. Then if $y_n = g^n(x_0)$, $n \in \mathbb{N}$, we have

$$d(y_n, z) \le (I - A)^{-1}(c) + A^n(I - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}.$$

Furthermore, there was presented a similar result for normal cone metric space, but instead of the requirement of positiveness and $r(A) < 1$, only requirement is $K\|A\| < 1$ where K is a normal constant. Also, this normal cone is not necessarily solid.

Theorem 2.3. *Let (X, d) be a complete cone metric space, $d : X \times X \mapsto E$, P a normal cone with a normal constant K, $A \in \mathcal{B}(E)$ and $K\|A\| < 1$. If the condition (2.2) holds for a mapping $f : X \mapsto X$, then f has a unique fixed point $z \in X$ and the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$, converges to z for any $x_0 \in X$.*

In Sections 11.3-11.5 of the classical monograph of Collatz [5] is given a general fixed point theorem in cone metric spaces, and in Section 12.1 it is considered a special case of this theorem. We note that that the first two parts of previous theorem can be obtained as a special case of Theorem 12.1 of [5].

Observe that with $\mathcal{B}(E)$ is denoted the set of all bounded linear operators on E and with $r(A)$ a spectral radius of an operator $A \in \mathcal{B}(E)$,

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|A^n\|^{1/n}.$$

If $r(A) < 1$, then the series $\sum\limits_{n=0}^{\infty} A^n$ is absolutely convergent, $I - A$ is invertible in $\mathcal{B}(E)$ and

$$\sum_{n=0}^{\infty} A^n = (I - A)^{-1}.$$

Also, if $\|A\| < 1$, then $I - A$ is invertible and

$$\|(I - A)^{-1}\| \le \frac{1}{1 - \|A\|}.$$

If X is a Banach space with a cone P and operator $A : E \mapsto E$, then:

(i) A is a positive operator if $A(P) \subseteq P$;

(ii) A is an increasing operator if $x \le y \implies A(x) \le A(y)$, for any $x, y \in X$.

If $A \in \mathcal{B}(E)$, then (i) and (ii) are equivalent ([6]).
Omitting the boundedness condition, we obtain the following result:

Theorem 2.4. *Let (X, d) be complete cone metric space with a solid cone P and $f : X \mapsto X$ a continuous mapping. If there exists an increasing operator $A : E \mapsto E$ such that $\lim\limits_{n\to\infty} A^n(e) = \theta$, $e \in E$, and, for any $x, y \in X$,*

$$d(f(x), f(y)) \le A(d(x, y)), \tag{2.3}$$

then a mapping f has a unique fixed point in X.

Conditions of Theorem 2.4 could be less strict asking for A to be increasing and tend to zero only on P.

Recent results in cone metric fixed point theory established some relation between b-metric spaces and normal cone metric spaces.

Definition 2.6. *Let X be a nonempty set and $s \ge 1$ be a given real number. A mapping $d \colon X \times X \to [0, +\infty)$ is said to be a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:*

(b_1) $d(x, y) = 0$ *if and only if* $x = y$;

(b_2) $d(x, y) = d(y, x)$;

(b_3) $d(x, z) \le s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space (with constant s).

Definitions of Cauchy and convergent sequence in a b-metric space, as well as completeness, go analogously as in a metric space.

3. Main Results

There were several papers ([3, 12, 16]) studying relations between cone metric spaces in general, and especially normal cone metric spaces, on one, and metric spaces on the other side. Many efforts are made in the attempt of reduction any cone metric space to a metric space. In the case that (X, d) is a normal cone metric space with a normal constant K, we may introduce a b-metric as presented in several recent papers.

Let (X, d) be a cone metric space, P a normal cone with a normal constant K. Define a function $D : X \times X \mapsto \mathbb{R}$,

$$D(x, y) = \|d(x, y)\|, \quad x, y \in X \tag{3.4}$$

Theorem 3.1. *A function D defined in (3.4) is a b-metric on X with a constant K.*

Proof. Let $x, y, z \in X$ be arbitrary points. From the definition of norm and (d_1) it easily follows that (b_1) holds. D is also a symmetric function since it directly follows from the symmetry of the norm. From the fact that d is a metric on X, (d_3) and since (X, d) is a normal cone metric space, we have

$$D(x, y) = \|d(x, y)\| \leq K (\|d(x, z)\| + \|d(z, y)\|) = K (D(x, z) + D(z, y)).$$

Thus, (X, D) is a b-metric space. □

If the normal constant K is equal to 1, then (X, D) is a metric space.

However, if (X, d) is a complete normal cone metric space, $\{x_n\}$ is Cauchy sequence in (X, d) if and only if $\lim\limits_{n,m\to\infty} \|d(x_n, x_m)\| = 0$ and $\lim\limits_{n\to\infty} x_n = x$ if and only if $\lim\limits_{n\to\infty} \|d(x_n, x)\| = 0$. Therefore, we may state the following corollary.

Theorem 3.2. *(X, d) is a complete cone metric space, P a normal cone with a normal constant K and D an b-metric defined as in (3.4) if and only if (X, D) is a complete b-metric space.*

We will give another proof of the generalization of Perov fixed point theorem in the setting of normal cone metric space.

Theorem 3.3. *Let (X, d) be a complete cone metric space, P a normal cone with a normal constant K and $f : X \mapsto X$ a self-mapping. If there exists an operator $A \in \mathcal{B}(E)$ such that $K\|A\| < 1$, for all $x, y \in X$,*

$$d(f(x), f(y)) \leq A(d(x, y)), \tag{3.5}$$

then f has a unique fixed point in X.

Proof. From the condition (3.5) and the fact that P is a normal cone, it follows

$$D(fx, fy) = \|d(f(x), f(y))\| \leq K\|A(d(x, y))\| \leq K\|A\|D(x, y), \ x, y \in X,$$

and f is a contraction in b-metric space and the existence of an unique fixed point follows by the generalization of Banach fixed point theorem in b-metric space presented in [7]. □

Observe that we can obtain the same result from Banach fixed point theorem (on complete metric spaces) by renorming, as presented in [14].

Theorem 3.4. *Let (X, d) be a cone metric space, $P \subseteq E$ a normal cone with a normal constant K where $(E, \|\cdot\|)$ is a Banach space. Then:*

(i) A function $\|\cdot\|_1 : E \mapsto \mathbb{R}$ defined with

$$\|x\|_1 = \inf\{\|u\| \mid x \leq u\} + \inf\{\|v\| \mid v \leq x\}, \ x \in E,$$

is a norm on E.

(ii) Norms $\|\cdot\|$ *and* $\|\cdot\|_1$ *are equivalent norms on* X.

(iii) If we observe P *as a cone in Banach space* $(E, \|\cdot\|_1)$*, then* (X, d) *is a normal cone metric space with a normal constant equal to 1.*

The equivalence of the norms allows us to determine the relation between $\|A\|$ and $\|A\|_1$.

Remark 3.5. *Based on the previously made observations regarding renorminization of a normal cone with a normal constant K and Theorem 3.3, we may conclude that existence of the unique fixed point Perov type contractions (including extended and more general contractive conditions) on normal cone metric spaces could be derived from analogous results on metric spaces.*

Focusing on just first two statements of Perov theorem, we may state the following result:

Theorem 3.6. *Perov theorem is a consequence of a Banach fixed point theorem.*

Proof. Notice that generalized metric space introduced by Perov is a type of normal cone metric space.
If $P = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ i = \overline{1,n}\}$, then P evidently determines a cone in a Banach space $\mathbb{R}^n$ with supremum norm, $\|x\| = \max_{i=1^n} |x_i|$, and $x \leq y$ if and only if $x_i \leq y_i$, $i = \overline{1,n}$. Since $\theta \leq x \leq y$, for $\theta = (0, 0, \dots, 0)$, $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$, implies $0 \leq x_i \leq y_i$, $i = \overline{1,n}$, then $\|x\| = \max_{i=\overline{1,n}} |x_i| \leq \max_{i=\overline{1,n}} |y_i| = \|y\|$ and P is a normal cone with a normal constant $K = 1$.
By taking into the account results of Theorem 3.1, it follows that for any generalized metric space (X, d) in the sense of Perov, the appropriate b-metric space (X, D) is a metric space.
Assume that the requirements of Perov theorem are fulfilled for some $A \in \mathcal{M}_{m,m}(\mathbb{R}^+)$ such that $A^n \to \Theta_m$, as $n \to \infty$. Since a matrix A converges to the zero matrix, then $\|A^n\| \to 0$, $n \to \infty$. Choose $n_0 \in \mathbb{N}$ such that $\|A^n\| < 1$ for any $n \geq n_0$. For such n,

$$d(f^n x, f^n y) \leq A^n(d(x, y)), \quad x, y \in \mathbb{R}^m,$$

and

$$D(f^n x, f^n y) \leq \|A^n\| D(x, y), \quad x, y \in \mathbb{R}^m. \tag{3.6}$$

If we apply Banach contraction principle for f^n and $q = \|A^n\| < 1$, f^n has a unique fixed point z in X. Since $f^n(fz) = fz$, it must be $fz = z$. If $fu = u$ for some $u \in X$, then $f^n u = u$, so $u = z$.
Hence, Perov theorem is a direct consequence of Banach contraction principle.
It is easy to observe that the iterative sequence $\{x_n\}$ is a Cauchy sequence, thus convergent, and since $\{f^{nk(x)}\}_{k \in \mathbb{N}}$ converges to z by Banach fixed point theorem, (ii) holds. □

Remark 3.7. *On the other hand, if* $n = 1$*, then generalized metric space is a metric space and a positive matrix* $A = [q]$ *tends to zero if and only if* $q < 1$*. Thus, Banach contraction principle is a Perov fixed point theorem for* $n = 1$*. However, remarks regarding distance presented in* (iii) *and* (iv) *(easily observed if we take* $g = f$*) could not be derived directly from Banach contraction principle since the inequality (3.6) do not imply (iii).*

Example 3.8. *Define a mapping* $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ *with* $f(x) = (\frac{x_1}{2} + x_2, \frac{x_2}{2})$, $x = (x_1, x_2) \in \mathbb{R}^2$*. Let*

$$A = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix},$$

then $\lim_{n\to\infty} A^n = \Theta_2$ *and*

$$d(f(x), f(y)) \leq A(d(x, y)),\ x, y \in \mathbb{R}^2.$$

Since $\|A\| = 1$, $D(f(x), f(y)) \leq D(x, y)$ *and if* $x = (0, 0)$, $y = (0, 1)$*, it follows that* f *is not a contraction in* $(\mathbb{R}^2, D)$*, but it is a Perov contraction and based on Perov theorem it possesses a unique fixed point* $(0, 0)$*.*

From the proof of Theorem 3.6 and the previous example, we may notice correlation between Perov theorem and well-known consequence of Banach theorem.

corollary 3.1. *Let (X, d) be a complete metric space, $f : X \mapsto X$ a mapping. If*

$$d(f^n(x), f^n(y)) \leq qd(x, y), \quad x, y \in X,$$

for some $n \in \mathbb{N}$ and $q \in [0, 1)$, then f has a unique fixed point in X.

The following example shows that Perov type theorems including requirement $r(A) < 1$ could not be derived directly from Banach theorem.

Example 3.9. *Let c_0 be the set containing all sequences of real numbers convergent to zero equipped with supremum norm $\|\cdot\|_\infty$ and define $A : E \mapsto E$ with*

$$A(x) = A(x_1, x_2, x_3, \ldots, x_n, \ldots) = (0, x_3, \frac{x_4}{2}, \ldots, \frac{x_{n+1}}{2}, \ldots), \quad x = \{x_n\} \in c_0.$$

Operator A is linear on Banach space $(c_0, \|\cdot\|_\infty)$ and also bounded since $\|Ax\|_\infty \leq \|x\|_\infty$. By choosing $e_3 = (0, 0, 1, 0, \ldots, 0, \ldots) \in c_0$, it follows $\|A\| = 1$ by taking into account previous inequality.
For any $m \in \mathbb{N}$,

$$A^m(x) = A^m(x_1, x_2, x_3, \ldots) = (0, \frac{x_{m+2}}{2^{m-1}}, \frac{x_{m+3}}{2^m}, \ldots), \quad x = \{x_n\} \in c_0,$$

therefore, observing $e_{m+2} \in c_0$ with all zeros except one on (m+2)-nd place (i.e., $(e_{m+2})_n = \delta_{n,m+2}$, $n \in \mathbb{N}$), we obtain $\|A^m\| = \frac{1}{2^{m-1}}$. Spectral radius of A is $\frac{1}{2}$, A is a positive operator, so all the conditions of Theorem 2.2 are satisfied since

$$d(A(x), A(y)) \leq A(d(x, y)), \; x, t \in c_0,$$

where $\leq$ is usual partial ordering on c_0, i.e. $x_n \leq y_n$, $n \in \mathbb{N}$, determining a normal cone and $d : c_0 \times c_0 \mapsto c_0$ defined by $d(x, y)(n) = |x(n) - y(n)|$, $n \in \mathbb{N}$ is a cone metric.
On the other hand, since normal constant and $\|A\|$ are equal to 1, norm inequality implies

$$D(A(x), A(y)) \leq D(x, y),$$

thus Banach theorem is not applicable (let $x = \theta$ and $y = e_3$).

We may also assume that $K = 1$ due to the renormization and the invariance of spectral radius in renormized space. It is important to notice that $r(A) < 1$ implies $\|A^n\| < 1$ for some $n \in \mathbb{N}$, so instead of Banach theorem, we should consider Consequence 3.1.

If the inequality (2.2) holds, then, since A is an increasing operator,

$$d(f^n(x), f^n(y)) \leq A^n(d(x, y)),$$

thus,

$$D(f^n(x), f^n(y)) \leq \|A^n\|(d(x, y)),$$

and existence and uniqueness of a fixed point for a mapping f follows directly from Consequence 3.1.

In Example 3.8 f^3 is a contraction in induced metric space, and in Example 3.9 f^2.

As presented in [6], the requirement that A contains only positive entries, as stated in Perov theorem, could be removed thanks to the normality of the defined cone in generalized metric space. This could be explained also by the fact that, from the definition of matrix norm, only absolute value of matrix entries has impact on the norm value. So Perov type theorems are applicable, regardless of the positivity of matrix elements, if all entries are less than 1.

Perov theorem has a wide range of application and estimations obtained by Perov theorem and generalized metric are better than by using usual metric spaces and some well-known theorems. In [25] coupled fixed point problem on Banach space was analyzed and, implementation of various metric and vector-valued metric in the sense of Perov, lead to the conclusion that results obtained by Perov theorem are better and unify other results. The comparison is made for Schauder, Krasnoselskii, Leray-Schauder and Perov theorem. We will discuss results obtained by Banach fixed point theorem and compare them in the case of metric space.

Example 3.10. *If (X, d) is a complete metric space and $T_i : X \times X \mapsto X$, $i = 1, 2$, solution of a system*

$$\begin{aligned} T_1(x, y) &= x \\ T_2(x, y) &= y, \end{aligned} \tag{3.7}$$

is a fixed point of a mapping $T : X \times X \mapsto X \times X$ defined with

$$T(x, y) = (T_1(x, y), T_2(x, y)),\ x, y \in X.$$

To apply Banach theorem, T should be a contraction on $X \times X$. Let D be a metric on $X \times X$ induced by d, then

$$D(F(x, y), F(u, v)) \leq qD((x, y), (u, v)),\ (x, y), (u, v) \in X \times X,$$

for some $q \in (0, 1)$.
If $D((x, y), (u, v)) = d(x, y) + d(u, v)$, $(x, y), (u, v) \in X \times X$, then

$$d(T_1(x, y), T_1(u, v)) + d(T_2(x, y), T_2(u, v)) \leq q(d(x, y) + d(u, v)), \tag{3.8}$$

for any $(x, y), (u, v) \in X \times X$, because of

$$d(T_i(x, y), T_i(u, v)) \leq \frac{q}{2}(d(x, y) + d(u, v)),\ i = 1, 2, \tag{3.9}$$

holds for any $(x, y), (u, v) \in X \times X$.
On the other hand, if Perov theorem would be applied, T_1 and T_2 should be such that

$$d(T_i(x, y), T_i(u, v)) \leq a_i d(x, u) + b_i d(y, v),\ (x, y), (u, v) \in X \times X, i = 1, 2,$$

for some nonnegative $a_i, b_i \geq 0$, $i = 1, 2$, and a matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

convergent to zero. This means that $r(A) < 1$ or, equivalently,

$$a_1 + b_2 + \sqrt{-2a_1b_2 + 4a_2b_1 + a_1{}^2 + b_2{}^2} < 2.$$

Considering (3.9), $\max\{a_1, a_2\}$, $\max\{b_1, b_2\}$ should be less than $\frac{1}{2}$, or in view of (3.8), $\max\{a_1, a_2\} + \max\{b_1, b_2\} < 1$. Anyway, this result is more strict than $r(A) < 1$.

If

$$A = \begin{bmatrix} \frac{2}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{2}{3} \end{bmatrix},$$

then $r(A) = \frac{7}{9}$, but neither of the inequalities (3.8) and (3.9) is satisfied.

Perov fixed point theorem found application in solving various systems of differential equations. But, in some cases like [29], it is possible to replace it with the Consequence 3.1.

Example 3.11. *Let (X_i, d_i), $i = \overline{1, m}$ be some complete metric spaces and define a generalized metric d on their Cartesian product $X = \prod_{i=1}^{m} X_i$ with*

$$d(x, y) = \begin{bmatrix} d_1(x_1, y_1) \\ d_2(x_2, y_2) \\ \vdots \\ d_m(x_m, y_m) \end{bmatrix},$$

for $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in X$. *As previously discussed,* (X, d) *is, as generalized metric space, also a normal cone metric space with a normal constant* $K = 1$.
Let (Y, τ) *be a Hausdorff topological space and* $f = (f_1, f_2) : X \times Y \mapsto X \times Y$ *an operator.Theorem 2.1 of [29] states that if* f *is continuous,* (Y, τ) *has a fixed point property (i.e., every continuous mapping* $g : Y \mapsto Y$ *has a fixed point) and there exists a matrix* $S \in \mathrm{R}^{m\times m}$ *convergent to zero matrix such that*

$$d(f_1(u, y), f_1(v, y)) \leq Sd(u, v), \quad u, v \in X, y \in Y, \tag{3.10}$$

then f *has a fixed point. Uniqueness is not guaranteed because of contractive condition based on the first coordinate. Instead of using Perov theorem, as presented in [29], observe that, since* $S^n \to \Theta$, $n \to \infty$, *then there exists some* $n \in \mathbb{N}$ *such that* $\|S^n\| = q < 1$, *where assumed norm is the supremum norm. For such chosen* n, *(3.10) implies*

$$d(f_1^n(u, y), f_1^n(v, y)) \leq S^n\left(d(u, v)\right), \quad u, v \in X, y \in Y,$$

so

$$d_\infty(f_1^n(u, y), f_1^n(v, y)) \leq q d_\infty(u, v), \quad u, v \in X, y \in Y,$$

where $d_\infty : X \times X \mapsto \mathbb{R}$ *is a maximum metric defined with*

$$d(u, v) = \max_{i=\overline{1,m}} d_i(u_i, v_i), \ u, v \in X.$$

Hence, Consequence 3.1 guarantees unique fixed point x^* *of a mapping* $f_1^n(\cdot, y) : X \mapsto X$ *for any* $y \in Y$. *As in the proof of Theorem 3.6,* x^* *is also unique fixed point of* $f_1(\cdot, y) : X \mapsto X$ *for a fixed* $y \in Y$. *The rest of the proof would follow analogously as in [29].*
As stated in this paper, Y *could be any compact convex subset of a Banach space. This results is applied in solving systems of functional-differential equations such as:*

$$\begin{aligned} x(t) &= \int_0^1 K(t, s, x(s), y(s))ds + g(t), \quad t \in [0, 1], \\ y(t) &= \int_0^1 H(t, s, x(s), y(s), y(y(s)))ds, \quad t \in [0, 1], \end{aligned}$$

where $x \in X$ *and* $y \in Y$, *continuous mappings* $K \in C\left([0, 1] \times [0, 1] \times \mathbb{R}^m \times [0, 1], \mathbb{R}^m\right)$, $g \in C\left([0, 1], \mathbb{R}^m\right)$ *and* $H \in C\left([0, 1] \times [0, 1] \times \mathbb{R}^m \times [0, 1] \times [0, 1], \mathbb{R}\right)$,

Under assumptions that codomain of H *is contained in* $[0, 1]$, *that* H *is a first coordinate Lipschitzian mapping with a constant* L *and* K *is a Perov generalized contraction, this system has at least one solution in* $X \times Y$ *for* $X = C([0, 1], \mathbb{R}^m) = \prod_{i=1}^{m} X_i$, $X_i = C[0, 1]$, $i = \overline{1, m}$ *and* Y *set of all Lipschitzian mappings on* $C([0, 1], [0, 1])$ *with a constant* L. *Observe that we could not use Banach theorem instead of Perov to obtain this conclusion due to the contractive condition for* K.

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