Filomat 31:12 (2017), 3761–3770
https://doi.org/10.2298/FIL1712761A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Conditions for Approaching the Origin without Intersecting the x-axis in the Liénard Plane

Asadollah Aghajani[a], Mohsen Mirafzal[a], Donald O'Regan[b]

[a] *School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16844-13114, Iran.*
[b] *School of Mathematics, Statistics and Applied Mathematics National University of Ireland, Galway, University Road, Galway, Ireland.*

Abstract. We consider the Liénard system in the plane and present general assumptions to obtain some new explicit conditions under which this system has or fails to have a positive orbit which starts at a point on the vertical isocline and approaches the origin without intersecting the x-axis. This arises naturally in the existence of homoclinic orbits and oscillatory solutions. Our investigation is based on the notion of orthogonal trajectories of orbits of the system.

1. Introduction

To study the qualitative theory of the Liénard equation

$$x'' + f(x)x' + g(x) = 0,$$

such as boundedness, oscillation and periodicity of the solutions, results are established by examining the corresponding planar system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y - F(x), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -g(x), \tag{1}$$

where $F(x) := \int_0^x f(u)\mathrm{d}u$. Asymptotic and qualitative behavior of this system was studied by many authors. In the literature, there are a considerable number of results on the existence and uniqueness of periodic orbits, homoclinic orbits, oscillation of solutions, center problem and existence of limit cycles for Liénard and other types of second order differential equations (see [1-22] and the references cited therein). In the study of the qualitative behavior of solutions of the Liénard system the notion "property (Z_1^+)" is very useful. We say that system (1) has property (Z_1^+) if there exists a point $P(x_0, y_0)$ with $y_0 = F(x_0)$ and $x_0 > 0$ such that the positive semitrajectory of (1) starting at P approaches the origin through only the first quadrant.

In this paper, we give some conditions on $F(x)$ and $g(x)$ under which the system (1) has or fails to have property (Z_1^+). We assume that F and g are continuous on an open interval I which contains 0 and satisfy smoothness conditions to guarantee the existence and uniqueness of solutions of the corresponding initial

2010 *Mathematics Subject Classification*. Primary 34C05; Secondary 70K05
Keywords. Liénard system, Homoclinic orbit, Oscillation, Limit cycle
Received: 13 December 2015; Accepted: 25 May 2016
Communicated by Jelena Manojlović
Email addresses: aghajani@iust.ac.ir (Asadollah Aghajani), mirafzal@iust.ac.ir (Mohsen Mirafzal), donal.oregan@nuigalway.ie (Donald O'Regan)

value problems. Also, we assume that $F(0) = 0$ and $xg(x) > 0$ for $x \neq 0$ which implies the origin is the unique equilibrium of (1), and hence closed orbits (if any) rotate clockwise around it. Also, throughout this paper in the results related to property (Z_1^+), we assume that $F(x) > 0$, for $x > 0$ sufficiently small, because if $F(x)$ has an infinite number of positive zeros clustering at $x = 0$, then the system (1) obviously fails to have property (Z_1^+). The following results for property (Z_1^+) are the following theorems of Hara [11] and Hara-Sugie [16].

Theorem 1.1. (*Hara*[*11*]) *Under the condition that*

$$\limsup_{x\to 0^+} \frac{F(x)}{\sqrt{2G(x)}} < 2, \quad \textit{where} \;\; G(x) := \int_0^x g(u)\mathrm{d}u,$$

the Liénard system (1) *fails to have property* (Z_1^+), *while it has property* (Z_1^+) *if*

$$F(x) \geq 2\sqrt{2G(x)}.$$

Theorem 1.2. (*Hara* − *Sugie*[*16*]) *Suppose that*

$$F(x) \leq 2\sqrt{2G(x)} - h(\sqrt{2G(x)}),$$

for $x > 0$ *sufficiently small, where* $h(\xi)$ *is a non-negative continuous function with*

$$\frac{h(\xi)}{\xi} \leq 2 \textit{ is a non-decreasing function for } \xi > 0 \textit{ sufficiently small},$$

and

$$\int_0^{\xi_0} \frac{h(\xi)}{\xi^2}\mathrm{d}\xi = \infty \quad \textit{for some } \xi_0 > 0.$$

Then the system (1) *fails to have property* (Z_1^+).

The following result gives a sufficient condition for system (1) to have property (Z_1^+).

Theorem 1.3. (*Hara* − *Sugie*[*16*]) *Suppose that*

$$F(x) \geq 2\sqrt{2G(x)} - h(\sqrt{2G(x)}),$$

for $x > 0$ *sufficiently small, where* $h(\xi)$ *is a non-negative continuous function such that for* $\xi > 0$ *sufficiently small*

$$\frac{h(\xi)}{\xi} \quad \textit{is non-decreasing},$$

and

$$a[H(\xi)]^2 \leq \frac{h(\xi)}{\xi} \quad \textit{for some } a > 4,$$

where $H(\xi) = \int_0^\xi \frac{h(u)}{u^2}\mathrm{d}u$. *Then the Liénard system* (1) *has property* (Z_1^+).

These results are extended by some authors to more general planar dynamical systems of Liénard type, see for example [1, 5, 14]. In this paper, we consider system (1) and present some new explicit necessary and sufficient conditions under which this system has or fails to have property (Z_1^+). Our results improve the above and the existing results in the literature.

2. A comparison theorem

In this section we give a comparison result that will be used repeatedly in the sequel. First, we state the following result of T. Hara [11] on property (Z_1^+).

Lemma 2.1. (*Hara*[*11*]) *Consider system* (1)*. For each point* $P = (x_0, y_0)$ *on the vertical isocline in the right half-plane, the positive semitrajectory* $\gamma^+(P)$ *approaches the origin, without intersecting the x-axis, if and only if there exist a constant* $\delta \geq x_0$ *and a continuous function* $\psi(x)$*, such that*

$$\psi(x) < F(x) \quad \textit{and} \quad \int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} \mathrm{d}\xi \leq \psi(x) \textit{for } 0 < x < \delta.$$

Using the above lemma, we present a simple comparison theorem.

Theorem 2.2. *Suppose that the system* (1) *corresponding to* F_1 *and* g_1 *has property* (Z_1^+)*. If we have*

$$F_2(x) \geq F_1(x) \quad \textit{and } g_2(x) \leq g_1(x),$$

for all $x > 0$*, then system* (1) *corresponding to* F_2 *and* g_2 *has this property, too.*

Proof. By assumption the system (1) corresponding to F_1 and g_1 has property (Z_1^+). Thus from the above lemma, there is a continuous function $\psi(x) < F_1(x)$ (hence $\psi(x) < F_2(x)$) and some $\delta > 0$ such that for $0 < x < \delta$ we have

$$\int_0^x \frac{g_2(\xi)}{F_2(\xi) - \psi(\xi)} \mathrm{d}\xi \leq \int_0^x \frac{g_1(\xi)}{F_1(\xi) - \psi(\xi)} \mathrm{d}\xi \leq \psi(x).$$

Now using again the above lemma we have that (1) has property (Z_1^+). □

Consider (1) with $g(x) = x$ and $F(x) = \lambda x$. Then, we have a linear system and, either by elementary results or by the above lemma, we easily find that system (1) has property (Z_1^+), for $\lambda \geq 2$, and that it fails to have property (Z_1^+), for $\lambda < 2$. Now consider the system (1) as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y - F(x)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x. \tag{2}$$

The above comparison theorem shows that if

$$\limsup_{x \to 0^+} \frac{F(x)}{x} < 2,$$

then system (2) fails to have property (Z_1^+), while it has property (Z_1^+) if

$$F(x) \geq 2x.$$

From Theorem 1.2 of Sugie-Hara [16], we can solve the problem in the case $\frac{F(x)}{x} \nearrow 2$ as $x \to 0^+$. But this result is restricted and cannot solve the problem, when in general we have $\frac{F(x)}{x} \to 2$ as $x \to 0^+$. As we shall see, the results of this paper solve this problem even in the case

$$\liminf_{x \to 0^+} \frac{F(x)}{x} \leq 2 \leq \limsup_{x \to 0^+} \frac{F(x)}{x}.$$

3. Main results

The main idea in proving the main results in this paper is to find the family of orthogonal trajectories of system (1). Indeed, if this family described by $u(x,y)=C$ and taking $w(t):=u\big(x(t),y(t)\big)$, where $\big(x(t),y(t)\big)$ is a solution of this system, initiate at a point on the vertical isocline and remains at the first quadrant approaching the origin then we must have $w'(t)\le 0$ for all t sufficiently large (see Theorem 3.2).

Lemma 3.1. *The orthogonal trajectories of* (1) *in the right half-plane are level curves of the function*

$$u(x,y)=\int_{x_0}^{x}\frac{F(\xi)}{g(\xi)}\ \exp(-\int_1^{\xi}\frac{d\eta}{g(\eta)})d\xi+y\ \exp(-\int_1^{x}\frac{d\eta}{g(\eta)}),\quad x>0, \tag{3}$$

where $x_0>0$ *is arbitrary.*

Proof. The orbits of system (1) satisfy the following differential equation

$$\frac{dy}{dx}=\frac{-g(x)}{y-F(x)}.$$

Therefore, their orthogonal trajectories have the equation

$$-\frac{dx}{dy}=\frac{-g(x)}{y-F(x)},$$

or equivalently

$$\big(y-F(x)\big)dx-g(x)dy=0. \tag{4}$$

In order to make the equation (4) integrable, we multiply it by the integration factor $\mu=\dfrac{1}{g(x)}\exp\big(-\int_1^x\dfrac{d\eta}{g(\eta)}\big)$, then we get the exact equation $Du(x,y)=0$ where $u=u(x,y)$ is given in (3) and the proof is complete. □

Theorem 3.2. *Suppose that for some* $\delta>0$ *we have*

$$\limsup_{x\to0^+}\int_{\delta}^{x}\big(\frac{F(\xi)}{g(\xi)}-2\big)\exp\big(-\int_1^{\xi}\frac{d\eta}{g(\eta)}\big)d\xi=+\infty. \tag{5}$$

Then the Liénard system (1) *fails to have property* (Z_1^+).

Proof. If we assume that the system (1) has property (Z_1^+) then there exists a solution

$$(x(t),y(t))\quad t_0\le t<\infty,\quad x(t_0)=x_0,$$

hat initiates at the point $(x_0,F(x_0))$ on the vertical isocline $y=F(x)$ $(x>0)$, remains at the first quadrant and approaches the origin. By calculating $\dfrac{du}{dt}$ along with $(x(t),y(t))$, $t_0\le t<\infty$ we get

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}=-\frac{1}{g(x)}\exp\big(-\int_1^x\frac{d\eta}{g(\eta)}\big)\big[(\frac{dx}{dt})^2+(\frac{dy}{dt})^2\big]. \tag{6}$$

Since $\frac{dx}{dt}<0$ and $\frac{dy}{dt}=-g(x)<0$, then (6) gives

$$\frac{du}{dt}\le-\frac{2}{g(x)}(\frac{dx}{dt})(\frac{dy}{dt})\exp\big(-\int_1^x\frac{d\eta}{g(\eta)}\big)=2(\frac{dx}{dt})\exp\big(-\int_1^x\frac{d\eta}{g(\eta)}\big),$$

and by integration from t_0 to t we arrive at

$$u\big(x(t), y(t)\big) - K \leq \int_{t_0}^{t} 2\frac{\mathrm{d}x}{\mathrm{d}\tau} exp\Big(-\int_{1}^{x(\tau)} \frac{\mathrm{d}\eta}{g(\eta)}\Big)\mathrm{d}\tau = 2\int_{x_0}^{x(t)} \exp\Big(-\int_{1}^{\xi}\frac{\mathrm{d}\eta}{g(\eta)}\Big)\mathrm{d}\xi, \tag{7}$$

where $K = F(x_0)\exp\big(-\int_1^{x_0}\frac{\mathrm{d}\eta}{g(\eta)}\big)$. Now, since $y(t) > 0$, (3) together (7) give

$$\int_{x_0}^{x(t)} \frac{F(\xi)}{g(\xi)} \exp\Big(-\int_1^{\xi}\frac{\mathrm{d}\eta}{g(\eta)}\Big)\mathrm{d}\xi - K \leq 2\int_{x_0}^{x(t)} \exp\Big(-\int_1^{\xi}\frac{\mathrm{d}\eta}{g(\eta)}\Big)\mathrm{d}\xi,$$

or equivalently

$$\int_{x_0}^{x(t)} \Big(\frac{F(\xi)}{g(\xi)} - 2\Big)\exp\Big(-\int_1^{\xi}\frac{\mathrm{d}\eta}{g(\eta)}\Big)\mathrm{d}\xi \leq K.$$

This contradicts (5) and the proof is complete. □

Theorem 3.3. *Suppose that*

$$\limsup_{x\to 0^+} \int_{\delta}^{x} \Big(\frac{F(\xi)}{\sqrt{2G(\xi)}} - 2\Big)\frac{g(\xi)}{G(\xi)}\mathrm{d}\xi = +\infty, \tag{8}$$

for some $\delta > 0$. *Then Liénard system* (1) *fails to have property* (Z_1^+).

Proof. Define for $x > 0$ the function

$$u = u(x) := \sqrt{2G(x)},$$

and the mapping $\Lambda : R^+ \times R \to \big(0, \sqrt{2G(\infty)}\big) \times R$ by

$$\Lambda(x, y) = (u(x), y) \equiv (u, v),$$

Then the mapping Λ is a diffeomorphism of the right half-plane onto $\big(0, \sqrt{2G(\infty)}\big)$ that transforms the system (1) to the following simpler form of the Liénard system

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = v - F^*(u) \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}\tau} = -u,$$

in which $\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{g(x)}{\sqrt{2G(x)}}$ and $F^*(u) = F\Big(G^{-1}\big(\frac{u^2}{2}\big)\Big)$. Consequently, we have only to determine whether the system above, instead of (1), fails to have property (Z_1^+). Applying Theorem 3.2 to this system, it suffices to have

$$\limsup_{\eta\to 0^+} \int_{\delta}^{\eta} \frac{F^*(u) - 2u}{u^2}\mathrm{d}u = +\infty.$$

Now, the change of variable $u = \sqrt{2G(x)}$ in the integral above and the fact that $\sqrt{2G(x)} \downarrow 0$ as $x \to 0^+$, give the equivalent condition (8). □

The following corollary is an improvement of Theorem 1.2.

Corollary 3.4. *Suppose that*

$$F(x) \leq 2\sqrt{2G(x)} - h(\sqrt{2G(x)}),$$

where $h(u)$ *is a continuous function such that for some* $\delta > 0$

$$\limsup_{x\to 0^+} \int_{x}^{\delta} \frac{h(u)}{u^2}\mathrm{d}u = +\infty.$$

Then the Liénard system (1) *fails to have property* (Z_1^+).

Proof. We have

$$\begin{aligned}\int_\delta^x \Big(\frac{F(\xi)}{\sqrt{2G(x)}}-2\Big)\frac{g(\xi)}{G(\xi)}\mathrm{d}\xi &= 2\int_\delta^x \Big(F(\xi)-2\sqrt{2G(\xi)}\Big)\frac{g(\xi)}{[2G(\xi)]^{\frac{3}{2}}}\mathrm{d}\xi\\ &\geq \int_\delta^x h\Big(\sqrt{2G(\xi)}\Big)\mathrm{d}\Big(\frac{1}{\sqrt{2G(\xi)}}\Big)\\ &= \int_{\sqrt{2G(\delta)}}^{\sqrt{2G(x)}} h(u)\mathrm{d}\Big(\frac{1}{u}\Big)\\ &= \int_{\sqrt{2G(x)}}^{\sqrt{2G(\delta)}} \frac{h(u)}{u^2}\mathrm{d}u.\end{aligned}$$

This, together with the fact that $\sqrt{2G(x)} \downarrow 0$ as $x \to 0^+$ gives us

$$\limsup_{x\to 0^+} \int_\delta^x \Big(\frac{F(\xi)}{\sqrt{2G(\xi)}}-2\Big)\frac{g(\xi)}{G(\xi)}\mathrm{d}\xi = \limsup_{x\to 0^+}\int_x^\delta \frac{h(u)}{u^2}\mathrm{d}u = +\infty.$$

Now, Theorem 3.3 completes the proof. □

Example 3.5. *Consider the Liénard system* (1) *with*

$$g(x) = x \quad \text{and} \quad F(x) = 3x - 2.2\, x\, sin^2\Big(\frac{1}{x}\Big).$$

Here, we have $G(x) = \frac{x^2}{2}$. *Writing* $F(x) = 2x - h(x)$, *where* $h(x) = -x + 2.2\, x\, sin^2\Big(\frac{1}{x}\Big)$, *then we have*

$$\begin{aligned}\int_0^\delta \frac{h(u)}{u^2} &= \int_0^\delta \frac{-1+2.2\, sin^2\Big(\frac{1}{u}\Big)}{u}\mathrm{d}u\\ &= \int_0^\delta \frac{0.1-1.1\, cos\Big(\frac{2}{u}\Big)}{u}\mathrm{d}u\\ &= 0.1\int_0^\delta \frac{\mathrm{d}u}{u} + 1.1\int_{\frac{1}{\delta}}^{+\infty}\frac{cos u}{u}\mathrm{d}u = +\infty.\end{aligned}$$

Therefore, from Corollary 3.4 *system* (1) *fails to have property* (Z_1^+). *Notice that, in this example the function* $\frac{h(x)}{x}$ *is not non-decreasing in any interval* $(0, \delta)$ $(\delta > 0)$, *hence we cannot use Theorem* 1.2.

Remark 3.6. *The method in which* $F(x)$ *is compared with* $\sqrt{2G(x)}$ *to obtain sufficient conditions under which the system* (1) *has or fails to have property* (Z_1^+) *(and other similar properties) was initiated by A.F. Filippov [7] and improved by J. Sugie and T. Hara [11, 16]; see also [1–5, 10, 12–15, 17–20]. In the next results we use a new approach and in parallel with the above results, we compare* $F(x)$ *with* $2g(x)$, *instead of* $\sqrt{2G(x)}$ *(of course under some additional assumptions on* $g(x)$*).*

We start with a simple observation based on Hara's lemma.

Proposition 3.7. *Suppose that for some* $\delta > 0$,

$$g(x) \geq x, \quad 0 < x < \delta. \tag{9}$$

If we have

$$F(x) > 2g(x), \quad 0 < x < \delta, \tag{10}$$

then the Liénard system (1) *has property* (Z_1^+).

Proof. Taking $\psi(x) = g(x)$ then by assumptions (9) and (10) we have $g(x) < F(x) - \psi(x)$ for $0 < x < \delta$, and hence

$$\int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} \mathrm{d}\xi \leq \int_0^x \mathrm{d}\xi = x \leq g(x) = \psi(x), \quad 0 < x < \delta.$$

Now, Hara's lemma gives the desired result. □

Proposition 3.8. *Suppose that for some $\delta > 0$,*

$$\int_0^\delta \exp\Big(-\int_1^x \frac{d\eta}{g(\eta)}\Big)\mathrm{d}x = +\infty. \tag{11}$$

If we have

$$\limsup_{x\to 0^+} \frac{F(x)}{g(x)} < 2, \tag{12}$$

then the Liénard system (1) *fails to have property* (Z_1^+).

Proof. From (12) we have $\dfrac{F(\xi)}{g(\xi)} - 2 \leq -m < 0$, for some $m \in (0, 2)$ and all $\xi > 0$ sufficiently small. Thus using (11), we get

$$\limsup_{x\to 0^+} \int_\delta^x \Big(\frac{F(\xi)}{g(\xi)} - 2\Big) \exp\Big(-\int_1^\xi \frac{\mathrm{d}\eta}{g(\eta)}\Big)\mathrm{d}\xi \geq m \int_0^\delta \exp\Big(-\int_1^x \frac{\mathrm{d}\eta}{g(\eta)}\Big)\mathrm{d}x = +\infty,$$

and from Theorem 3.2 the proof is complete. □

Theorem 3.9. *Suppose that F and g satisfy*

$$\int_0^1 \frac{\mathrm{d}\eta}{g(\eta)} = +\infty, \tag{13}$$

and

$$F(x) \leq 2g(x) - h\Big(\exp \int_1^x \frac{\mathrm{d}\eta}{g(\eta)}\Big),$$

where h is a nonnegative function on $(0, \delta)$ for some $\delta > 0$, and satisfies

$$\limsup_{x\to 0^+} \int_x^\delta \frac{h(u)}{u^2}\mathrm{d}u = +\infty.$$

Then the Liénard system (1) *fails to have property* (Z_1^+).

Proof. Using the above assumptions we have

$$\begin{aligned}
\int_\delta^x \Big(\frac{F(\xi)}{g(\xi)} - 2\Big)\exp\Big(-\int_1^\xi \frac{\mathrm{d}\eta}{g(\eta)}\Big)\mathrm{d}\xi &= \int_\delta^x \big(F(\xi) - 2g(\xi)\big)\Big(\frac{1}{g(\xi)}\Big)\exp\Big(-\int_1^\xi \frac{\mathrm{d}\eta}{g(\eta)}\Big)\mathrm{d}\xi \\
&\geq \int_\delta^x h\Big(\exp\Big(\int_1^\xi \frac{\mathrm{d}\eta}{g(\eta)}\Big)\Big)\mathrm{d}\Big(\exp\Big(-\int_1^\xi \frac{\mathrm{d}\eta}{g(\eta)}\Big)\Big) \\
&= \int_{u(\delta)}^{u(x)} h(u)\mathrm{d}\Big(\frac{1}{u}\Big) \\
&= \int_{u(x)}^{u(\delta)} \frac{h(u)}{u^2}\mathrm{d}u,
\end{aligned}$$

where $u(x) = \exp\big(\int_1^x \frac{d\eta}{g(\eta)}\big)$. From (13) we also have $u(x) \downarrow 0$ as $x \to 0^+$, thus

$$\limsup_{x\to 0^+} \int_\delta^x \Big(\frac{F(\xi)}{g(\xi)} - 2\Big)\exp\Big(-\int_1^\xi \frac{d\eta}{g(\eta)}\Big)d\xi = \limsup_{x\to 0^+} \int_x^\delta \frac{h(u)}{u^2}du = +\infty,$$

and the proof is complete from Theorem 3.2. □

Theorem 3.10. *Suppose that for $x > 0$ sufficiently small, g is differentiable,*

$$1 \le g'(x) \le k - 1, \tag{14}$$

$$\int_0^1 \frac{d\eta}{g(\eta)} = +\infty, \tag{15}$$

and

$$g(x)\exp\Big(-\int_1^x \frac{d\eta}{g(\eta)}\Big) > l > 0. \tag{16}$$

If we have

$$F(x) \ge 2g(x) - h\Big(\exp\int_1^x \frac{d\eta}{g(\eta)}\Big), \tag{17}$$

where $h(u)$ is a non-negative continuous function such that for $u > 0$ sufficiently small

$$\frac{h(u)}{u} \quad \text{is non-decreasing}, \tag{18}$$

and

$$a[H(u)]^2 \le \frac{h(u)}{u} \quad \text{for some } a > \frac{k^2}{l}, \tag{19}$$

where $H(\xi) = \int_0^\xi \frac{h(u)}{u^2}du$, then the Liénard system (1) *has property* (Z_1^+).

Proof. Let $\rho = \lim_{x\to 0^+} \frac{h(x)}{x}$. Then by (18) $\rho \ge 0$ and $\frac{h(x)}{x} \ge \rho$ for $x > 0$. If $\rho > 0$, then for some $x_0 > 0$ we have $H(x_0) \ge \rho \int_0^{x_0} \frac{d\eta}{\eta} = \infty$, which contradicts (19). Therefore we have $\frac{h(x)}{x} \to 0$ and $H(x) \to 0$ as $x \to 0^+$. Define $u(x) = exp\big(\int_1^x \frac{d\eta}{g(\eta)}\big)$. From (15), $u(x) \downarrow 0$ as $x \to 0^+$. Let σ be chosen so that

$$0 < \sigma < 1 - \frac{k^2}{la}.$$

Then there exist $\delta > 0$ such that for $0 < x < \delta$,

$$\frac{g(x)}{u(x)} > l\,, \quad \frac{h(u(x))}{u(x)} < \frac{\sigma}{2k}l \quad \text{and} \quad H(u(x)) < \frac{\sigma}{2k(k+1)}l. \tag{20}$$

In Hara's lemma take $\psi(x) = g(x) + ku(x)H(u(x))$. Then we have

$$F(x) - \psi(x) \ge u(x)\Big[\frac{g(x)}{u(x)} - \frac{h\big(u(x)\big)}{u(x)} - kH\big(u(x)\big)\Big] > 0,$$

for $0 < x < \delta$. Define the function $\phi(x)$ by

$$\phi(x) = \psi(x) - \int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} \mathrm{d}\xi, \quad x > 0.$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi(x) \geq \frac{h(u(x))}{r(x)}\Big[1 - k(k+1)\Big(\frac{u(x)}{g(x)}\Big)H(x) - k\Big(\frac{u(x)}{g(x)}\Big)\frac{h\big(u(x)\big)}{u(x)} - k^2\Big(\frac{u(x)}{g(x)}\Big)\frac{H^2\big(u(x)\big)u(x)}{h\big(u(x)\big)}\Big],$$

hence by (20),

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi(x) \geq \frac{h\big(u(x)\big)}{r(x)}\Big(1 - \frac{\sigma}{2} - \frac{\sigma}{2} - \frac{k^2}{al}\Big) > 0, \quad 0 < x < \delta,$$

where $r(x) = g(x) - h(u(x)) - ku(x)H(u(x))$. Therefore, $\phi(x) > 0$ for $0 < x < \delta$ and Hara's lemma completes the proof. □

Example 3.11. *To present an example of a function g that satisfies all the required assumptions appearing in Theorem* 3.10, *consider the Liénard system* (1) *with* $g(x) = \Big(1 + \frac{1}{\ln^2(x)}\Big)x$, *on* $(0, 1)$. *Here, we have* $g(x) > x$ *and*

$$1 < g'(x) = 1 + \frac{1}{\ln^2(x)} - \frac{2}{\ln^3(x)} < 6, \quad \textit{for } x \in (0, \frac{1}{e}).$$

Also we have

$$\int_1^x \frac{\mathrm{d}\eta}{g(\eta)} = \ln x - \arctan(\ln x).$$

Therefore,

$$\lim_{x\to 0^+} g(x)\exp\Big(-\int_1^x \frac{\mathrm{d}\eta}{g(\eta)}\Big) = e^{-\frac{\pi}{2}} > 0,$$

$$\int_0^1 \frac{\mathrm{d}\eta}{g(\eta)} = +\infty \quad \textit{and} \quad \int_0^\delta \exp\Big(-\int_1^x \frac{\mathrm{d}\eta}{g(\eta)}\Big)\mathrm{d}x = +\infty,$$

so assumptions (14), (15) *and* (16) *are satisfied. Hence, for any function F which satisfies* (17), (18) *and* (19) *with a suitable h, Liénard system* (1) *has property* (Z_1^+).

References

[1] RP. Agarwal, A. Aghajani, V. Roomi, Existence of Homoclinic orbits for general planer dynamical system of Liénard type, Dyn. Contin. Discret. I. Series A: Mathematical Analysis 18 (2012) 267-284.

[2] A. Aghajani, A. Moradifam, The Generalized Liénard Equations, Glasgow Mathematical Journal 51, 03 (2009) 605-617.

[3] A. Aghajani, A. Moradifam, Intersection with the vertical isocline in the Liénard plane, Nonlinear Analysis, 68 (2008) 3475-3484.

[4] A. Aghajani, D. O'Regan, V. Roomi, Oscillation of solutions to second-order nonlinear differential equations of generalized Euler type, Electr. J. Differ. Equ. 2013, 185 (2013) 1-13.

[5] A. Aghajani, A. Moradifam, On the homoclinic orbits of the generalized Liénard equations, Applied Mathematics Letters 20, 3 (2007) 345-351.

[6] L. A. Cherkas, Conditions for a Liénard equation to have a center, Differential Equations, 12, 2 (1977) 201-206

[7] A.F. Filippov, Sufficient conditions for the existence of stable limit cycles of second order equations, Mat. Sbornik 30 (1952) 171-180.

[8] J.R. Graef, Boundedness and oscillation of solutions of the Liénard equation, Bull. Amer. Math. Soc. 77 (1971) 418-421.

[9] S.R. Grace, Oscillation theorems for nonlinear differential equations of second order, J. Math. Anal. Appl. 171 (1992), 220-241.

[10] M. Gyllenberg, P. Yan, New conditions for the intersection of orbits with the vertical isocline of the Liénard system, Mathematical and Computer Modelling 49, 56 (2009) 906-911.

[11] T. Hara, Notice on the Vinograd type theorems for Liénard system, Nonlinear Anal. 22 (1994) 1437-1444.

[12] T. Hara, T. Yoneyama, On the global center of generalized Liénard equation and its application to stability problems, Funkcial. Ekvac. 28 (1985) 171-192.
[13] P. Hasil, M. Vesel, Non-oscillation of half-linear differential equations with periodic coefficients, Electronic Journal of Qualitative Theory of Differential Equations 1 (2015) 1-21.
[14] M. Hayashi, Homoclinic orbits for the planar system of Liénard-type, Qualitative Theory of Dynamical Systems, 12, 2 (2013) 315-322.
[15] L. Herrmann, Oscillations for Liénard type equations, Journal de Mathématiques Pures et Appliquées 90, 1 (2008) 60-65.
[16] J. Sugie, T. Hara, Existence and non-existence of homoclinic trajectories of the Liénard system, Discrete and Continuous Dynam. Systems 2 (1996), 237-254.
[17] J. Sugie, T. Hara, Nonlinear oscillations of second-order differential equations of Euler type, Proc. Amer. Math. Soc. 124 (1996) 3173-3181.
[18] J. Sugie, N. Yamaoka, Oscillation of solutions of second-order nonlinear self-adjoint differential equations, J. Math. Anal. Appl. 291 (2004) 387-405.
[19] G. Villari and F. Zanolin, On a dynamical system in the Liénard plane. Necessary and sufficient conditions for the intersection with the vertical isocline and applications, Funkcial. Ekvac. 33 (1990) 19-38.
[20] X. Yang, YI. Kim, K. Lo, Sufficient conditions for the intersection of orbits with the vertical isocline of the Liénard system, Mathematical and Computer Modelling 57, 910 (2013) 2374-2377.
[21] H. Zeghdoudi, L. Bouchahed, R. Dridi, A complete Classification of Liénard equation, European Journal of Pure and Applied Mathematics 6, 2 (2013) 126-136.
[22] Z. Zhifen, D. Tongren, H. Wenzao and D. Zhenxi, Qualitative Theory of Differential Equations, Translations of Math. Monographs, Vol. 101, Amer. Math. Soc, Providence, 1992.