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Characteristic Properties of Scattering Data of a Boundary Value Problem

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Abstract. Consider the differential equation

$$-y'' + q(x)y = \lambda^2\rho(x)y, \quad 0 < x < \infty \tag{1}$$

with boundary condition

$$-(\alpha_1 y(0) - \alpha_2 y'(0)) = \lambda^2(\beta_1 y(0) - \beta_2 y'(0)). \tag{2}$$

Here $q(x)$ is a real valued function such that

$$\int_0^\infty (1+x)|q(x)|dx < \infty$$

and $\rho(x)$ is a real valued piecewise continuous function. It is known that the boundary value problem (3)-(4) has only finite number of simple negative eigenvalues $-\mu_1^2, \cdots, -\mu_n^2, (\mu_j > 0)$ and the half axis constitutes absolutely continuous spectrum. For normalized eigenfunctions of the problem (3)-(4) we have the asymptotic formulae as $x \to \infty$

$$u_j(x) \sim m_j e^{-\mu_j x}, \qquad j = 1, \dots, n,$$
$$u(\lambda, x) \sim e^{-i\lambda x} - S(\lambda)e^{i\lambda x}, \qquad -\infty < \lambda < \infty.$$

So at infinity behaviour of the radial waves is defined by $\{S(\lambda)\ (-\infty < \lambda < \infty), -\mu_k^2,\ m_k\ (k = 1 \dots n)\}$. These are called scattering data of the (3)-(4) boundary value problem. In this work characteristic properties of the scattering data will be investigated.

1. Introduction

Consider the differential equation

$$-y'' + q(x)y = \lambda^2\rho(x)y, \quad 0 < x < \infty \tag{3}$$

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with boundary condition

$$-(\alpha_1 y(0) - \alpha_2 y'(0)) = \lambda^2(\beta_1 y(0) - \beta_2 y'(0)). \tag{4}$$

Here α_i, β_i are real numbers and $\gamma = \alpha_1\beta_2 - \alpha_2\beta_1 > 0$, $q(x)$ is a real valued function such that

$$\int_0^\infty (1+x)|q(x)|dx < \infty$$

and $\rho(x)$ is a real valued piecewise continuous function such that

$$\rho(x) = \begin{cases} \alpha^2, & 0 \le x < a, \\ 1, & x \ge a, \end{cases} \tag{5}$$

In the case $\beta_1 = \beta_2 = 0$, namely when the spectral parameter does not appear in the boundary condition, the inverse scattering problem for the boundary value problem (3)-(4) when $\rho(x) \equiv 1$ was completely solved in [16, 17, 26, 27]. When $\rho(x) \equiv 1$ in (3) with the spectral parameter appearing in the boundary conditions, the inverse problem on the half-line was considered by Pocheykina-Fedotova [29] according to spectral function, by Yurko [31–33] according to Weyl function and by Mamedov [18, 19] according to scattering data. For $\rho(x) \neq 1$, this problem was studied in [1, 2, 8, 15, 18]. Spectral analysis of the problem on the half line was studied by Fulton [11]. Physical applications of the problem with the linear spectral parameter appearing in the boundary conditions on the finite interval was also given by Fulton [12]. In finite interval, inverse spectral problems for Sturm-Liouville operators with linear or nonlinear dependence on the spectral parameter in the boundary conditions were studied by Chernozhukova and Freiling [5], Chugunova [6], Rundell and Sacks [30], Guliyev [14], Mamedov and Cetinkaya [21–23], Binding and Browne [3], Browne and Sleeman [4], McCarthy and Rundell [28] .

The discontinuous version was studied by Gasymov [13] and Darwish [10]. In these papers, solution of inverse scattering problem on the half line $[0, +\infty)$ was reduced to solution of two inverse problems on the intervals $[0, a]$ and $[a, +\infty)$. This type boundary condition arises from a varied assortment of physical problems and other applied problems such as the study of heat conduction by Cohen [7] and wave equation by Yurko [31, 32].

It turns out that the discontinuity of the function $\rho(x)$ strongly influences the structure of the representation of the Jost solution and the basic equation of the inverse problem. Similar situation do not arise for the system of Dirac equations with discontinuous coefficients see, [9, 24, 25].

The function

$$f_0(x,\lambda) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^+(x)} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^-(x)} \tag{6}$$

is the Jost solution of (3) when $q(x) \equiv 0$, where $\mu^\pm(x) = \pm x\sqrt{\rho(x)} + a(1 \mp \sqrt{\rho(x)})$.

It is known from [15, 19] that, for all λ from the closed upper half-plane, (3)-(4) has a unique Jost solution $f(x,\lambda)$ which satisfies the condition

$$\lim_{x\to+\infty} f(x,\lambda)e^{-i\lambda x} = 1 \tag{7}$$

and it can be represented in the form

$$f(x,\lambda) = f_0(x,\lambda) + \int_{\mu^+(x)}^{+\infty} K(x,t)e^{i\lambda t}dt, \tag{8}$$

where the kernel $K(x,t)$ satisfies the inequality

$$\int_{\mu^+(x)}^{+\infty} |K(x,t)|\,dt \le C\left(\exp\left(\int_x^{+\infty} t\left|q(t)\right|dt\right)\right), \quad 0 < C = constant \tag{9}$$

and $K(x,t)$ has first order partial derivatives with respect to both variables. Moreover, as $x \to \infty$ we have the asymptotic formula

$$u_j(x) \sim m_j e^{-\mu_j x}, \qquad j = 1, \dots, n,$$
$$u(\lambda, x) \sim e^{-i\lambda x} - S(\lambda) e^{i\lambda x}, \qquad -\infty < \lambda < \infty,$$

in which the scattering function $S(\lambda)$ is given by

$$S(\lambda) = \frac{(\alpha_2 + \beta_2\lambda^2)\overline{f'(0,\lambda)} - (\alpha_1 + \beta_1\lambda^2)\overline{f(0,\lambda)}}{(\alpha_2 + \beta_2\lambda^2)f'(0,\lambda) - (\alpha_1 + \beta_1\lambda^2)f(0,\lambda)} = \frac{\overline{E(\lambda)}}{E(\lambda)} \tag{10}$$

where $f(x,\lambda)$ is the Jost solution of (3) given in [20], $i\mu_k$ $(k = 1, \dots, n)$ are the zeros of the function $E(\lambda)$, and m_k $(k = 1, \dots, n)$ are the normalized or normalizing numbers given by

$$m_k^{-2} = \int_0^\infty \rho(x) \left| f(x, i\mu_k) \right|^2 dx + \frac{1}{\gamma} \left[\beta_2 f'(0, i\mu_k) - \beta_1 f(0, i\mu_k) \right]^2$$

So at infinity behaviour of the radial waves is defined by $\{S(\lambda)\ (-\infty < \lambda < \infty), -\mu_k^2,\ m_k\ (k = 1 \dots n)\}$. These are called scattering data of the (3)-(4) boundary value problem. This scattering data is uniquely determines the potential function $q(x)$.

According to Lemma 2.2 in [20], the equation $E(\lambda) = 0$ has only finite number of simple roots in the half plane $\Im\lambda > 0$; moreover, these roots lie on the imaginary axis.

The aim of this work is to investigate the continuity of the scattering function $S(\lambda)$ and derive Levinson formula for the boundary value problem (3), (4)

2. Continuity of the Scattering Function $S(\lambda)$

We will use the fundamental equation to show that the scattering function $S(\lambda)$ is continuous on $(-\infty, \infty)$. Equation (11) is called *the fundamental equation* of the inverse problem of the scattering theory for the boundary value problem (3)-(4). This equation is different from the classic equation of Marchenko and we call equation (11) the *modified Marchenko equation*. As it is seen below that, the discontinuity of the function $\rho(x)$ strongly influences the structure of the fundamental equation of the boundary value problem (3)-(4).

$$F(x,y) + \int_{\mu^+(x)}^{+\infty} K(x,t) F_0(t+y) dt + K(x,y) + \frac{1 - \sqrt{\rho(x)}}{1 + \sqrt{\rho(x)}} K(x, 2a - y) = 0, \tag{11}$$

where

$$F_0(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [S_0(\lambda) - S(\lambda)] e^{-i\lambda x} d\lambda + \sum_{k=1}^{n} m_k^2 e^{-\lambda_k x}, \tag{12}$$

$$F(x,y) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right) F_0(y + \mu^+(x)) + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right) F_0(y + \mu^-(x)), \tag{13}$$

$$S_0(\lambda) = \begin{cases} \frac{\overline{f_0(0,\lambda)}}{f_0(0,\lambda)} = e^{-2i\lambda a} \frac{1 + \tau e^{-2i\lambda a\alpha}}{e^{-2i\lambda a\alpha} + \tau}, & \beta_2 = 0, \\ \\ \frac{\overline{f_0'(0,\lambda)}}{f_0'(0,\lambda)} = -e^{-2i\lambda a} \frac{1 - \tau e^{-2i\lambda a\alpha}}{e^{-2i\lambda a\alpha} - \tau}, & \beta_2 \neq 0 \end{cases}$$

and $\tau = (\alpha - 1)/(\alpha + 1)$.

Theorem 2.1. *The function $S(\lambda)$ is continuous at all real points λ and*

$$S(0) = \begin{cases} 1, & E(0) \neq 0, \\ -1, & E(0) = 0. \end{cases} \tag{14}$$

Proof. Since $E(\lambda) \neq 0$ for $\lambda \neq 0$, $S(\lambda) = \frac{\overline{E(\lambda)}}{E(\lambda)}$ is continuous for all $\lambda \neq 0$. When $E(0) \neq 0$, $S(\lambda)$ is continuous for $\lambda = 0$ and $S(0) = 1$

Now let's consider the case $E(0) = 0$

$$\begin{aligned} E(0) &= \alpha_2 f'(0,0) - \alpha_1 f(0,0) \\ &= \alpha_2 \left[-\alpha K(0, a(1-\alpha)) + \int_{a(1-\alpha)}^{\infty} K_x(0,t)dt \right] - \alpha_1 \left[1 + \int_{a(1-\alpha)}^{\infty} K(0,t)dt \right] = 0 \end{aligned} \tag{15}$$

Writing $x = 0$ in the fundamental equation (11), we obtain

$$K(0,y) + \frac{1-\alpha}{1+\alpha} K(0, 2a-y) + F(0,y) + \int_{a(1-\alpha)}^{\infty} K(0,t) F_0(t+y)dt = 0 \tag{16}$$

Taking derivative of the fundamental equation with respect to x and writing $x = 0$, we obtain

$$K_x(0,y) + \frac{1-\alpha}{1+\alpha} K_x(0,2a-y) + \int_{a(1-\alpha)}^{\infty} K_x(0,t)F_0(t+y)dt - \alpha K(0,a(1-\alpha))F_0(a(1-\alpha)+y) + F_x(0,y) = 0 \tag{17}$$

Multiplying (16) with α_1 and (17) with α_2 and subtracting them, we obtain

$$\begin{aligned} &\alpha_2 K_x(0,y) - \alpha_1 K(0,y) + \frac{1-\alpha}{1+\alpha} \left(\alpha_2 K_x(0,2a-y) - \alpha_1 K(0,2a-y) \right) + \\ &+ \int_{a(1-\alpha)}^{\infty} \left(\alpha_2 K_x(0,t) - \alpha_1 K(0,t) \right) F_0(t+y)dt + \alpha_2 F_x(0,y) - \alpha_1 F(0,y) \\ &- \alpha\alpha_2 K(0,a(1-\alpha))F_0(a(1-\alpha)+y) = 0 \end{aligned} \tag{18}$$

Integrating (18) with respect to y from z to ∞

$$\begin{aligned} &\int_z^{\infty} \left(\alpha_2 K_x(0,y) - \alpha_1 K(0,y) \right) dy + \frac{1-\alpha}{1+\alpha} \int_z^{\infty} \left(\alpha_2 K_x(0,2a-y) - \alpha_1 K(0,2a-y) \right) dy + \\ &+ \int_{a(1-\alpha)}^{\infty} \left(\int_z^{\infty} \left(\alpha_2 K_x(0,s) - \alpha_1 K(0,s) \right) ds \right) F_0(t+z)dt \\ &- \frac{\alpha_1}{2}(\alpha+1)F_0(z+a(1-\alpha)) + \frac{\alpha_1}{2}(\alpha-1)F_0(z+a(1+\alpha)) = 0 \end{aligned} \tag{19}$$

Let $K_1(z) = \int_z^{\infty} \left(\alpha_2 K_x(0,t) - \alpha_1 K(0,t) \right) dt$

We can rewrite (19) as follows

$$K_1(z) - \int_{a(1-\alpha)}^{\infty} K_1(t)F_0(t+z)dt = \alpha\phi(z), \quad (0 \leq z < \infty) \tag{20}$$

where

$$\begin{aligned} \phi(z) &= -\frac{1-\alpha}{1+\alpha} \int_{a(1-\alpha)}^{2a-z} \left(\alpha_2 K_x(0,y) - \alpha_1 K(0,y) \right) dy \\ &+ \frac{\alpha_1}{2} \left[(\alpha+1)F_0(z+a(1-\alpha)) - (\alpha-1)F_0(z+a(1+\alpha)) \right] \end{aligned}$$

Hence when $E(0)=0$, $K_1(z)$ is bounded solution of the equation (20). Bounded solution of this equation is integrable on the half axis $[a(1-\alpha),\infty)$. Now we have

$$\begin{aligned}E(\lambda)&=i\lambda\left\{\frac{\alpha_1}{2i\lambda}\left(1-\frac{1}{\alpha}\right)\left[e^{i\lambda a(1-\alpha)}-e^{i\lambda a(1+\alpha)}\right]+\int_{a(1-\alpha)}^{\infty}K_1(t)e^{i\lambda t}dt+\alpha_2\frac{1+\alpha}{2}e^{i\lambda a(1-\alpha)}\right.\\&-\alpha_2\frac{\alpha-1}{2}e^{i\lambda a(1+\alpha)}+i\lambda\beta_2\alpha K(0,a(1-\alpha))e^{i\lambda a(1-\alpha)}+\frac{1}{2}i\lambda\beta_1\left(1+\frac{1}{\alpha}\right)e^{i\lambda a(1+\alpha)}\\&+\frac{1}{2}i\lambda\beta_2\left(1-\frac{1}{\alpha}\right)e^{i\lambda a(1+\alpha)}-i\lambda\int_{a(1-\alpha)}^{\infty}\left(\beta_2K_x(0,t)-\beta_1K(0,t)\right)e^{i\lambda t}dt\\&\left.+\lambda^2\beta_2\frac{\alpha+1}{\alpha}e^{i\lambda a(1-\alpha)}-\lambda^2\beta_2\frac{\alpha-1}{\alpha}e^{i\lambda a(1+\alpha)}\right\}=i\lambda\hat{K}(\lambda)\end{aligned}\tag{21}$$

where

$$\begin{aligned}\hat{K}(\lambda)&=\frac{\alpha_1}{2i\lambda}\left(1-\frac{1}{\alpha}\right)\left[e^{i\lambda a(1-\alpha)}-e^{i\lambda a(1+\alpha)}\right]+\int_{a(1-\alpha)}^{\infty}K_1(t)e^{i\lambda t}dt+\alpha_2\frac{1+\alpha}{2}e^{i\lambda a(1-\alpha)}\\&-\alpha_2\frac{\alpha-1}{2}e^{i\lambda a(1+\alpha)}+i\lambda\beta_2\alpha K(0,a(1-\alpha))e^{i\lambda a(1-\alpha)}+\frac{1}{2}i\lambda\beta_1\left(1+\frac{1}{\alpha}\right)e^{i\lambda a(1+\alpha)}\\&+\frac{1}{2}i\lambda\beta_2\left(1-\frac{1}{\alpha}\right)e^{i\lambda a(1+\alpha)}-i\lambda\int_{a(1-\alpha)}^{\infty}\left(\beta_2K_x(0,t)-\beta_1K(0,t)\right)e^{i\lambda t}dt\\&+\lambda^2\beta_2\frac{\alpha+1}{\alpha}e^{i\lambda a(1-\alpha)}-\lambda^2\beta_2\frac{\alpha-1}{\alpha}e^{i\lambda a(1+\alpha)}\end{aligned}$$

Similarly we obtain

$$E(-\lambda)=-i\lambda\hat{K}(-\lambda)$$

Consequently,

$$S(\lambda)=-\frac{\hat{K}(-\lambda)}{\hat{K}(\lambda)}\tag{22}$$

From Lemma 2.1 in [20], we have the identity

$$\frac{2i\lambda w(x,\lambda)}{(\alpha_2+\beta_2\lambda^2)f'(0,\lambda)-(\alpha_1+\beta_1\lambda^2)f(0,\lambda)}=\overline{f(x,\lambda)}-S(\lambda)f(x,\lambda)\tag{23}$$

holds for all real $\lambda\neq 0$. Using (21) and (23), we can write

$$2w(x,\lambda)=\hat{K}(\lambda)[\overline{f(x,\lambda)}-S(\lambda)f(x,\lambda)]$$

and from this we can clearly see that, $\hat{K}(\lambda)\neq 0$. Otherwise $w(x,0)=0$, but this is not possible. So $S(\lambda)$ is continuous at $\lambda=0$ and $S(0)=-1$. □

3. Levinson Formula

Now, we give the Levinson type formula that expresses the relation between the increment of argument of $S(\lambda)$ and the number of eigenvalues of boundary value problem (3)-(4)

Theorem 3.1. *The following formula is valid:*

$$\frac{\ln S(+0)-\ln S(+\infty)}{2\pi i}+C(\beta_2)-\frac{1-S(0)}{4}=n,\tag{24}$$

where

$$C(\beta_2)=\begin{cases}\frac{3}{2}, & \beta_2\neq 0,\\ 1, & \beta_2=0.\end{cases}\tag{25}$$

Proof. Let us apply argument principle to the $E(\lambda)$ function. This function is regular on the upper half plane and continuous on the closed half plane $\Im\lambda \geq 0$. When moving from $-\infty$ to ∞ on the whole real axis and passing origin from top along with half circle with radius ε, the change in the argument of $E(\lambda)$ is equal to number of its pole points times -2π:

$$\Delta_{\Gamma_{R,\varepsilon}} arg E(\lambda) = 2\pi n$$

here $\Gamma_{R,\varepsilon} = C_R^+ \cup [-R,-\varepsilon] \cup C_\varepsilon^- \cup [\varepsilon, R]$, C_R^+ and C_ε^- are circles centered at the origin with radius R and ε respectively. Orientation on the C_R^+ is positive and on the C_ε^- is negative. On $\Im\lambda \geq 0$ and for $\lambda \to \infty$

$$E(\lambda) \sim C_0 \lambda^{2C(\beta_2)} \tag{26}$$

here C_0 is constant and $C(\beta_2)$ is given by (25). From (26), on the half plane $\Im\lambda \geq 0$, for $\lambda \to \infty$

$$\frac{1}{2\pi}\left\{\arg E(\lambda)\Big|_{-\infty}^{-\varepsilon} + \arg E(\lambda)\Big|_{-\varepsilon}^{\varepsilon} + \arg E(\lambda)\Big|_{\varepsilon}^{\infty}\right\} + C(\beta_2) = n \tag{27}$$

On the other hand, on the half plane $\Im\lambda \geq 0$, for $\lambda \to 0$

$$E(\lambda) \sim \begin{cases} C_1, & E(0) \neq 0, \\ C_2\lambda, & E(0) = 0. \end{cases} \tag{28}$$

here C_0 and C_1 are constants. Using this relation

$$\lim_{\varepsilon \to 0} \arg E(\lambda)\Big|_{-\varepsilon}^{\varepsilon} = \begin{cases} 0, & E(0) \neq 0, \\ -\pi, & E(0) = 0. \end{cases} \tag{29}$$

For all real λ, $E(\lambda) = \overline{E(-\lambda)}$. From this and (27), for $\varepsilon \to 0$

$$\frac{1}{\pi} \arg E(\lambda)\Big|_{+0}^{\infty} = n - C(\beta_2) + \begin{cases} \frac{1}{2}, & E(0) \neq 0, \\ 0, & E(0) = 0. \end{cases} \tag{30}$$

or

$$\frac{\arg E(\lambda)}{\pi}\Big|_{+0}^{\infty} = n - C(\beta_2) + \frac{1 - S(0)}{4} \tag{31}$$

Since $|S(\lambda)| = 1$, $\arg S(\lambda) = -2 \arg E(\lambda)$ and $\ln S(\lambda) = i \arg S(\lambda) = -2i \arg E(\lambda)$,

$$-\frac{1}{2\pi} \arg S(\lambda)\Big|_{+0}^{\infty} = n - C(\beta_2) + \frac{1 - S(0)}{4}$$

or

$$\frac{1}{2\pi i}\{\ln S(+0) - \ln S(\infty)\} = n - C(\beta_2) + \frac{1 - S(0)}{4}$$

From that, (24) is obtained. This concludes the proof of the theorem. □

(24) is called Levinson formula for the boundary value problem (3)-(4).

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