Filomat 31:12 (2017), 3741–3748
https://doi.org/10.2298/FIL1712741W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Note on a Competitive Lotka-Volterra Model with Lévy Noise

Sheng Wang[a,b], Linshan Wang[c], Tengda Wei[b]

[a] *School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, Henan 454003, PR China*
[b] *College of Oceanic and Atmospheric Sciences, Ocean University of China, Qingdao, Shandong 266071, PR China*
[c] *School of Mathematics, Ocean University of China, Qingdao, Shandong 266071, PR China*

Abstract. In this note, stochastic permanence for a competitive Lotka-Volterra model with Lévy noise (which can be used to describe sudden environmental perturbations) is studied by using stochastic analytical techniques. Moreover, some numerical simulations are provided to support the results.

1. Introduction

Recently, stochastic Lotka-Volterra models driven by white noise have been received great attention and have been studied extensively (see e.g. [1–8]). However, in the real world population systems often suffer sudden environmental perturbations, such as earthquakes, hurricanes, planting, harvesting, etc (see e.g. [9–11]). These phenomena can not be described by white noise [12]. Bao et al. [13, 14] pointed out that one may use Lévy jump processes to describe these phenomna and they studied the following n-dimensional competitive Lotka-Volterra model with Lévy noise:

$$dX_i(t) = X_i(t^-)\left[\left(a_i(t) - \sum_{j=1}^{n} b_{ij}(t)X_j(t^-)\right)dt + \sigma_i(t)dW(t) + \int_{\mathbb{Z}} \gamma_i(t,\mu)\widetilde{N}(dt,d\mu)\right],\ 1 \le i \le n, \tag{1}$$

where $X_i(t^-)$ is the left limit of $X_i(t)$, $W(t)$ is a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\ge0}$. N is a Poisson counting measure with characteristic measure λ on a measurable subset $\mathbb{Z}$ of $[0, +\infty)$ with $\lambda(\mathbb{Z}) < +\infty$ and $\widetilde{N}(dt, d\mu) = N(dt, d\mu) - \lambda(d\mu)dt$. It is well known that permanence means the long time survival in a population dynamics and thus has its theoretical and practical significance. So, in this note we study the stochastic permanence of system (1). To the best of authors' knowledge, this is the first attempt to investigate stochastic permanence for the general competitive Lotka-Volterra model with Lévy noise.

2010 *Mathematics Subject Classification*. Primary 60H10; Secondary 60J75, 60J28.
Keywords. Lotka-Volterra model; Lévy noise; Stochastic permanence.
Received: 03 December 2015; Accepted: 20 August 2016
Communicated by Miljana Jovanović
This work is supported by National Natural Science Foundation of China (No. 11171374) and Natural Science Foundation of Shandong Province (No. ZR2011AZ001).
Email addresses: wangpengzhan2012@163.com (Sheng Wang), wangls@ouc.edu.cn (Linshan Wang), tdwei123@163.com (Tengda Wei)

2. Main Results

For convenience, let

$$\begin{cases} A = \min_{1\le i\le n}\left\{\inf_{t\ge 0}\left[a_i(t) - \frac{\sigma_i^2(t)}{2} - \int_{\mathbb{Z}} (\gamma_i(t,\mu) - \ln(1+\gamma_i(t,\mu)))\,\lambda(d\mu)\right]\right\}, \\ B_* = \min_{1\le i\le n}\left\{\inf_{(t,\mu)\in[0,+\infty)\times\mathbb{Z}} \gamma_i(t,\mu)\right\},\ B^* = \max_{1\le i\le n}\left\{\sup_{t\ge 0}\sigma_i(t)\right\}, \\ P_i(t) = \int_{\mathbb{Z}} \gamma_i(t,\mu)\lambda(d\mu),\ H_i(t) = \int_{\mathbb{Z}} \ln(1+\gamma_i(t,\mu))\lambda(d\mu),\ 1\le i\le n, \\ X(t) = (X_1(t),...,X_n(t))^{\mathrm{T}},\ X_0 = (X_1(0),...,X_n(0))^{\mathrm{T}},\ |X(t)| = \left[\sum_{i=1}^n X_i^2(t)\right]^{\frac{1}{2}}. \end{cases} \tag{2}$$

In this note, we always assume that $W(t)$ and N are independent and
$(\mathcal{A})$ For $1 \le i,j \le n$, $i \ne j$, $a_i(t) > 0$, $b_{ii}(t)$, $b_{ij}(t) \ge 0$, $\sigma_i(t)$, $\gamma_i(t,\mu) > -1$ are bounded functions and $\inf_{t\ge 0} b_{ii}(t) > 0$.
$(\mathcal{B})$ $A > 0$.

Definition 2.1. *(see Bao et al. [13]) System (1) is said to be stochastically permanent, if, for any $\epsilon > 0$, there exist $\delta_* = \delta_*(\epsilon) > 0$ and $\delta^* = \delta^*(\epsilon) > 0$ such that*

$$\liminf_{t\to+\infty} P\{|X(t)| \ge \delta_*\} \ge 1-\epsilon,\ \liminf_{t\to+\infty} P\{|X(t)| \le \delta^*\} \ge 1-\epsilon. \tag{3}$$

Remark 2.2. *The stochastic permanence definition of multi-population systems was first proposed by Li et al. [5] and has been intensively applied (see e.g. [13, 15–17]).*

Lemma 2.3. *(see Bao et al. [13]) Under assumption $(\mathcal{A})$, for any initial value $X_0 \in \mathbb{R}_+^n$, system (1) has a unique global solution $X(t) \in \mathbb{R}_+^n$ for $t \ge 0$ a.s.*

Lemma 2.4. *(see Bao et al. [13]) Under assumption $(\mathcal{A})$, for any $p \in [0,1]$, there is a constant K such that*

$$\sup_{t\in\mathbb{R}_+} \mathbb{E}[|X(t)|^p] \le K. \tag{4}$$

Lemma 2.5. *Assume that $X(t)$ is the solution to system (1) with initial value $X_0 \in \mathbb{R}_+^n$, then*

$$\sum_{i\ne j}^n \sigma_i(t)\sigma_j(t)X_i(t)X_j(t) - \sum_{i\ne j}^n \sigma_i^2(t)X_i(t)X_j(t) \le 0. \tag{5}$$

Proof. We use the mathematical induction. For $n = 2$,

$$\sum_{i\ne j}^n \sigma_i(t)\sigma_j(t)X_i(t)X_j(t) - \sum_{i\ne j}^n \sigma_i^2(t)X_i(t)X_j(t) = -(\sigma_1(t)-\sigma_2(t))^2 X_1(t)X_2(t) \le 0. \tag{6}$$

Assume that for $n = k$, (5) is true, that is

$$\sum_{i\ne j}^k \sigma_i(t)\sigma_j(t)X_i(t)X_j(t) - \sum_{i\ne j}^k \sigma_i^2(t)X_i(t)X_j(t) \le 0. \tag{7}$$

From (7) we have

$$\begin{aligned}
&\sum_{i\neq j}^{k+1}\sigma_i(t)\sigma_j(t)X_i(t)X_j(t)-\sum_{i\neq j}^{k+1}\sigma_i^2(t)X_i(t)X_j(t)\\
=&\left\{\sum_{i\neq j}^{k}\sigma_i(t)\sigma_j(t)X_i(t)X_j(t)+2\sum_{i=1}^{k}\sigma_i(t)\sigma_{k+1}(t)X_i(t)X_{k+1}(t)\right\}\\
&-\left\{\sum_{i\neq j}^{k}\sigma_i^2(t)X_i(t)X_j(t)+\sum_{i=1}^{k}(\sigma_i^2(t)+\sigma_{k+1}^2(t))X_i(t)X_{k+1}(t)\right\}\\
\leq& 2\sum_{i=1}^{k}\sigma_i(t)\sigma_{k+1}(t)X_i(t)X_{k+1}(t)-\sum_{i=1}^{k}(\sigma_i^2(t)+\sigma_{k+1}^2(t))X_i(t)X_{k+1}(t)\\
=&-\sum_{i=1}^{k}(\sigma_i(t)-\sigma_{k+1}(t))^2X_i(t)X_{k+1}(t)\leq 0.
\end{aligned} \tag{8}$$

In conclusion, the proof is complete. □

Theorem 2.6. *Under assumptions ($\mathcal{A}$) and ($\mathcal{B}$), system (1) is stochastically permanent.*

Proof. Let $V(X)=\sum_{i=1}^{n}X_i$. Applying Itô's formula to $V(X)$ leads to

$$dV=\beta(t)dt+\sigma(t)dW(t)+\int_{\mathbb{Z}}\gamma(t,\mu)\widetilde{N}(dt,d\mu), \tag{9}$$

where

$$\beta(t)=\sum_{i=1}^{n}X_i\left(a_i(t)-\sum_{j=1}^{n}b_{ij}(t)X_j\right),\ \sigma(t)=\sum_{i=1}^{n}\sigma_i(t)X_i,\ \gamma(t,\mu)=\sum_{i=1}^{n}\gamma_i(t,\mu)X_i. \tag{10}$$

From Lemma 2.3 we have $P\{X_i(t)>0\ for\ t\geq 0\}=1$. Define $U(X)=\frac{1}{V(X)}$. By Itô's formula, we obtain

$$dU=\widetilde{\beta(t)}dt+\widetilde{\sigma(t)}dW(t)+\int_{\mathbb{Z}}\widetilde{\gamma(t,\mu)}\widetilde{N}(dt,d\mu), \tag{11}$$

where

$$\begin{cases}\widetilde{\beta(t)}=-U^2\beta(t)+U^3\sigma^2(t)+\int_Z\left\{\frac{1}{V+\gamma(t,\mu)}-U+U^2\gamma(t,\mu)\right\}\lambda(d\mu),\\ \widetilde{\sigma(t)}=-U^2\sigma(t),\ \widetilde{\gamma(t,\mu)}=\frac{1}{V+\gamma(t,\mu)}-U.\end{cases} \tag{12}$$

In the light of (2), we deduce that for any $t\geq 0$ and $1\leq i\leq n$,

$$a_i(t)\geq A+\frac{\sigma_i^2(t)}{2}+P_i(t)-H_i(t). \tag{13}$$

Consider the following auxiliary function:

$$G(\theta)=A\theta-\frac{\theta^2}{2}(B^*)^2-\int_{\mathbb{Z}}\left\{\left(\frac{1}{1+B_*}\right)^{\theta}-1-\theta\ln\left(\frac{1}{1+B_*}\right)\right\}\lambda(d\mu),\ 0\leq\theta\leq 1. \tag{14}$$

Then $G(0) = 0$ and

$$G'(\theta) = A - (B^*)^2\theta - \int_{\mathbb{Z}} \left\{ \left(\frac{1}{1+B_*} \right)^{\theta} - 1 \right\} \ln\left(\frac{1}{1+B_*} \right) \lambda(d\mu), \ 0 \le \theta \le 1. \tag{15}$$

According to (15) and ($\mathcal{B}$), we obtain

$$G'(0) = A > 0. \tag{16}$$

Thus, there exist $\theta > 0$ and $k > 0$ such that

$$A\theta - \frac{\theta^2}{2}(B^*)^2 - \int_{\mathbb{Z}} \left\{ \left(\frac{1}{1+B_*} \right)^{\theta} - 1 - \theta \ln\left(\frac{1}{1+B_*} \right) \right\} \lambda(d\mu) > k > 0. \tag{17}$$

Applying Itô's formula to $[\mathrm{e}^{kt}(1+U)^{\theta}]$ yields

$$\begin{aligned} d[\mathrm{e}^{kt}(1+U)^{\theta}] =& \mathcal{L}\left[\mathrm{e}^{kt}(1+U)^{\theta}\right] dt + \mathrm{e}^{kt}\theta(1+U)^{\theta-1}\widetilde{\sigma(t)}dW(t) \\ &+ \int_{\mathbb{Z}} \left[\mathrm{e}^{kt}\left(1+U+\widetilde{\gamma(t,\mu)}\right)^{\theta} - \mathrm{e}^{kt}(1+U)^{\theta} \right] \widetilde{N}(dt, d\mu), \end{aligned} \tag{18}$$

where

$$\begin{aligned} \mathcal{L}\left[\mathrm{e}^{kt}(1+U)^{\theta}\right] =& k\mathrm{e}^{kt}(1+U)^{\theta} + \mathrm{e}^{kt}\theta(1+U)^{\theta-1}\widetilde{\beta(t)} + \tfrac{\theta(\theta-1)}{2}\mathrm{e}^{kt}(1+U)^{\theta-2}\left[\widetilde{\sigma(t)}\right]^2 \\ &+ \int_{\mathbb{Z}} \left[\mathrm{e}^{kt}\left(1+U+\widetilde{\gamma(t,\mu)}\right)^{\theta} - \mathrm{e}^{kt}(1+U)^{\theta} - \mathrm{e}^{kt}\theta(1+U)^{\theta-1}\widetilde{\gamma(t,\mu)} \right] \lambda(d\mu). \end{aligned} \tag{19}$$

Substituting inequality (13) into (19) gives

$$\begin{aligned} \mathcal{L}\left[\mathrm{e}^{kt}(1+U)^{\theta}\right] \le& k\mathrm{e}^{kt}(1+U)^{\theta} + \tfrac{\theta(\theta-1)}{2}\mathrm{e}^{kt}U^2(1+U)^{\theta-2}\left[U\sum_{i=1}^{n}\sigma_i(t)X_i\right]^2 \\ &+ \theta\mathrm{e}^{kt}(1+U)^{\theta-1}U\left[U\sum_{i=1}^{n}\sigma_i(t)X_i\right]^2 + \theta\mathrm{e}^{kt}U(1+U)^{\theta-1}\int_{\mathbb{Z}} U\sum_{i=1}^{n}\gamma_i(t,\mu)X_i\lambda(d\mu) \\ &+ \int_{\mathbb{Z}} \mathrm{e}^{kt}\left[\left(1+\frac{1}{V+\sum_{i=1}^{n}\gamma_i(t,\mu)X_i}\right)^{\theta} - (1+U)^{\theta}\right]\lambda(d\mu) \\ &- A\theta\mathrm{e}^{kt}U(1+U)^{\theta-1} - \theta\mathrm{e}^{kt}U(1+U)^{\theta-1}U\sum_{i=1}^{n}\tfrac{\sigma_i^2(t)X_i}{2} \\ &- \theta\mathrm{e}^{kt}U(1+U)^{\theta-1}U\sum_{i=1}^{n}P_i(t)X_i + \theta\mathrm{e}^{kt}U(1+U)^{\theta-1}U\sum_{i=1}^{n}H_i(t)X_i \\ &+ \theta\mathrm{e}^{kt}(1+U)^{\theta-1}U^2\sum_{i=1}^{n}X_i\sum_{j=1}^{n}b_{ij}(t)X_j \triangleq \mathrm{e}^{kt}\left[O(U^{\theta})U^{\theta} + G(U)\right], \end{aligned} \tag{20}$$

where $\lim_{U\to+\infty}\frac{G(U)}{U^{\theta}} = 0$. Since

$$0 \le U^2\sum_{i=1}^{n}X_i\sum_{j=1}^{n}b_{ij}(t)X_j = \sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}(t)\frac{X_iX_j}{(\sum_{i=1}^{n}X_i)^2} \le \frac{1}{2}\sum_{i,j=1}^{n}\sup_{t\ge0}b_{ij}(t), \tag{21}$$

we derive that

$$\begin{aligned}\mathcal{O}(U^{\theta}) =& k + \frac{\theta(\theta-1)}{2}\left(U\sum_{i=1}^{n}\sigma_i(t)X_i\right)^2 + \theta\left(U\sum_{i=1}^{n}\sigma_i(t)X_i\right)^2 \\ &+ \theta\int_{\mathbb{Z}} U\sum_{i=1}^{n}\gamma_i(t,\mu)X_i\lambda(d\mu) + \int_{\mathbb{Z}}\frac{V^{\theta}}{\left[\sum_{i=1}^{n}(1+\gamma_i(t,\mu))X_i\right]^{\theta}}\lambda(d\mu) \\ &- \int_{\mathbb{Z}}\lambda(d\mu) - A\theta - \frac{\theta}{2}U\sum_{i=1}^{n}\sigma_i^2(t)X_i - \theta U\sum_{i=1}^{n}P_i(t)X_i + \theta U\sum_{i=1}^{n}H_i(t)X_i \\ =& k - A\theta - \int_{\mathbb{Z}}\lambda(d\mu) + \frac{\theta(\theta+1)}{2}\frac{(\sum_{i=1}^{n}\sigma_i(t)X_i)^2}{(\sum_{i=1}^{n}X_i)^2} - \frac{\theta}{2}\frac{(\sum_{i=1}^{n}\sigma_i^2(t)X_i)(\sum_{i=1}^{n}X_i)}{(\sum_{i=1}^{n}X_i)^2} \\ &+ \int_{\mathbb{Z}}\left(\frac{\sum_{i=1}^{n}X_i}{\sum_{i=1}^{n}(1+\gamma_i(t,\mu))X_i}\right)^{\theta}\lambda(d\mu) + \theta\int_{\mathbb{Z}}\frac{\sum_{i=1}^{n}\ln(1+\gamma_i(t,\mu))X_i}{\sum_{i=1}^{n}X_i}\lambda(d\mu).\end{aligned} \tag{22}$$

On the basis of Lemma 2.5, we get

$$\begin{aligned}&\theta(\theta+1)\left(\sum_{i=1}^{n}\sigma_i(t)X_i(t)\right)^2 - \theta\left(\sum_{i=1}^{n}\sigma_i^2(t)X_i(t)\right)\left(\sum_{i=1}^{n}X_i(t)\right) \\ =&\theta^2\left(\sum_{i=1}^{n}\sigma_i(t)X_i(t)\right)^2 + \theta\left(\sum_{i\neq j}\sigma_i(t)\sigma_j(t)X_i(t)X_j(t) - \sum_{i\neq j}\sigma_i^2(t)X_i(t)X_j(t)\right) \le \theta^2\left(\sum_{i=1}^{n}\sigma_i(t)X_i(t)\right)^2.\end{aligned} \tag{23}$$

In a view of (22), (23), Jensen's inequality, $e^x = \sum_{n=0}^{+\infty}\frac{x^n}{n!}$ and (17), we obtain

$$\begin{aligned}\mathcal{O}(U^{\theta}) \le& k - A\theta - \int_{\mathbb{Z}}\lambda(d\mu) + \frac{\theta^2}{2}\left(\frac{\sum_{i=1}^{n}\sigma_i(t)X_i}{\sum_{i=1}^{n}X_i}\right)^2 \\ &+ \int_{\mathbb{Z}}\left(\frac{\sum_{i=1}^{n}X_i}{\sum_{i=1}^{n}(1+\gamma_i(t,\mu))X_i}\right)^{\theta}\lambda(d\mu) + \theta\int_{\mathbb{Z}}\ln\left(\frac{\sum_{i=1}^{n}(1+\gamma_i(t,\mu))X_i}{\sum_{i=1}^{n}X_i}\right)\lambda(d\mu) \\ =& k - A\theta + \frac{\theta^2}{2}\left(\frac{\sum_{i=1}^{n}\sigma_i(t)X_i}{\sum_{i=1}^{n}X_i}\right)^2 + \int_{\mathbb{Z}}\sum_{n=2}^{+\infty}\frac{\theta^n}{n!}\left[\ln\left(\frac{\sum_{i=1}^{n}X_i}{\sum_{i=1}^{n}(1+\gamma_i(t,\mu))X_i}\right)\right]^n\lambda(d\mu) \\ \le& k - A\theta + \frac{\theta^2}{2}(B^*)^2 + \int_{\mathbb{Z}}\left\{\left(\frac{1}{1+B_*}\right)^{\theta} - 1 - \theta\ln\left(\frac{1}{1+B_*}\right)\right\}\lambda(d\mu) < 0.\end{aligned} \tag{24}$$

On the basis of (18), (20) and (24), there exists $\mathcal{K} > 0$ such that

$$\mathbb{E}[e^{kt}(1+U)^{\theta}] - [1+U(X_0)]^{\theta} \le \int_0^t \mathcal{K}e^{ks}ds = \tfrac{\mathcal{K}}{k}(e^{kt}-1). \tag{25}$$

From (25), we have

$$\limsup_{t\to+\infty}\mathbb{E}[U^{\theta}] \le \tfrac{\mathcal{K}}{k}. \tag{26}$$

In the light of $\frac{1}{|X(t)|^{\theta}} \le n^{\frac{\theta}{2}}U^{\theta}$ and (26), we get

$$\limsup_{t\to+\infty}\mathbb{E}\left[\tfrac{1}{|X(t)|^{\theta}}\right] \le n^{\frac{\theta}{2}}\tfrac{\mathcal{K}}{k}. \tag{27}$$

Based on Chebyshev's inequality, for any $\epsilon > 0$, there exists $\delta_* = \frac{\sqrt{n}}{n}\left(\frac{k\epsilon}{\mathcal{K}}\right)^{\frac{1}{\theta}} > 0$ such that

$$\limsup_{t\to+\infty} P\{|X(t)| < \delta_*\} = \limsup_{t\to+\infty} P\left\{\frac{1}{|X(t)|} > \frac{1}{\delta_*}\right\} \leq (\delta_*)^{\theta} \limsup_{t\to+\infty} \mathbb{E}\left[\frac{1}{|X(t)|^{\theta}}\right] \leq \epsilon. \tag{28}$$

Therefore,

$$\liminf_{t\to+\infty} P\{|X(t)| \geq \delta_*\} \geq 1 - \epsilon. \tag{29}$$

The second part of (3) follows from combining Lemma 2.4 with Chebyshev's inequality. That is, system (1) is stochastically permanent. □

Remark 2.7. *For $n = 1$, system (1) becomes*

$$dX(t) = X(t^-)\left[(a(t) - b(t)X(t^-))\,dt + \sigma(t)dW(t) + \int_{\mathbb{Z}} \gamma(t,\mu)\widetilde{N}(dt,d\mu)\right]. \tag{30}$$

From Theorem 2.6, system (30) is stochastically permanent, if

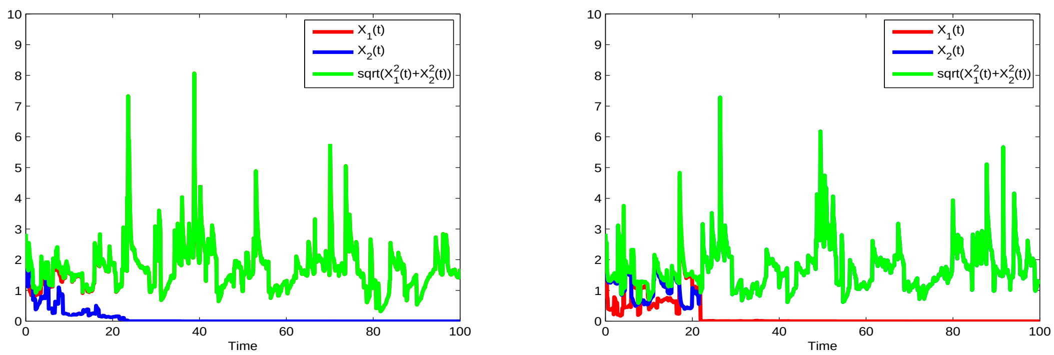
$$\inf_{t\geq 0}\left[a(t) - \frac{\sigma^2(t)}{2} - \int_{\mathbb{Z}} (\gamma(t,\mu) - \ln(1+\gamma(t,\mu)))\,\lambda(d\mu)\right] > 0. \tag{31}$$

Thus, Theorem 2.6 includes Theorem 1 in [18] as a special case.

3. An Example

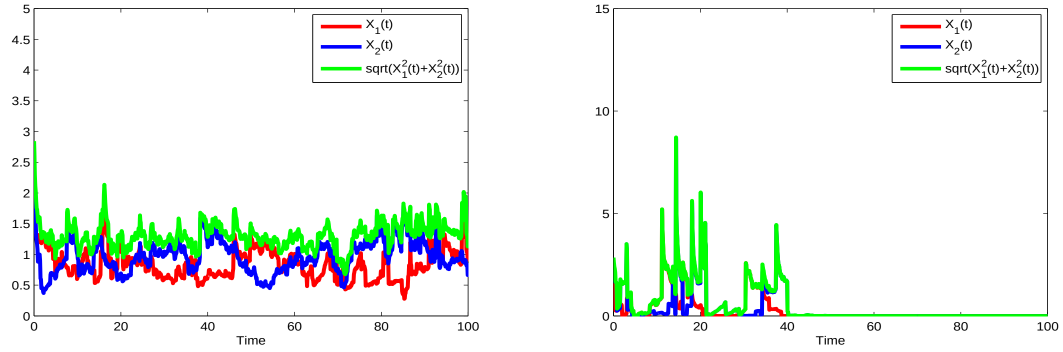
By the method in [19], for $\lambda(\mathbb{Z}) = 1.0$ and step size $\Delta t = 0.1$ we numerically simulate the solutions of the following system to support our results:

$$\begin{cases} dX_1(t) = X_1(t^-)\,[0.81 - 0.51X_1(t^-) - 0.39X_2(t^-)]\,dt + X_1(t^-)\left\{0.1dW(t) + \int_{\mathbb{Z}} \gamma_1(t,\mu)\widetilde{N}(dt,d\mu)\right\}, \\ dX_2(t) = X_2(t^-)\,[0.79 - 0.41X_1(t^-) - 0.49X_2(t^-)]\,dt + X_2(t^-)\left\{0.12dW(t) + \int_{\mathbb{Z}} \gamma_2(t,\mu)\widetilde{N}(dt,d\mu)\right\}, \\ X_1(0) = 1.8,\ X_2(0) = 2.2. \end{cases} \tag{32}$$



(a) $\gamma_1(t,\mu) = 0.48,\ \gamma_2(t,\mu) = 0.92.$

(b) $\gamma_1(t,\mu) = 1.2,\ \gamma_2(t,\mu) = 0.51.$



(c) $\gamma_1(t,\mu) = 0.25,\ \gamma_2(t,\mu) = 0.2$ (d) $\gamma_1(t,\mu) = 2.3,\ \gamma_2(t,\mu) = 2.5$

Figure 1: Stochastic permanence of system (32).

For system (32), we introduce some mathematical notations as follows (as in [18]):

$$\begin{cases} b_i(t) = a_i(t) - \frac{1}{2}\sigma_i^2(t) - \int_{\mathbb{Z}} (\gamma_i(t,\mu) - \ln(1+\gamma_i(t,\mu)))\lambda(d\mu),\ i = 1,2, \\ \Delta(t) = b_{11}(t)b_{22}(t) - b_{12}(t)b_{21}(t),\ \Delta_1(t) = b_1(t)b_{22}(t) - b_2(t)b_{12}(t),\ \Delta_2(t) = b_2(t)b_{11}(t) - b_1(t)b_{21}(t). \end{cases} \tag{33}$$

(I) For $\gamma_1(t,\mu) = 0.48$, $\gamma_2(t,\mu) = 0.92$, we have (Figure 1(a)):

$$b_1(t) = 0.71704,\ b_2(t) = 0.51513,\ \Delta(t) = 0.09,\ \Delta_1(t) = 0.15045,\ \Delta_2(t) = -0.03142. \tag{34}$$

By Theorem 2.6, system (32) is stochastically permanent. From Theorem 4(i) in [18], $X_1(t)$ is persistent in mean while $X_2(t)$ is extinctive a.s.
(II) For $\gamma_1(t,\mu) = 1.2$, $\gamma_2(t,\mu) = 0.51$, we have (Figure 1(b)):

$$b_1(t) = 0.39346,\ b_2(t) = 0.68491,\ \Delta(t) = 0.09,\ \Delta_1(t) = -0.07432,\ \Delta_2(t) = 0.18799. \tag{35}$$

In view of Theorem 2.6, system (32) is stochastically permanent. Based on Theorem 4(ii) in [18], $X_1(t)$ is extinctive while $X_2(t)$ is persistent in mean a.s.
(III) For $\gamma_1(t,\mu) = 0.25$, $\gamma_2(t,\mu) = 0.2$, we have (Figure 1(c)):

$$b_1(t) = 0.77814,\ b_2(t) = 0.76512,\ \Delta(t) = 0.09,\ \Delta_1(t) = 0.08289,\ \Delta_2(t) = 0.07117. \tag{36}$$

According to Theorem 2.6, system (32) is stochastically permanent. By Theorem 4(iii) in [18], both $X_1(t)$ and $X_2(t)$ are persistent in mean a.s.
(IV) For $\gamma_1(t,\mu) = 2.3$, $\gamma_2(t,\mu) = 2.5$, we have (Figure 1(d)):

$$b_1(t) = -0.30108,\ b_2(t) = -0.46444. \tag{37}$$

On the basis of Theorem 4.6 in [13], both $X_1(t)$ and $X_2(t)$ are extinctive a.s. Hence system (32) is not stochastically permanent. In other words, if ($\mathcal{B}$) is false, then system (1) may be not stochastically permanent.
All mentioned above can be confirmed by Figure 1.

Remark 3.1. *(see Liu et al. [18]) Consider the autonomous case of system (1). For $n = 2$ and $\Delta(t) > 0$,*
(i) if $\Delta_1(t) > 0$, $\Delta_2(t) < 0$, then $X_1(t)$ is persistent in mean while $X_2(t)$ is extinctive a.s.
(ii) if $\Delta_1(t) < 0$, $\Delta_2(t) > 0$, then $X_1(t)$ is extinctive while $X_2(t)$ is persistent in mean a.s.
(iii) if $\Delta_i(t) > 0$, then $X_i(t)$ is persistent in mean a.s., $1 \le i \le 2$.

4. Conclusions and Future Directions

In this note, sufficient conditions for stochastic permanence of a competitive Lotka-Volterra model with Lévy noise are established.

Some interesting topics deserve further investigation. To begin with, it is interesting to study "stochastic persistence in probability" (see e.g. [20, 21]) of system (1). The motivation is that multi-population systems may remain stochastically permanent, although some species are extinctive (see Figure 1(a) and Figure 1(b)).

Next, we could investigate more realistic and complex systems in lieu of system (1), for instance, hybrid population systems with Lévy noise. The motivation is that parameters in population systems may suffer abrupt changes (see e.g. [1, 22]). One can use a continuous-time Markov chain with a finite state space to describe these abrupt changes (see e.g. [9, 23]).

Motivated by the works in [6, 11, 21, 24–26], we may also study the optimization problem of harvesting for stochastic delay population systems with Lévy noise. We leave these investigations for future work.

References

[1] C. Zhu, G. Yin, On competitive Lotka-Volterra model in random environments, J. Math. Anal. Appl. 357 (2009) 154-170.
[2] Y. Hu, F. Wu, C. Huang, Stochastic Lotka-Volterra models with multiple delays, J. Math. Anal. Appl. 375 (2011) 42-57.
[3] L. Wan, Q. Zhou, Stochastic Lotka-Volterra model with infinite delay, Statist. Probab. Lett. 79 (2009) 698-706.
[4] M. Liu, K. Wang, Global asymptotic stability of a stochastic Lotka-Volterra model with infinite delays, Commun. Nonlinear Sci. Numer Simulat. 17 (2012) 3115-3123.
[5] X. Li, X. Mao, Population dynamical behavior of non-autonomous Lotka-Volterra competitive system with random perturbation, Discrete Contin. Dyn. Syst. 24 (2009) 523-545.
[6] M. Liu, H. Qiu, K. Wang, A remark on a stochastic predator-prey system with time delays, Appl. Math. Lett. 26 (2013) 318-323.
[7] D. Jiang, C. Ji, X. Li, Analysis of autonomous Lotka-Volterra competition systems with random perturbation, J. Math. Anal. Appl. 390 (2012) 582-595.
[8] Y. Huang, Q. Liu, Y. Liu, Global asymptotic stability of a general stochastic Lotka-Volterra system with delays, Appl. Math. Lett. 26 (2013) 175-178.
[9] M. Liu, K. Wang, Dynamics of a Leslie-Gower Holling-type II predator-prey system with Lévy jumps, Nonlinear Anal. 85 (2013) 204-213.
[10] X. Zhang, W. Li, M. Liu, K. Wang, Dynamics of a stochastic Holling II one-predator two-prey system with jumps, Physica A. 421 (2015) 571-582.
[11] X. Zou, K. Wang, Optimal harvesting for a stochastic regime-switching logistic diffusion system with jumps, Nonlinear Anal. Hybrid Syst. 13 (2014) 32-44.
[12] M. Liu, M. Deng, B. Du, Analysis of a stochastic logistic model with diffusion, Appl. Math. Comput. 266 (2015) 169-182.
[13] J. Bao, X. Mao, G. Yin, C. Yuan, Competitive Lotka-Volterra population dynamics with jumps, Nonlinear Anal. 74 (2011) 6601-6616.
[14] J. Bao, C. Yuan, Stochastic population dynamics driven by Lévy noise, J. Math. Anal. Appl. 391 (2012) 363-375.
[15] M. Liu, K. Wang, Analysis of a stochastic autonomous mutualism model, J. Math. Anal. Appl. 402 (2013) 392-403.
[16] M. Liu, K. Wang, Population dynamical behavior of Lotka-Volterra cooperative systems with random perturbations, Discrete Contin. Dyn. Syst. 33 (2013) 2495-2522.
[17] Ky Tran, G. Yin, Stochastic competitive Lotka-Volterra ecosystems under partial observation: Feedback controls for permanence and extinction, J. Franklin Inst. 351 (2014) 4039-4064.
[18] M. Liu, K. Wang, Stochastic Lotka-Volterra systems with Lévy noise, J. Math. Anal. Appl. 410 (2014) 750-763.
[19] Paul Classerman, Monte Carlo Methods in Financial Engineering, Columbia University, Springer-Verlag, 2003.
[20] Sebastian J. Schreiber, Persistence for stochastic difference equations: a mini-review, J. Differ. Equ. Appl. 18 (2012) 1381-1403.
[21] M. Liu, C. Bai, Analysis of a stochastic tri-trophic food-chain model with harvesting, J. Math. Biol. (2016) 1-29. DOI 10.1007/s00285-016-0970-z.
[22] Q. Luo, X. Mao, Stochastic population dynamics under regime switching II, J. Math. Anal. Appl. 355 (2009) 577-593.
[23] C. Zhu, G. Yin, On hybrid competitive Lotka-Volterra ecosystems, Nonlinear Anal. 71 (2009) e1370-e1379.
[24] J. Bao, Z. Hou, C. Yuan, Stability in distribution of neutral stochastic differential delay equations with Markovian switching, Stat. Probab. Lett. 79 (2009) 1663-1673.
[25] M. Liu, C. Bai, Optimal harvesting of a stochastic logistic model with time delay, J. Nonlinear Sci. 25 (2015) 277-289.
[26] M. Liu, Optimal harvesting policy of a stochastic predator-prey model with time delay, Appl. Math. Lett. 48 (2015) 102-108.