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A Remark on the Ball-Covering Property of Product Spaces

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Abstract. In this paper, we prove that Banach spaces X and Y have the ball-covering property (BCP) if and only if $(X \times Y, \|\cdot\|_p)$ have the BCP, where $1 \le p \le \infty$.

1. Introduction

The study of geometric and topological properties of unit balls of normed spaces has played an important role in the geometry of Banach spaces. Almost all properties of Banach spaces, such as convexity, smoothness, reflexivity and the Radon-Nikodym property, can be viewed as the corresponding properties of their unit ball.

Starting with a different viewpoint, Cheng [1] introduced a notion of a ball-covering property for Banach spaces: A Banach space is said to have the ball-covering property (BCP, in short) if its unit sphere can be contained in the union of countably many closed (open) balls off the origin. Since every open ball is a countable union of closed balls, and every closed ball off origin is contained in an open ball off the origin with almost the same radius, it does not matter much to consider ball-coverings by open balls or by closed ones. In this paper we will speak about ball-coverings by closed balls, and symbol $B(x,r)$ ($B^{\circ}(x,r)$, respectively) denote the closed (open, respectively) ball with center x and radius r.

Clearly, a collection of countable balls $\left\{B(x_n, r_n)\right\}$ forms a ball-covering if and only if it satisfies: (1)$S_X \subset \bigcup B(x_n, r_n)$ and (2) $\|x_n\| > r_n$ for all n. In the recent years, the BCP of Banach spaces and its applications have been intensively studied (see, for instance, [1–13]).

It is easy to see that all separable spaces possess the BCP, but the converse version is not true. l_∞ [1] is a typical example of non-separable space with BCP. In [3], Cheng, Cheng and Liu constructed many equivalent norms on l_∞ such that l_∞ does not admit the BCP under these norms. It is well known [1] that if X has the BCP, then X^* is w^*-separable. Furthermore, Cheng, Shi and Zhang [6], Fonf and Zanco [8] showed independently that if X^* is w^*-separable, then for every $\varepsilon > 0$ there is a $(1+\varepsilon)$-equivalent norm on X such that X has the BCP. All these results imply that BCP is not invariant under isomorphic mappings.

For Gâteaux differentiable spaces (GDS) X, Cheng [2] proved that X has the BCP if and only if there exists a sequence $\{x_n^*\}$ of w^*-exposed points of B_{X^*} such that $\{x_n^*\}$ positively separates points of X. When considering product spaces $X\times Y$, by studying the sequence of w^*-exposed points which positively separates points of $X \times Y$, Shang [13] proved that Gâteaux differentiable spaces X and Y have the BCP if and only if

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$(X \times Y, \|\cdot\|_p)$ have the BCP, where $1 \le p \le \infty$. This means that for Gâteaux differentiable spaces X and Y, the BCP of $X \times Y$ is invariant under all the norms $\|\cdot\|_p$, where $1 \le p \le \infty$. In this paper, by direct constructions of countable ball-coverings, we point out that the assumption "Gâteaux differentiable space" in the Shang's theorem can be removed. The paper is organized as follows. In Section 1, some definitions and notations are collected. In Section 2, we prove that Banach spaces X and Y have the BCP if and only if $(X \times Y, \|\cdot\|_p)$ have the BCP, where $1 \le p \le \infty$.

In all the text below, X and Y are real Banach spaces and the unit sphere of X is denoted by S_X.

2. Main result

First, we give two lemmas.

Lemma 1. Suppose that $0 < t$ and $s < r$. Let $t' > 0$ with $|t - t'|$ sufficiently small, then $|t - t'| + t's < t'r$.

Proof. We only need to note that the inequality is equivalent to $\frac{|t-t'|}{t'} < r - s$, and that $\lim_{t' \to t} \frac{|t-t'|}{t'} = 0$.

Lemma 2. Let $0 \le \alpha_1 < \beta_1$ and $0 \le \alpha_2 < \beta_2$, then

$$\inf_{t \in [0,1]} \Big(\|(t\beta_1, (1-t^p)^{\frac{1}{p}}\beta_2)\|_p - \|(t\alpha_1, (1-t^p)^{\frac{1}{p}}\alpha_2)\|_p \Big) > 0,$$

where $1 \le p < \infty$ and $\|(a_1, a_2)\|_p = \Big(|a_1|^p + |a_2|^p\Big)^{\frac{1}{p}}$ for scalars a_1 and a_2.

Proof. Let $f(t) = \|(t\beta_1, (1-t^p)^{\frac{1}{p}}\beta_2)\|_p - \|(t\alpha_1, (1-t^p)^{\frac{1}{p}}\alpha_2)\|_p$. By noting that $f(t) > 0, t \in [0,1]$ and $f(t)$ is continuous on the closed interval [0,1] which is compact, we can complete the proof.

The following is our main theorem.

Theorem 1. Suppose that X and Y are Banach spaces. Then the following statements are equivalent:
(1) X and Y have the BCP;
(2) Prodcut spaces $(X \times Y, \|\cdot\|_p)$ have the BCP, where $\|(x,y)\|_p = (\|x\|^p + \|y\|^p)^{\frac{1}{p}}, 1 \le p < \infty$ and $\|(x,y)\|_\infty = \max\{\|x\|, \|y\|\}$.

Proof. (2)$\Longrightarrow$ (1). Let $\Big\{B\Big((x_n, y_n), r_n\Big)\Big\}$ be a countable ball-covering of $S_{(X \times Y, \|\cdot\|_p)}$. Since $\Big\{B\Big((x_n, y_n), r_n\Big)\Big\}$ dose not contain the origin, we have that for every n, $\|(x_n, y_n)\|_p > r_n$. Next, we will divide the proof into two cases.

Case I. $p = \infty$. First, for those n with $\|y_n\| = \|(x_n, y_n)\|_\infty$, we have

$$B\Big((x_n, y_n), r_n\Big) \bigcap \Big(X \times \{0\}\Big) = \emptyset.$$

Indeed, under this condition, $\|(x, 0) - (x_n, y_n)\|_\infty = \max\{\|x - x_n\|, \|y_n\|\} \ge \|y_n\| = \|(x_n, y_n)\|_\infty > r_n$. Hence, $(x, 0) \notin B\Big((x_n, y_n), r_n\Big)$.

On the other side, if $\|x_n\| > \|y_n\|$ and $(x, 0) \in B\Big((x_n, y_n), r_n\Big)$, then we have

$$\|x - x_n\| \le \|(x, 0) - (x_n, y_n)\|_\infty \le r_n.$$

This means that $x \in B(x_n, r_n)$ and $\|x_n\| = \|(x_n, y_n)\|_\infty > r_n$.

Now, let $A = \{n : \|x_n\| > \|y_n\|\}$. By the above discussion, we have that $\Big\{B(x_n, r_n), n \in A\Big\}$ is a countable ball-covering of S_X. Therefore X has the BCP. Similarly, we obtain that Y has the BCP.

Case II. $1 \le p < \infty$. As in Case I, for those n with $\|y_n\|^p > r_n^p$, we have

$$B\Big((x_n, y_n), r_n\Big) \bigcap \Big(X \times \{0\}\Big) = \emptyset.$$

Indeed, under this condition, $\|(x,0)-(x_n,y_n)\|_p^p = \|x-x_n\|^p + \|y_n\|^p > r_n^p$. Hence, $(x,0) \notin B\Big((x_n,y_n),r_n\Big)$.

Moreover, for those n with $\|y_n\|^p \leq r_n^p$, we have that $0 \leq r_n^p - \|y_n\|^p < \|x_n\|^p$. Hence, we can pick some r_n' such that

$$0 \leq r_n^p - \|y_n\|^p < {r_n'}^p < \|x_n\|^p.$$

Now, if $(x,0) \in B\Big((x_n,y_n),r_n\Big)$, *i.e.* $\|(x,0)-(x_n,y_n)\|_p^p = \|x-x_n\|^p + \|y_n\|^p \leq r_n^p$, we obtain that

$$\|x-x_n\|^p \leq r_n^p - \|y_n\|^p < {r_n'}^p,$$

which means that $x \in B(x_n, r_n')$.

Let $A = \{n : \|y_n\|^p \leq r_n^p\}$ and it is easy to see that $\Big\{B(x_n,r_n'), n \in A\Big\}$ is a countable ball-covering of S_X. Therefore X has the BCP. Similarly, Y has the BCP.

(1)$\Longrightarrow$ (2). Suppose that X and Y have the BCP, we will prove that $(X \times Y, \|\cdot\|_p)$ also have the BCP for $1 \leq p \leq \infty$. Let $\Big\{B(x_n,r_n)\Big\}$ and $\Big\{B(y_m,s_m)\Big\}$ be countable ball-coverings of S_X and S_Y respectively. Since for every n and m, $\|x_n\| > r_n, \|y_m\| > s_m$, we may assume that $S_X \subset \bigcup B^\circ(x_n,r_n)$ and $S_Y \subset \bigcup B^\circ(y_m,s_m)$ by taking r_n and s_m a little bigger. First, Let $\{t_k\} \supset \{0,1\}$ be a dense sequence in $[0,1]$. Next, we will divide the proof into two cases.

Case I. $p = \infty$. For every n, m and k, Let

$$B_{n,m,k} = B\Big((x_n, t_k y_m), \max\{r_n, t_k s_m\}\Big),$$

and

$$B^*_{n,m,k} = B\Big((t_k x_n, y_m), \max\{t_k r_n, s_m\}\Big).$$

Note that

$$\|(x_n, t_k y_m)\|_\infty = \max\{\|x_n\|, t_k\|y_m\|\} > \max\{r_n, t_k s_m\},$$

and

$$\|(t_k x_n, y_m)\|_\infty = \max\{t_k\|x_n\|, \|y_m\|\} > \max\{t_k r_n, s_m\}.$$

This implies that the countable closed ball collection $\Big\{B_{n,m,k}, B^*_{n,m,k}\Big\}$ dose not contain the origin of $X \times Y$. Moreover, we will show that $S_{(X\times Y,\|\cdot\|_\infty)} \subset \bigcup \Big\{B_{n,m,k}, B^*_{n,m,k}\Big\}$.

Without loss of generality, we only prove that $(x,ty) \in \bigcup B_{n,m,k}$, where $\|x\| = \|y\| = 1$ and $0 \leq t \leq 1$. By hypothesis, there exist some n and m such that $\|x-x_n\| < r_n$ and $\|y-y_m\| < s_m$. For $t=0$, we have that

$$\|(x,0)-(x_n, 0\cdot y_m)\|_\infty = \|x-x_n\| < r_n = \max\{r_n, 0\cdot s_m\},$$

i.e. $(x,0) \in B_{n,m,0}$. For $0 < t \leq 1$, by Lemma 1, we can take some t_k with $|t-t_k|$ sufficiently small such that

$$|t-t_k| + t_k\|y-y_m\| < t_k s_m.$$

Thus

$$\begin{aligned}\|(x,ty)-(x_n,t_k y_m)\|_\infty &= \max\{\|x-x_n\|, \|ty - t_k y_m\|\} \\ &\leq \max\{\|x-x_n\|, |t-t_k| + t_k\|y-y_m\|\} \\ &< \max\{r_n, t_k s_m\}.\end{aligned}$$

This implies that $(x,ty) \in B_{n,m,k}$.

Case II. $1 \leq p < \infty$. For every n, m and k, Let

$$\theta_{n,m,k} = (t_k x_n, (1-t_k^p)^{\frac{1}{p}} y_m),\ \gamma_{n,m,k} = \Big(t_k^p r_n^p + (1-t_k^p)s_m^p\Big)^{\frac{1}{p}},$$

and

$$B_{n,m,k} = B\Big(\theta_{n,m,k}, \gamma_{n,m,k}\Big).$$

Note that

$$\|\theta_{n,m,k}\|_p^p = t_k^p\|x_n\|^p + (1-t_k^p)\|y_m\|^p > \gamma_{n,m,k}^p.$$

This implies that the countable closed ball collection $\Big\{B_{n,m,k}\Big\}$ dose not contain the origin of $X \times Y$. And, we will show that $S_{(X\times Y,\|\cdot\|_p)} \subset \bigcup\Big\{B_{n,m,k}\Big\}$.

Suppose that $\theta = \Big(tx, (1-t^p)^{\frac{1}{p}}y\Big) \in S_{(X\times Y,\|\cdot\|_p)}$, where $\|x\| = \|y\| = 1$ and $0 \le t \le 1$. By hypothesis, there exist some n and m such that $\|x - x_n\| < r_n$ and $\|y - y_m\| < s_m$. Then we can take some t_k with $|t - t_k|$ sufficiently small such that

$$\begin{aligned}
\|\theta - \theta_{n,m,k}\|_p &\le \|\theta - \Big(t_kx, (1-t_k^p)^{\frac{1}{p}}y\Big)\|_p + \|\Big(t_kx, (1-t_k^p)^{\frac{1}{p}}y\Big) - \theta_{n,m,k}\|_p \\
&= \Big(|t-t_k|^p + \Big|(1-t^p)^{\frac{1}{p}} - (1-t_k^p)^{\frac{1}{p}}\Big|^p\Big)^{\frac{1}{p}} + \Big(t_k^p\|x-x_n\|^p + (1-t_k^p)\|y-y_m\|^p\Big)^{\frac{1}{p}} \\
&< \Big(t_k^pr_n^p + (1-t_k^p)s_m^p\Big)^{\frac{1}{p}} \qquad\qquad (\star) \\
&= \gamma_{n,m,k},
\end{aligned}$$

where the inequality $(\star)$ can be deduced from Lemma 2 by letting $\alpha_1 = \|x-x_n\|, \beta_1 = r_n, \alpha_2 = \|y-y_m\|, \beta_2 = s_m$. Thus, we complete the proof.

Remark 1. It can be seen from Theorem 1 that the BCP of $X\times Y$ is invariant under all the norms $\|\cdot\|_p$, where $1 \le p \le \infty$. But this result does not hold any more if we consider all equivalent norms on $X \times Y$ other than all the norms $\|\cdot\|_p$. To see this, We can take $X = Y = (l_\infty, \|\cdot\|_\infty)$. Then X and Y have the BCP (see [1]). However, it is well known [3] that there are many equivalent norms on $X \times Y = l_\infty$ such that l_∞ does not possess the BCP under these norms.

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