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Non-linear Water Waves (KdV) Equation by Painlevé Property and Schwarzian Derivative

Miodrag Mateljević[a], Attia .A.H Mostafa[a]

[a]*Faculty of Mathematics, University of Belgrade, Studentski trg 16, P. P. 550, 11000 Belgrade, Republic of Serbia*

Abstract. The Korteweg-de Vries (KdV) equation, a nonlinear partial differential equation which describes the motion of water waves, has been of interest since John Scott Russell (1834) [4]. In present work we study this kind of equation and through our study we found that the KdV equation passes Painleve's test, but we could not locate the solution directly, so we used Schwarzian derivative technique. Therefore, we were able to find two new exact solutions to the KdV equation. Also, we used the numerical method of Modified Zabusky-Kruskal to describe the behavior of these solutions.

1. Introduction

A nonlinear differential equation can be used to describe most scientific phenomena. It is normally a consequence of natural phenomena, such as water waves created by tides, wind or others [1, 3, 5, 10, 12, 13].In particular, by means of the Exp-function method and its generalization, Aslan [3], reports further exact traveling wave solutions, in a concise form, to the Schwarzian Korteweg-de Vries equation which admits physical significance in applications. Not only solitary and periodic waves but also rational solutions are observed. The goal of this study is to find the solution for the nonlinear wave (KdV) equation, even though definite solution is normally very complicated to find [11]. However, the analytic solution is possible to find by using the Painlevé analysis and Schwarzian derivative. The solution will be of great value for physicists, meteorologists and oceanographers for the purpose of reaching a better understanding.

2. Painlevé Analysis

For the Painlevé property see example [1, 5, 12, 13] and the literature cited there. A PDE has the Painlevé property when the solutions of the PDE are single valued about the movable, singularity manifolds. However, in the context of PDEs with d independent (complex) variables $z_1, ..., z_d$, the singularities of the solution no longer occur at isolated points but rather on an analytic hypersurface S of codimension one, defined by an equation $\phi(z) = 0$, $z = (z_1, ..., z_d) \in C^d$, where ϕ is analytic in the neighborhood of S. The hypersurface where the singularities lie is known as the singular manifold, and it can be used to define a natural extension of the Painlevé property for PDEs, which we state here in the form given by Ward.

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Email addresses: miodrag@matf.bg.ac.rs (Miodrag Mateljević), attia716@gmail.com (Attia .A.H Mostafa)

Definition: The Painlevé property for PDEs: If S is an analytic non-characteristic complex hypersurface in C^d, then every solution of the PDE which is analytic on $C^d \setminus S$ is meromorphic on C^d. With the above definition in mind, it is natural to look for the solutions of the PDE in the form of a Laurent-type expansion near $\phi(z) = 0$:

$$u(z) = \phi(z)^{-p} \sum_{j=0}^{\infty} u_j(z)\phi(z)^j . \tag{S1}$$

If the PDE has the Painlevé property, then the leading order exponent p appearing in the denominator of (S1) should be a positive integer, with the expansion coefficients u_j being analytic near the singular manifold $\phi(z) = 0$, and sufficiently many of these must be arbitrary functions together with the arbitrary non-characteristic function ϕ.

In this section we first outline that the KdV equation has the Painlevé property and then apply Painlevé property on the KdV equation:

$$u_t + \beta u^n u_x + \mu u_{xxx} = 0, \tag{S2}$$

where $n = \mu = 1$ and $\beta \in \mathbb{R} \setminus \{0\}$,
To verify that KdV has the Painlevé property we use a method for expanding a solution of a nonlinear PDE (here KdV) about a movable singularity manifolds (here a curve $\phi(x, t) = 0$).

The series solution of the partial differential equation is in the form [11]:

$$u = \sum_{j=0}^{\infty} u_j \phi^{j-p}, \tag{S3}$$

where ϕ is an analytic function that defines a non-characteristic hypersurface $S : \phi(x, t) = 0, (x, t) \in C^2$. First of all, observe that if $\phi_x \neq 0$ then locally we can apply the implicit function theorem and solve the equation $\phi(x, t) = 0$ for x. Thus we set $\phi(x, t) = x + \psi(t)$.

Some authors to determine whether equation (S2) satisfies Painlevé property use this simplified condition

$$\phi(x, t) = x + \psi(t) = 0, \tag{S4}$$

where ψ is an arbitrary function (ϕ is a characteristic of (S2) if $\partial\phi/\partial x \neq 0$) [12]. Then we can take the coefficients in the expansion (S1) to be functions of t only; this is referred to as the ,,reduced ansatz" of Kruskal. To find a value of equilibrium point p, by substituting (S3) into the equation (S2), where $u_t(t, x) = \partial u(t, x)/\partial t$, $u_x(t, x) = \partial u(t, x)/\partial x$ and $u_{xxx}(t, x) = \partial^3 u(t, x)/\partial x^3$ and by comparing the lowest powers in the produced series, we find $p = 2$. In the neighborhood of the singularity manifold (S4), the series solution (S3) becomes: $u = \sum_{j=0}^{\infty} u_j \phi^{j-2}$ where $u_0, u_1, \ldots,$ are arbitrary functions.
By associating the summation, we find the recursion relation,

$$\begin{aligned}(j-4)[j^2 - 5j + (6-\beta)]u_0^2 u_j = F_j(&\phi_t, \phi_x, \phi_{xx}, \phi_{xxx}, u_t,\\ &u_{0,x}, u_{1,x}, \ldots, u_{j-1,x}, u_{0,xx}, \ldots, u_{j-1,xx}, u_{0,xxx}, \ldots, u_{j-1,xxx}),\end{aligned} \tag{S5}$$

the coefficients of u_j are $(j-4)$ and $[j^2 - 5j + (6-\beta)]$, then, in the prevalence of the integer, resonance point is $j = 4$, and the another resonance points depend on the value of β. For instance, if $\beta = 12$ the resonance points will be $j = -1, 4, 6$, and if $\beta = 6$, the resonance points will be $0, 4, 5$.
Now, to find the value of u_j where $j = 0, 1, 2, \ldots$ from the series (S5) [10]:

$$\text{At } j = 0 \text{ then, } \quad u_0 = -\frac{12}{\beta}\phi_x^2. \tag{S6}$$

$$\text{At } j = 1 \text{ then, } \quad u_1 = \frac{12}{\beta}\phi_{xx}. \tag{S7}$$

$$\text{At } j=2 \text{ then, } \quad u_2 = -\frac{1}{\beta}\frac{\phi_t}{\phi_x} - \frac{4}{\beta}\frac{\phi_{xxx}}{\phi_x} + \frac{3}{\beta}\left(\frac{\phi_{xx}}{\phi_x}\right)^2. \tag{S8}$$

Since $p=2$, by using the truncation technique, and let $u_j = 0$, for all $j > 2$. Then $u = \sum_{j=0}^{\infty} u_j \phi^{j-2}$, becomes:

$$u = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2. \tag{S9}$$

This is the series solutions.

$$\text{At } j=3 \text{ then, } \quad u_3 = \frac{1}{\beta}\frac{\phi_{xt}}{\phi_x^2} + \frac{\phi_{xx}u_2}{\phi_x^2} + \frac{1}{\beta}\frac{\phi_{xxxx}}{\phi_x^2}. \tag{S10}$$

At $j=4$ and since $u_j = 0$ for all $j > 2$, we get $u_4 = 0$. Then the KdV equation (S2) satisfies the Painlevé property [12].
At $j=5$ in the equation (S5) and $u_j = 0$ for all $j > 2$, we get:

$$u_{2,t} + \beta u_2 u_{2,x} + u_{2,xxx} = 0. \tag{S11}$$

Then u_2 is also a solution of the KdV equation (S2)

3. Analytic Solution

In this section we use Schwarzian derivative technique so, we follow the steps to derive analytic solution [1]. They are invariant under the transformation,

$$H : \phi \longrightarrow \frac{a\phi + b}{c\phi + d} \quad \text{where} \quad ad - bc \neq 0.$$

Schwartzian derivative

$$S(\phi) = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2, \tag{S12}$$

dimension of velocity,

$$C(\phi) = -\frac{\phi_t}{\phi_x}. \tag{S13}$$

Furthermore, we define,

$$L(\phi) = -\frac{\phi_{xx}}{2\phi_x}. \tag{S14}$$

The relations,

$$L_t = CL_x - LC_x + \frac{1}{2}C_{xx} \text{ and } L_x = -L^2 - \frac{1}{2}S. \tag{S15}$$

The compatibility of C and S is given by,

$$S_t + C_{xxx} + 2C_x S + CS_x = 0. \tag{S16}$$

Now, by using the equations (S8) and (S10), we obtain:

$$\beta\phi_x u_3 = \frac{\phi_{xt}}{\phi_x} + \frac{\beta\phi_{xx}}{\phi_x}\left[-\frac{1}{\beta}\frac{\phi_t}{\phi_x} - \frac{4}{\beta}\frac{\phi_{xxx}}{\phi_x} + \frac{3}{\beta}\left(\frac{\phi_{xx}}{\phi_x}\right)^2\right] + \frac{\phi_{xxxx}}{\phi_x}.$$

Since, $u_j = 0$ for all $j > 2$, we get:

$$\frac{\phi_t\phi_{xx}}{\phi_x^2} - \frac{\phi_{xt}}{\phi_x} = \frac{\phi_{xxxx}}{\phi_x} - \frac{4\phi_{xx}\phi_{xxx}}{\phi_x^2} + 3\left(\frac{\phi_{xx}}{\phi_x}\right)^3. \tag{S17}$$

Then, by comparing both sides of the equation (S17) with equations (S12) and (S13), we observe:

$$C_x = S_x. \tag{S18}$$

Now, by using the equations (S12), (S13) and (S14) the equation (S8) becomes:

$$u_2 = \frac{1}{\beta}C - \frac{4}{\beta}S - \frac{12}{\beta}L^2. \tag{S19}$$

We derive the equation (S19), to find $u_{2,t}, \beta u_2 u_{2,x}$ and $u_{2,xxx}$ and substitute them into the equation (S11), and by using the relations (S15), and equation (S18) [10], we get:

$$C_t + C_{xxx} + 2C_xS + CC_x = 0. \tag{S20}$$

By comparing the equations (S16) with (S20), and using the equation (S18), we get:

$$C_t = S_t. \tag{S21}$$

Then by the equations (S18) and (S21), we get, $C = S + K$ where K is constant. For $K = 0$, we get:

$$C = S. \tag{S22}$$

By substituting $S = C$, into the equation (S16), we get:

$$S_t + 3SS_x + S_{xxx} = 0. \tag{S23}$$

This is Korteweg-de Vries like equation KdV.

4. The Schwarzian KdV Equation, Solutions

The Korteweg de Vries equation (KdV equation for short) is a mathematical model of waves on shallow water surfaces. It is particularly notable as the prototypical example of an exactly solvable model, that is, a non-linear partial differential equation whose solutions can be exactly and precisely specified. KdV can be solved by means of the inverse scattering transform. The mathematical theory behind the KdV equation is a topic of active research. The KdV equation was first introduced by Boussinesq (1877, footnote on page 360) and rediscovered by Diederik Korteweg and Gustav de Vries (1895)

(A1) Note that the coefficients in front of the three terms are somewhat arbitrary and were chosen for future notational simplicity, since we can rescale our coordinates by letting $u = a\tilde{u}$, $x = b\tilde{x}$, and $t = c\tilde{t}$ and the equation becomes $ac\tilde{u}_{\tilde{t}} + a^2b\tilde{u}\tilde{u}_{\tilde{x}} + ab^3\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0$. Hence by suitably choosing a, b, and c, we can obtain any real coefficients. Another observation that should be made about the KdV equation is that it is

(A2) Galilean invariant, meaning that if $u(x;t)$ is a solution, then so is $u(x-ct,t)+c$ for any $c \in \mathbb{R}$, as can be easily verified. Thus we obtain a one parameter family of (S2) solutions.

We can ask if there are any permanent wave solutions of the KdV equation of the form $u(x;t) = f(x-ct)$. Substituting this into equation (S2), in special case (where $\beta = 6$) , we obtain $f''' + 6ff' - cf' = 0$ which can be immediately integrated once to get $f'' + 3f^2 - cf = m$ for some constant m. Now multiplying by f' and integrating once more, $1/2(f')^2 + f^3 - c/2f^2 - 2mf = n$ for some other constant of integration n. This can actually be used to solve for f implicitly in terms of elliptic integrals, but we will consider only solutions where $f(x)$ decays sufficiently rapidly, which forces m and n to be zero.

The differential equation for f can now be integrated directly, and the result is, (see for example Karigiannis [7]): a soliton solution of (S2) is $u(x,t) = 3c\,sech^2(\frac{\sqrt{c}}{2}(x - ct - x_0)$. Note that since the solution

exists only for a wave speed $c > 0$, these solitary wave solutions always travel to the right, and the propagation speed is proportional to the wave amplitude, with larger waves moving faster. Numerical experiments in (1965) by Kruskal and Zabusky (see [2]), showed that when two solitary wave solutions pass through each other, they emerge with their shape unchanged and a relative phase shift. Since their interactions were particle like, these solutions were named solitons. Because nonlinear equations do not obey a superposition principle, special solutions like solitons were not expected to play a special role. However, the experiments by Kruskal and Zabusky also revealed that any solution of the KdV equation which vanishes asymptotically must in some sense be made up of a finite number of solitons.

Proposition-1. Solutions of the KdV equation that decay sufficiently rapidly ($u(\pm L, t) = u_x(\pm L, t) = u_{xx}(\pm L, t)$, $-L \leq x \leq L$, $0 \leq t$) are uniquely determined by the initial data. Let u, v, be two solutions of equation (S2), and let w = u-v. Substitution easily yields the equation $w_t + 6uw_x + 6wv_x + w_{xxx} = 0$. Note that $\int wu_{xxx} = 0$.

Set $E(t) = \int w^2$, $R(x) = 6\int(v_x - u_x/2)w^2$. If we multiply the equation by w and integrate over all x, then after integrating by parts on the second term and using the fact that w and all its x-derivatives go to zero, we obtain $E(t) = -R(x)$. Setting $M = sup|6v_x - 3u_x| < \infty$, we have $E(t) \leq E(0)e^{Mt}$. Since $E(0) = 0$, we have $E(t) \equiv 0$ and hence $w \equiv 0$, so $u = v$ for all times t. It is known that by (S9) we can find a solution of (S2) in the form $u = u_0\phi^{-2} + u_1\phi^{-1} + u_2$, where u_0, u_1, u_2 and ϕ are functions of (x, t), $\phi = \phi(x, t)$, such that u_2 is a solution of (S2).

In the relation (S9) we denote: $u = u_\phi = u_0\phi^{-2} + u_1\phi^{-1} + u_2$, where u_0, u_1 and u_2 are given respectively in (S6), (S7) and (S8).

Schwarzian is defined by $S(\phi) = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}(\frac{\phi_{xx}}{\phi_x})^2$. We write also $\{f; x\}$ instead of $S(f)(x)$.

Theorem. If $u = u_\phi$ is a solution of (S2), then $S = S(\phi)$ satisfies Korteweg-de Vries like equation KdV (S23).

Proposition-2. Let $A(w) = \frac{aw+b}{cw+d}$. If f is a solution of (S23) $C(f) = S(f)$ then $A \circ f$ is a solution of (S23) $C(A \circ f) = S(A \circ f)$. It is known that

(1) $S(A \circ f) = S(f)$.

Since $(A \circ f)_x = A'f_x$ and $(A \circ f)_t = A'f_x$,

(2) $C((A \circ f)) = C(f)$.

Thus (1) and (2) together provide us with a proof.

Proposition-3. In the next section. By using (S22) we consider $S = C = \pm 2\lambda^2$.

In the case $C = 2\lambda^2$, we show that there are two independent solutions $v_1(x) = e^{\lambda x}$ and $v_2(x) = e^{-\lambda x}$. Therefore, we will use Lemma-1 and Lemma-2 to be able to define that:

$\phi(x, t) = \frac{E(t)e^{\lambda x} + F(t)e^{\lambda x}}{M(t)e^{\lambda x} + N(t)e^{\lambda x}}$ is solution of $S(\phi) = 2\lambda^2$. Where $E(t)$, $F(t)$, $M(t)$ and $N(t)$ are arbitrary functions of t.

Moreover, if S is independent of t then (S23) is reduced to:

$$3SS_x + S_{xxx} = 0, \tag{S24}$$

there is a particular solution of this form: $f(x) = -4x^{-2}$ then $ff' = -32x^{-5}$ and $f'''(x) = 96x^{-5}$. Hence $3ff' + f'''$=0.

Then $S(\phi) = -4x^{-2}$ is a solution of (S24).

So we can find two independent solutions $v_3(t) = x^2$ and $v_4(t) = \frac{1}{x}$. Therefore, we define $\phi(x, t) = \frac{E(t)x^2 + F(t)x^{-1}}{M(t)x^2 + N(t)x^{-1}}$ is solution of $S(\phi) = -4x^{-2}$.

Conjecture . If $f = f(x, t)$ is a solution of (S23) and $\phi = \phi(x, t)$, which is continuous in t, for every fixed t, solution of $S(\phi(\cdot, t)) = f(\cdot, t)$, such that ϕ has partial derivative $\phi_t(x, t)$ and $S = C$, that $\phi_t(x, t) = f(x, t)\phi_x(x, t)$, then $u = u_\phi$ given by (S9) is a solution of (S2).

Note if we set $t = a(x, t) = \int \frac{dx}{f(x,t)}$, then using $-t = \int \frac{dx}{f(x,t)}$, we get that $t = a(x, t)$ is a solution of (S24).

5. Exact Solution

The constant functions $S = \pm 2\lambda^2$ where λ a constant, are solutions of the KdV like equation (S23).

Lemma-1: Let D be a simply connected domain in complex plan and P holomorphic functions on D. For given point $z_0 \in D$, the equation

$$\frac{d^2v}{dy^2} + P(y)v = 0, \tag{S25}$$

has a unique holomorphic solution V if we prescribe $v(z_0)$ and $v'(z_0)$. Let v_1 and v_2 be pair linearly independent solutions of the equation (S25), which are defined and holomorphic on simply connected domain D in complex plane, then $W(y) = v_1(y)/v_2(y)$ satisfies the equation [11],

$$\{W, Y\} = 2P(y), \tag{S26}$$

Conversely, if W(y) is a solution of (S26) at each point of D, then one can find pair linearly holomorphic independent solutions v_1 and v_2 of (S25) such that $W(y) = v_1(y)/v_2(y)$ in some neighborhood of $y_0 \in D$.

By the substitution $y = W''/W'$ equation (S26) transform to Riccati equation $y' - y^2/2 = P$. From this we obtain by substitution $y = -2v'/v$ equation (S25).

Lemma-2: The Schwarzian derivative is invariant under fractional linear transformation acting on the first argument [11], namely,

$$\left\{\frac{aW+b}{cW+d}; y\right\} = \{W; y\} \quad \text{where} \quad ad - bc \neq 0,$$

where a, b, c and d are constants. For $S = -2\lambda^2$, we have, $S = \{v, x\} = -2\lambda^2$. Hence $P(x) = -\lambda^2$ in (S26), and two linearly independent solutions are:

$$\Psi_1 = E(t)e^{\lambda x} + F(t)e^{-\lambda x} \quad , \quad \Psi_2 = M(t)e^{\lambda x} + N(t)e^{-\lambda x}$$

Therefore by Lemma -1 and Lemma -2, we get:

$$\phi(t, x) = \frac{E(t)e^{\lambda x} + F(t)e^{-\lambda x}}{M(t)e^{\lambda x} + N(t)e^{-\lambda x}} \quad \text{where} \quad EN - FM \neq 0, \tag{S27}$$

By using the equations (S13) and (S22), then:

$$C = S = -\frac{\phi_t}{\phi_x} = -2\lambda^2, \tag{S28}$$

Now, to find the equation of coefficients $E(t)$, $F(t)$, $M(t)$ and $N(t)$, we derive $\phi(x, t)$ in the equation (S27) to find $\phi_x(x, t)$ and $\phi_t(x, t)$ and substituting them into the equation (S28), the equation (S28) becomes:

$$C = \frac{[M(t)E'(t) - E(t)M'(t)]e^{2\lambda x} + [N(t)F'(t) - F(t)N'(t)]e^{-2\lambda x}}{-2\lambda[N(t)E(t) - M(t)F(t)]}$$

$$+\frac{M(t)F'(t) - F(t)M'(t) + N(t)E'(t) - E(t)N'(t)}{-2\lambda[N(t)E(t) - M(t)F(t)]} = -2\lambda^2.$$

This leads to the nonlinear ODE system in coefficients $E(t), F(t), M(t)$ and $N(t)$, then:

(I) $GE' - EM' = 0$

(II) $NF' - FN' = 0$

(III) $(MF' - FM') + (NE' - EN') = 4\lambda^3(NE - MF)$

The particular solutions of (I) and (II) are:

$$E(t) = A_1M(t) \quad \text{and} \quad F(t) = A_2N(t)$$

where A_1 and A_2 are real arbitrary constants. By substituting these into (III), we get:

$$A_2(M(t)N^{'}(t) - N(t)M^{'}(t)) + A_1(N(t)M^{'}(t) - M(t)N^{'}(t)) \\ = 4\lambda^3 N(t)M(t)(A_1 - A_2),$$

then:

$$\frac{N^{'}(t)}{N(t)} - \frac{M^{'}(t)}{M(t)} = -4\lambda^3.$$

By integrating the above, we get:

$$\frac{N(t)}{M(t)} = \exp(-4\lambda^3 t),$$

Then the equation (S27) becomes:

$$\phi(t,x) = \frac{A_1 M(t)\exp(\lambda x) + A_2 M(t)\exp(-4\lambda^3 t - \lambda x)}{M(t)\exp(\lambda x) + M(t)\exp(-4\lambda^3 t - \lambda x)},$$

which leads to:

$$\phi(t,x) = \frac{(A_1 + A_2)\cosh\lambda\eta + (A_1 - A_2)\sinh\lambda\eta}{2\cosh\lambda\eta},$$

where $\eta = x + 2\lambda^2 t$.

$$\text{Then:} \quad \phi(t,x) = K_1 + K_2\tanh\lambda\eta, \tag{S29}$$

where K_1 and K_2 are arbitrary constants, and $K_1 = (A_1 + A_2)/2$ and $K_2 = (A_1 - A_2)/2$. For $K_1 = 0$, and by substituting the equation (S29) into the equation (S8), we obtain:

$$\begin{aligned} u_2 = & -\frac{1}{\beta}\frac{2K_2\lambda^3\mathrm{sech}^2\lambda\eta}{K_2\lambda\mathrm{sech}^2\lambda\eta} + \frac{3}{\beta}\frac{4K_2^2\lambda^4\mathrm{sech}^4\lambda\eta\tanh^2\lambda\eta}{K_2^2\lambda^2\mathrm{sech}^4\lambda\eta} \\ & -\frac{4}{\beta}\frac{-2K_2\lambda^3\mathrm{sech}^4\lambda\eta + 4K_2\lambda^3\mathrm{sech}^2\lambda\eta\tanh^2\lambda\eta}{K_2\lambda\mathrm{sech}^2\lambda\eta}. \end{aligned}$$

Then:

$$u_2 = \frac{12\lambda^2}{\beta}\left(\mathrm{sech}^2\lambda\eta - \frac{1}{2}\right), \quad \text{where} \quad \eta = x + 2\lambda^2 t. \tag{S30}$$

Hence $u_2(x,t)$ is the first exact solution for KdV equation (S2).

Now, by the equations (S6), (S7), (S9) and (S29) we obtain:

$$\begin{aligned} u = & \frac{-12}{\beta}\frac{\phi_x^2}{\phi^2} + \frac{12}{\beta}\frac{\phi_{xx}}{\phi} + u_2, \\ = & \frac{-12}{\beta}\frac{K_2^2\lambda^2\mathrm{sech}^4\lambda\eta}{K_2^2\tanh^2\lambda\eta} - \frac{24}{\beta}\frac{K_2\lambda^2\mathrm{sech}^2\lambda\eta\tanh\lambda\eta}{K_2\tanh\lambda\eta} + u_2. \end{aligned}$$

Then:

$$u = -\frac{12\lambda^2}{\beta}\left(\mathrm{csch}^2\lambda\eta + \frac{1}{2}\right), \quad \text{where} \quad \eta = x + 2\lambda^2 t. \tag{S31}$$

Hence $u(x,t)$ is the second exact solution for KdV equation (S2).

6. Numerical Results

In this section, we will focus studies in the nonlinear part of equation (S2). Where β is the coefficient of $\beta u u_x$ and $\beta \in \mathbb{R} \setminus \{0\}$. We will discuss the behavior of the soliton solution when β takes different values on several times, to verify extent of their impact in the wave movement.

So, we compute the numerical solutions (S30) and (S31) by using modified Zabusky-Kruskal Finite Difference Method [2], and we applied them to the MATLAB program [6].
Consider the KdV equation (S2):

$$u_t + \beta u u_x + u_{xxx} = 0, \quad x \in [-L, L], \ \ t > 0, \tag{S32}$$

Case I

In this case we use the first solution (S30) for the above KdV equation.

$$u_2(x,t) = \frac{12\lambda^2}{\beta}\left[\operatorname{sech}^2\lambda(x + 2\lambda^2 t) - \frac{1}{2}\right],$$

where $\lambda = 1$ and $\beta \in \mathbb{R} \setminus \{0\}$,
and the initial condition:

$$u_2(x,0) = \frac{12\lambda^2}{\beta}\left[\operatorname{sech}^2\lambda(x) - \frac{1}{2}\right].$$

Figures 1 and 2 show the soliton solution, where $\beta = 1$ and $\beta = -1$ respectively, at the time $T = 1$ where $N = 2^{13}$ and $\Delta t = 3.8641e - 004$
Figure 3 shows the four soliton solutions at the times $T = 1$, $T = 3$, $T = 5$ and $T = 7$ and $\beta = 1$ where $N = 2^{13}$ and $\Delta t = 3.8641e - 004$.
Figure 4 shows four the soliton solutions, when ($T = 1$, $\beta = 1$), ($T = 3$, $\beta = 6$), ($T = 5$, $\beta = 12$) and ($T = 7$, $\beta = 24$), where $N = 2^{13}$ and $\Delta t = 3.8641e - 004$.

Case II

In this case we use the second solution (S31) for the equation (S32).

$$u(x,t) = -\frac{12\lambda^2}{\beta}\left[\operatorname{csch}^2\lambda(x + 2\lambda^2 t) + \frac{1}{2}\right],$$

where $\lambda = 1$ and $\beta \in \mathbb{R} \setminus \{0\}$,
and the initial condition:

$$u(x,0) = -\frac{12\lambda^2}{\beta}\left[\operatorname{csch}^2\lambda(x) + \frac{1}{2}\right],$$

Figures 5 and 6 show the soliton solution, where $\beta = -6$ and $\beta = 6$ respectively, at the time $T = 1$, where $N = 2^{13}$ and $\Delta t = 3.8641e - 004$.
Figures 7, 8 and 9 show the four soliton solutions at the times $T = 0.5$, $T = 1.0$, $T = 1.5$, $T = 2.0$ and $\beta = -1$, $\beta = -6$ and $\beta = -12$, respectively, where $N = 2^{13}$ and $\Delta t = 3.8641e - 004$.
Figure 10 shows the four soliton solutions, when ($T = 0.5$, $\beta = -24$), ($T = 1$, $\beta = -12$), ($T = 1.5$, $\beta = -6$) and ($T = 2$, $\beta = -1$). where $N = 2^{13}$ and $\Delta t = 3.8641e - 004$.

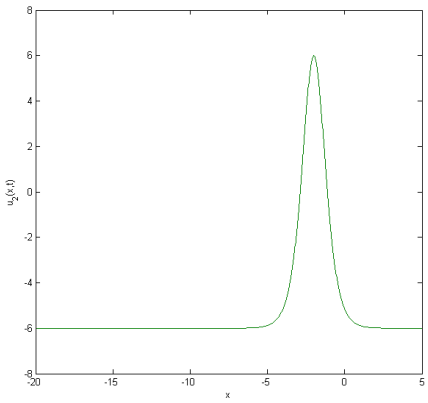

Figure 1: CASE I, where the amplitude soliton $\beta = 1$, at the time $T = 1$

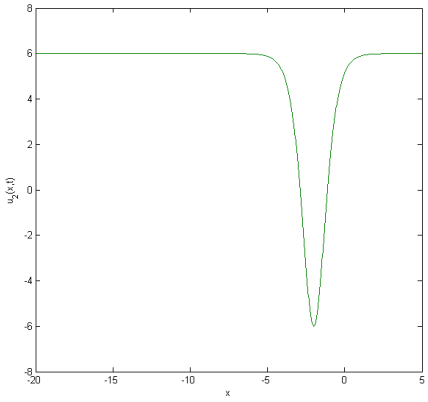

Figure 2: CASE I, where the amplitude soliton $\beta = -1$, at the time $T = 1$

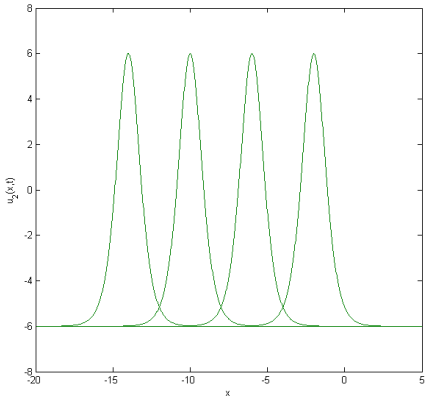

Figure 3: CASE I, where the amplitude soliton $\beta = 6$, at the times $T = 1$, $T = 3$, $T = 5$ and $T = 7$

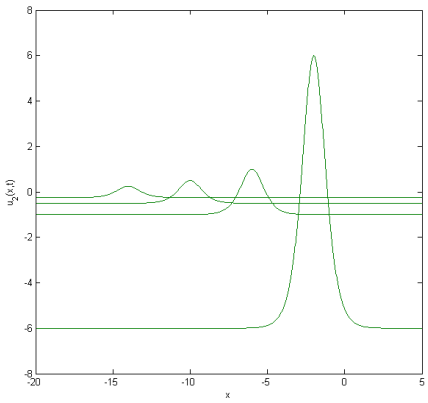

Figure 4: CASE I, where the amplitude soliton $\beta = 1,\ 6,\ 12,\ 24$, at the times $T = 1,\ T = 3,\ T = 5$ and $T = 7$

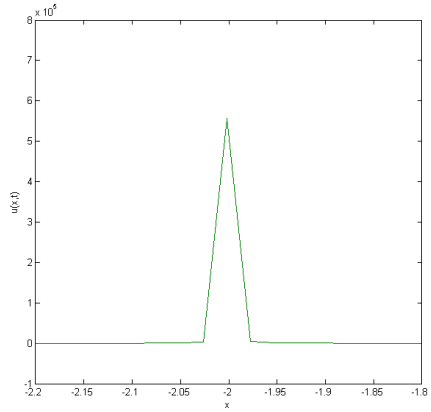

Figure 5: CASE II, where the amplitude soliton $\beta = -6$ at the time $T = 1$

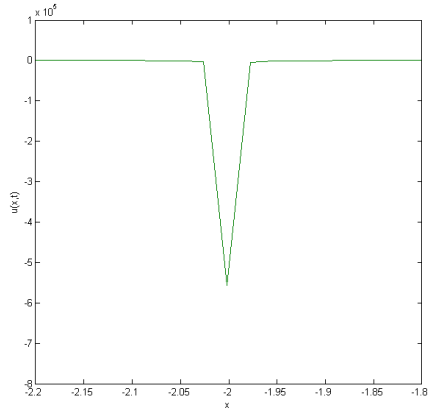

Figure 6: CASE II, where the amplitude soliton $\beta = 6$, at the time $T = 1$

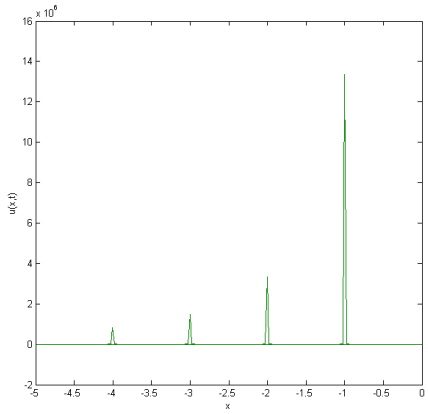

Figure 7: CASE II, where the amplitude soliton $\beta = -1$, at the times $T = 0.5,\ T = 1.0,\ T = 1.5$, and $T = 2.0$

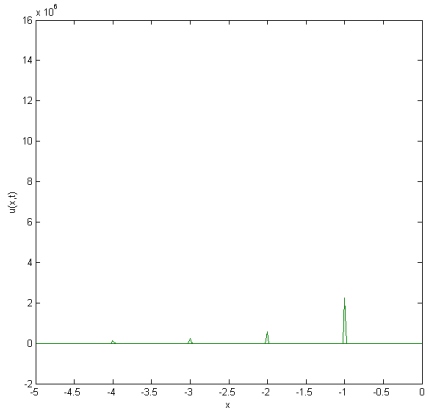

Figure 8: CASE II, where the amplitude soliton $\beta = -6$, at the times $T = 0.5,\ T = 1.0,\ T = 1.5$, and $T = 2.0$

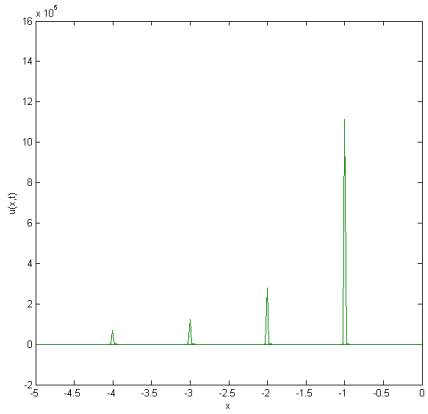

Figure 9: CASE II, where the amplitude soliton $\beta = -12$ at the times $T = 0.5,\ T = 1.0,\ T = 1.5$, and $T = 2.0$

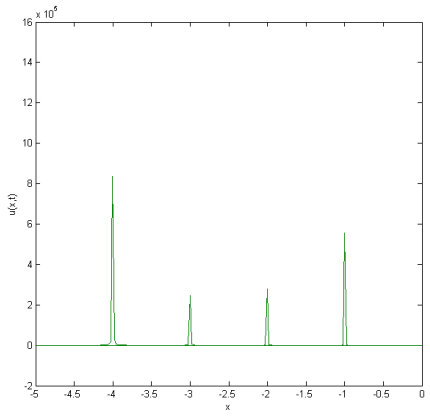

Figure 10: CASE II, where the amplitude soliton $\beta = -24, -12, -6, -1$, at the times $T = 0.5$, $T = 1.0$, $T = 1.5$, and $T = 2.0$

.

Conclusion

1- In (S28) when taking $S(\phi) = 2\lambda^2$ and following the previous steps, we get:

$$u = -\frac{12\lambda^2}{\beta}\left(\sec^2\lambda\eta_1 - \frac{1}{2}\right), \quad \text{where} \quad \eta_1 = x - 2\lambda^2 t,$$

and

$$u = -\frac{12\lambda^2}{\beta}\left(\csc^2\lambda\eta_1 - \frac{1}{2}\right), \quad \text{where} \quad \eta_1 = x - 2\lambda^2 t.$$

These are third and fourth exact solutions for KdV equation (S2).
In addition, when taking $S(\phi) = -\frac{4}{x^2}$ in (S28), we got new results.
2- We note that, in the first case: the waves are moderate and they maintain their shape with the alteration of time. On the other hand, these waves are affected by the value of the coefficient of the nonlinear part "β". When β increases, then the solitons increase. And when β takes a negative value, then the solitons appear in the inverted form.
In the second case: the soliton waves are tall and they take sharp a curve form. Also, these solitons are affected by the value of β as well as the time T. When β takes a positive value, then the solitons appear in the inverted form. (i.e The both cases are affected by the sign and value of β. Moreover, the CASE II is affected by T, too). We recommend the use of CASE II when $\beta \in \mathbb{R}^-$, and CASE I when $\beta \in \mathbb{R}^+$.

Future work

We have discussed the nonlinear part of equation (S2), when the coefficient $\beta \in \mathbb{R} \setminus \{0\}$ and the power $n = 1$.
In next studies, our goal will be to discuss the situation when the power of equation $n = 1, 2, \cdots$. And in the dispersal part μu_{xxx}, when $\mu \in \mathbb{R} \setminus \{0\}$. So we can observe behavior of soliton and wave motion on several options.

Appendix

Note that the coefficients in front of the three terms can be chosen arbitrary and for future notational simplicity, since we can rescale our coordinates by letting $u = a\tilde{u}$, $\tilde{x} = bx$, and $\tilde{t} = ct$, where a, b and c are

constants different from 0. So we have $u(x,t) = a\tilde{u}(bx, ct)$ and the equation becomes $ac\tilde{u}_{\tilde{t}} + a^2b\tilde{u}\tilde{u}_{\tilde{x}} + ab^3\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0$. If $a \neq 0$ we have $c\tilde{u}_{\tilde{t}} + ab\tilde{u}\tilde{u}_{\tilde{x}} + b^3\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0$. Hence by suitably choosing a, b, and c, we can obtain any real coefficients. Consider the modified Kortewegde Vries equation (mKdV)
$u_t + 6u_x^2 + u_x''' = 0$
which arises in many physical problems (e.g. plasma physics, fluid dynamics, non-linear optics). As t tends ∞, the solution decomposes into finite amplitude waves, called solitons (cf. Soliton), and a dispersive tail. The tail is dominated by the similarity solution of the modified Kortwegde Vries equation:
$u = (3t)^{-1/3}w(z), z = (3t)^{-1/3}x$,
where $w = w(z)$ satisfies: $w_z''' + 6w^2w_z' - (zw_z' + w) = 0$. The latter equation integrates to the second Painlevé transcendent P_{II} discussed above. he latter equation integrates to the second Painlevé transcendent discussed above.

In fact, Ablowitz and Segur showed that many of the well-known integrable non-linear wave equations solvable by the inverse scattering transform have similarity reductions which reduce to equations of Painlevé type. For example, a similarity solution of the Boussinesq equation reduces to the first Painlevétranscendent, and a similarity solution of the sine-Gordon equation can be transformed to the third Painlevé equation. Subsequently, researchers have shown that all of the classical second-order Painlevé equations (cf. also Painlevé equation) can be obtained as reductions of integrable partial differential equations (a large number of references may be found in encyclopediaofmath).

The connections are much deeper than to the classical Painlevé equations. In fact, the integrable partial differential equations solvable by the inverse scattering transform have two general features that result. Firstly, their ordinary differential equation reductions have the Painlevé property, i.e. they do not have movable critical points. Although there is not a general proof of this statement available, nevertheless it can be proven that: 1) a subclass of solutions obtained via the linear integral equations of the inverse scattering transform satisfy this property once an ordinary differential equation is obtained there is a test, called the Painlevé test or, more generally, Painlevé analysis, which can be applied to establish that there is a full-dimensional family of solutions which have Laurent series representations in a neighborhood of pole-type singularities (or sometimes more general singularities). This analysis can be generalized to apply, formally speaking, to partial differential equations .

6.1. Schwarzian

The Schwarzian plays an important role in the theory of univalent functions, conformal mapping and Teichmüller spaces, cf. [8, 9]. In this subsection we only outline some properties of Schwarzian. Let $f : \mathbb{H} \to \mathbb{C}$ be conformal, then the *Schwarzian derivative* is

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

for $f' \neq 0$. Sometimes we write $\hat{S}(f)$, $S[f]$ or simply $[f]$ if it is convenient.

$S_f \equiv 0 \Leftrightarrow f$ is a Möbius transformation.

If A is Möbius transformation, a direct computation shows that $S_A = 0$.

If $S_f = 0$, by the substitution $y = f''/f'$, the equation is transformed to the Riccati equation $y' - y^2/2 = 0$. From this, we obtain, by the standard substitution $y' = -2w'/w$, the linear second order equation $w'' = 0$. Solving this differential equation shows that f is Möbius transformation. If we introduce

$$T_f = \frac{f''}{f'},$$

we have

$$S_f = T_f' - \frac{1}{2}T_f^2 = \left(\frac{f'''}{f'}\right) - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

Let f be holomorphic in a neighborhood of the origin and suppose $f'(0) \neq 0$. Let L be the best approximating Möbius transformation to f at the origin. By this we mean that $L^{-1} \circ f$ has a Taylor expansion of the form $L^{-1} \circ f(z) = z + b_3 z^3 + b_4 z^4 + \cdots$. Then $S[f](0) = b_3$.

Let first assume that f is holomorphic in a domain G in the complex plane and $f'(z) \neq 0$ in G. If, in addition, $f(z) \neq 0$, we find $\hat{S}(f) = \hat{S}(1/f)$. We use this formula to define Schwarzian derivative for meromorphic locally injective functions.

Consider $F = f \circ g$, where g is conformal. Direct computation gives the transformation rule
$\hat{S}(f \circ g) = (\hat{S}(f) \circ g)(g')^2 + \hat{S}(g)$.
Hence, for $f = A$, a linear transformation, $\hat{S}(A \circ g) = \hat{S}(g)$ and for $f = A$, we have
$\hat{S}(f \circ A) = (\hat{S}(f) \circ A)(A')^2$.

Lemma 6.1. *Let φ be a holomorphic in a simply connected domain D in the complex plane. Hence there is a meromorphic function f in D such that $S_f = \varphi$. The solution is unique up to an arbitrary Mobius transformation.*

By the substitution $y = f''/f'$, the equation is trasformed to the Riccati equation $y' - y^2/2 = \varphi$. From this, we obtain, by the standard substitution $y' = -2w'/w$, the linear second order equation

$$w'' + \frac{1}{2}\varphi w = 0\,. \tag{S33}$$

For given a point $z_0 \in D$, equation (S33) has a unique holomorphic solution w locally (in a neighborhood of z_0), once we prescribe the values $w(z_0)$ and $w'(z_0)$. We left to reader to verify this directly using power series. Since D simply connected, an application of the Monodromy theorem gives a global solution of (S33) by analytic continuation.

Let w_1 and w_2 be two linearly independent solution of (S33).

Since $w_1 w_2'' - w_1'' w_2 = 0$, we have $w_1 w_2' - w_1' w_2 = c = const$. Set $f = w_1/w_2$. Direct computation yields

$f' = (w_1 w_2' - w_1' w_2)/w_2^2 = c w_2^{-2}$, $f''/f' = -2w_2'/w_2$, and so $S_f = -2w_2''/w_2$.

If f and g are solutions, we can define locally $A = g \circ f^{-1}$. Since $S_A = 0$, we get that $g = A \circ f$ locally, where A is a Möbius transformation, but then it is true in D, and the uniqueness part of the theorem is proved.

Further, we need here only a corollary of Bieberbach area inequality: If $F(\zeta) = \zeta + \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} + \cdots$ is univalent in $\mathbb{E} = \{|\zeta| > 1\}$, then

(I-1) $|b_1|^2 + 2|b_2|^2 + \cdots + n|b_n|^2 + \cdots \leq 1$.

An immediate corollary of (I-1), is

(I-2) $|b_1| \leq 1$.

Remark: The Bieberbach conjecture (proved by De Branges (1985)) states that if $f : \mathbb{D} \to \mathbb{C}$ conformal(holomorphic and injective), $f(0) = 0$, $f'(0) = 1$, then $|a_n| \leq n$, where $|a_n|$ are the Taylor coefficients of f. Bieberbach (1916) proved $|a_2| \leq 2$.

By $\mathbb{H}^- = \{z = x + iy : y < 0\}$ we denote the upper -half plane. Let $Q(\mathbb{H}^-)$ denote the space of all function φ holomorphic in $\mathbb{H}^-$ for which the hyperbolic sup norm $\|\varphi\|_\infty = \|\varphi\|_{hyp,\infty} \sup y^2|\varphi(z)|$, $z + x + iy$, is finite.

Theorem 6.2 (Nehari). *If $f : \mathbb{H}^- \to \mathbb{C}$ conformal, $\|S_f\|_\infty \leq 3/2$, $S_f \in Q(\mathbb{H}^-)$.*

Proof. Substituting expression for f', F'' and F''' in $[F]$, one gets

$[F](\zeta)\zeta^4 = -6b_1 + o(1)$, when $\zeta \to \infty$. Hence

(I-3) $\lim_{\zeta\to\infty} |[F](\zeta)\zeta^4| \leq 6$.

Let $z_0 = x_0 + iy_0$, $y_0 < 0$. In order to evaluate $[f](z_0)$, set $\zeta = Lz = \frac{z-\overline{z_0}}{z-z_0}$. L maps $\mathbb{H}^-$ onto $\mathbb{E}$ and therefore L^{-1} maps $\mathbb{E}$ onto $\mathbb{H}^-$.

Consider $F = f \circ L^{-1}$, which is univalent on $\mathbb{E}$.

Then $f = F \circ L$ and

$[f] = ([F] \circ L)L'^2$. Since, $L = 1 + \frac{2iy_0}{z-z_0}$, here we have

$$L' = -\frac{2iy_0}{(z-z_0)^2},$$

$$L \sim \frac{2iy_0}{z-z_0}, (z \to z_0) \text{ and}$$

$$L'^2 \sim -\frac{1}{4y_0^2} L^4 \, (z \to z_0).$$

Since $z \to z_0$ corresponds to $\zeta = Lz \to \infty$, by the limit we find

$$[f](z_0) = -\frac{1}{4y_0^2} \lim_{z \to z_0} \left(L^4(z)[F](L(z)) \right) = -\frac{1}{4y_0^2} \lim_{\zeta \to \infty} [F]\zeta^4$$

and then, by (I-3), we have $y^2|[f](z)| \leq 3/2$ for $z = x + iy$. □

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