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Uniqueness Part of the Schwarz Lemma at the Boundary

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Abstract. In this paper, a boundary version of the uniqueness (or, rigidity) part of the Schwarz lemma should be investigated. Also, new results related to inner functions, inner capacities, and bilogaritmic concave majorants are obtained.

1. Introduction

The classical Schwarz lemma gives information about the behavior of a holomorphic function on the unit disc $D = \{z : |z| < 1\}$ at the origin, subject only to the relatively mild hypotheses that the function maps the unit disc to the disc and the origin to the origin. In its most basic form, the familiar Schwarz lemma says this:

Let f be a holomorphic function in the unit disc D, $f(0) = 0$ and $\left|f(z)\right| < 1$ for $|z| < 1$. Then, for any point z in the disc D, we have $\left|f(z)\right| \leq |z|$ and $\left|f'(0)\right| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = \lambda z$, $|\lambda| = 1$ ([4], p.329). For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [16]). Also, the similar to considered problem is studied in ([15]).

In recent years, a boundary version of Schwarz lemma was investigated in Daniel M. Burns and Steven G. Krantz ([1]), Dov Chelts ([2]), M. Mateljević ([9], [10], [11], [12] and [13]) and a few other authors' papers. They studied the uniqueness(or, rigidity) part of the Schwarz lemma. Also, the similar to considered problem is studied in ([15]).

The uniqueness part of the boundary Schwarz lemma was established in 1994 by Daniel M. Burns and Steven G. Krantz ([1]).

Theorem 1.1. *Let $f : D \to D$ be a holomorphic function from the unit disc to itself such that*

$$f(z) = z + O\left((z-1)^4\right) \tag{1.1}$$

as $z \to 1$. Then $f(z) = z$ on the disc.

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The theorem presented in ([1]) has no such hypothesis. The exponent 4 is sharp: simple geometric arguments show that the function

$$f(z) = z + \frac{1}{10}(z-1)^3$$

satisfies the conditions of the theorem with 4 replaced by 3. Note also that it follows from the proof that $O\left((z-1)^4\right)$ can be replaced by $o\left((z-1)^3\right)$.

The Burns-Krantz Theorem was improved in 1995 by Thomas L. Kriete and Barbara D. MacCluer ([7]), who replaced f with its real part and considered the radial limit in $o\left((z-1)^3\right)$ instead of the unrestricted limit. Here is a more precise statement of their result.

Theorem 1.2. *Let $f : D \to D$ be a holomorphic function with radial limit $f(1) = 1$ and angular derivative $f'(1) = 1$. If*

$$\liminf_{r \to 1^-} \frac{Re\left(f(r) - r\right)}{(1-r)^3} = 0,$$

then $f(z) = z$.

In 2001, Dov Chelst ([2]), in turn, established the following conditions on the local behavior of f near a finite set of boundary points which ensure that f is a finite Blaschke product.

Theorem 1.3. *Let $f : D \to D$ be a holomorphic function from the unit disc to itself. In addition, let $\phi : D \to D$ be a finite Blaschke product which equals $\tau \in \partial D$ on a finite set $A_f \subset \partial D$. If (i) for a given $\gamma_0 \in A_f$,*

$$f(z) = \phi(z) + O\left((z-1)^4\right), \text{as } z \to \gamma_0,$$

and (ii) for all $\gamma \in A_f - \{\gamma_0\}$,

$$f(z) = \phi(z) + O\left((z-1)^{k_\gamma}\right), \text{ for some } k_\gamma \geq 2 \text{ as } z \to \gamma,$$

then $f(z) = \phi(z)$ on the disc.

In 2015, Miodrag Mateljević improved Theorem 1.3 and obtained the following theorem ([12]).

Theorem 1.4. *Let $f : D \to D$ be a holomorphic function. Let B be an inner function which equals 1 precisely on a set $A \subset \partial D$. Suppose the following condition are satisfied (a) for all $a \in A$*

$$f(e^{it}) = B(e^{it}) + o\left((e^{it} - a)^2\right), \ e^{it} \in \partial D, \ e^{it} \to a,$$

(b) there is a $a_0 \in A$, such that

$$f(e^{it}) = B(e^{it}) + o\left((e^{it} - a_0)^3\right), \ e^{it} \in \partial D, \ e^{it} \to a_0.$$

Then $f \equiv B$ on all of D.

Let $\mathcal{M}$ be a class of functions $\mu : (0, +\infty) \to (0, +\infty)$ for each of which $\log \mu(x)$ is concave with respect to $\log x$. For each function $\mu \in \mathcal{M}$ the limit

$$\mu_0 = \lim_{x \to 0} \frac{\log \mu(x)}{\log x},$$

exists, and $-\infty < \mu_0 \leq +\infty$. Here, the function $\mu \in \mathcal{M}$ is called bilogaritmic concave majorant and μ_0 is called the order of μ ([5]). For example, the power function $\mu(x)=x^a$, $a \in \mathbb{R}$, belongs $\mathcal{M}$ with $\mu_0=a$.

We use the following assertions for the proofs of our theorems:

Lemma 1.5 (Hopf's lemma on the disc). *Let u be a nonconstant real-valued harmonic function in D, Let $\gamma \in \partial D$ be such that: (i) u is continuous at γ; (ii) $u(\gamma) \geq u(z)$ for all $z \in \partial D$. Then the outher normal derivative $\frac{\partial u}{\partial v}$ of u at γ, if it exists, satisfies the strict inequality*

$$\frac{\partial u}{\partial v}(\gamma) > 0$$

([3],p.34).

Remark 1.6. *Let $f : D \to D$ be a holomorphic function and have a continuous limit at some $\gamma \in \partial D$, and let $f(\gamma) = 1$. Then f is not $o(z-\gamma)$ ([2]).*

2. Main Results

In this paper, the more general majorants will be taken instead of power majorants in conditions (i), (ii) and (1.1). Also, new results related to inner functions, inner capacities, and bilogaritmic concave majorants are obtained. This type of results were first announced ([14]). Let $d(z, G)$ be a distance from G to the point z and $U(z, r)$ be an open disc with centre z and radius r, respectively. Let $\mathfrak{M}$ be the class of sets with zero inner capacities ([8], p.13-14).

Theorem 2.1. *Let $\mu \in \mathcal{M}$, $\mu_0 > 3$ and f be a holomorphic function in the unit disc that is continuous on $\overline{D} \cap \mathbb{U}$ $(1, \eta_0)$ for some $\eta_0 > 0$ and $\left|f(z) - \alpha\right| < \alpha$ for $|z| < 1$, where α is a positive real number. Suppose the condition*

$$f(z) = \alpha(1+z) + O(\mu(|z-1|)),\ z \in \partial D,\ z \to 1. \tag{1.2}$$

Then $f(z) = \alpha(1+z)$.

Proof. Consider the function

$$h(z) = \frac{f(z)-\alpha}{\alpha}.$$

Therefore, we obtain

$$h(z) = \frac{f(z)-\alpha}{\alpha} = \frac{\alpha(1+z) + O(\mu(|z-1|)) - \alpha}{\alpha}$$

and

$$h(z) = z + O(\mu(|z-1|))$$

So, there is a number $b_1 > 0$ such that

$$|h(z) - z| \leq b_1 \mu(|z-1|),\ \ \forall z \in \partial D \cap \mathbb{U}(1, \eta_0).$$

We will write s and b_2 as follows;

$$s = \sup_{\substack{|z-1|=\eta_0 \\ z \in D}} |h(z) - z|$$

and

$$b_2 = \max\left\{\frac{s}{\mu(\eta_0)}, b_1\right\}.$$

Obviously,

$$|h(z) - z| \leq b_2 \mu(|z-1|)$$

inequality is satisfied at every boundary point of the set of $D \cap \mathbb{U}(1, \eta_0)$. Thus, it is seen that the same inequality from Theorem3 of ([5]) is satisfied at the set of $D \cap \mathbb{U}(1, \eta_0)$. That is,

$$|h(z) - z| \leq b_2 \mu(|z-1|), \ \forall z \in D \cap \mathbb{U}(1, \eta_0). \tag{1.3}$$

From the hypothesis $\mu_0 > 3$ follows that there is a some positive constant $\varepsilon > 0$ such that

$$\frac{\log \mu(x)}{\log x} \geq 3 + \varepsilon, \quad \forall x \in (0,1)$$

and

$$\log \mu(x) \leq (3+\varepsilon) \log x, \ \ \forall x \in (0,1).$$

That is to say,

$$\mu(x) \leq x^{3+\varepsilon}, \ \ \forall x \in (0,1). \tag{1.4}$$

From (1.3) and (1.4) we obtain

$$|h(z) - z| \leq c_2 |z-1|^{3+\varepsilon}. \tag{1.5}$$

Consider the harmonic function k defined as

$$k(z) = Re\left(\frac{1+h(z)}{1-h(z)}\right) - Re\left(\frac{1+z}{1-z}\right).$$

The function

$$\frac{1+h(z)}{1-h(z)}$$

maps the disc D to the right half plane and hence the first term of $k(z)$ is nonnegative, the second term is zero on $\partial D \backslash \{1\}$. That is,

$$Re\left(\frac{1+z}{1-z}\right) = 0.$$

Therefore, the boundary values of $k(z)$ function is not negative at its every point except for point 1 in the unit disc. In other words,

$$\liminf_{z \to \varsigma, z \in D} k(z) \geq 0, \ \ \forall \varsigma \in \partial D \backslash \{1\}.$$

Let us now examine our function at point 1. Let $\omega(z) = h(z) - z$.

From the definition of $k(z)$, we take

$$k(z) = Re\left(\frac{2\omega(z)}{(1-h(z))(1-z)}\right).$$

According to Remark 1.6, the denominator of the last fraction can decrease no more quickly than $O\left(|z-1|^2\right)$ at the point 1. From (1.5), the numerator approaches zero as $O\left(|z-1|^{3+\varepsilon}\right)$. Thus, $k(z)$ is $O\left(|z-1|^{1+\varepsilon}\right)$ in some neighborhood of 1. We obtain from the maximum principle ([6], p.48) either $k(z) > 0$, $\forall z \in D$ or $k \equiv 0$. If k is not a constant, it takes minimum at the point $z = 1$, and it is $O\left(|z-1|^{1+\varepsilon}\right)$ there, as well. This contradicts with Hopf's lemma statement. Consequently, $k \equiv 0$. This $h(z) = z$ and $f(z) = \alpha(1+z)$. $\square$

Theorem 2.2. *Let $\mu \in \mathcal{M}$ be a bilogaritmic concave majorant, $P \in \mathfrak{M}$, $\mu_0 > 3$; f be a holomorphic function in the unit disc and $\left|f(z) - \alpha\right| < \alpha$ for $|z| < 1$, where α is a positive real number, which satisfies the following condition*

$$\limsup_{z\to\varsigma,\ z\in D} \left|f(z) - \alpha(1+z)\right| = O\left(\mu(|\varsigma - 1|)\right),\ \forall\varsigma \in (\partial D\backslash P) \cap \mathbb{U}(1, \eta_0), \tag{1.6}$$

for some $\eta_0 > 0$. Then $f(z) = \alpha(1+z)$.

Proof. Let

$$\omega(z) = \frac{f(z) - \alpha}{\alpha}.$$

So, we have

$$\omega(z) - z = \frac{f(z) - \alpha}{\alpha} - z = \frac{f(z) - \alpha - \alpha z}{\alpha} = \frac{f(z) - \alpha(1+z)}{\alpha} = O\left(\mu(|\varsigma - 1|)\right.$$

and

$$\limsup_{z\to\varsigma,\ z\in D} |\omega(z) - z| = O\left(\mu(|\varsigma - 1|)\right). \tag{1.7}$$

From (1.7), there exists $c_1 > 0$ that the inequality

$$\limsup_{z\to\varsigma,\ z\in D} |\omega(z) - z| \le c_1\mu(|\varsigma - 1|, \quad \forall\varsigma \in (\partial D\backslash P) \cap \mathbb{U}(1, \eta_0)$$

satisfied as the proof of Theorem 2.1. Let us make a marking as follow:

$$a_1 = \sup_{\substack{|z-1|=\eta_0 \\ z\in D}} |\omega(z) - z|$$

and

$$a_2 = \max\left\{\frac{a_1}{\mu(\eta_0)}, c_1\right\}.$$

The inequality

$$\limsup_{z\to\varsigma,\ z\in D} |\omega(z) - z| \le a_2\mu(|\varsigma - 1|$$

is satisfied in every z boundary point of $D \cap \mathbb{U}(1, \eta_0)$. So, from Theorem 2.1, same inequality is also obtained in $D \cap \mathbb{U}(1, \eta_0)$. Then, we take that $f(z) = \alpha(1+z)$ similar to the proof of Theorem 2.1. □

In the following theorem, we shall show that certain conditions on the growth of the boundary function (which is defined out of a set of zero inner capacity) in a neighborhood of a given point yields the uniqueness.

Theorem 2.3. *Let ϕ be an inner function which equal $\tau \in \partial D$ on a finite set $A_f \subset \partial D$. Let f be a holomorphic function in the unit disc and $\left|f(z) - \alpha\right| < \alpha$ for $|z| < 1$, where α is a positive real number. Assume that $P \in \mathfrak{M}$, $\mu^1, \mu^2 \in \mathcal{M}$, $\mu_0^1 > 3$, $\mu_0^2 > 2$, where μ_0^1 and μ_0^2 are the orders of μ^1 and μ^2, respectively. Suppose the following conditions are satisfied (i) for a given $\gamma_0 \in A_f$*

$$\limsup_{z\to\varsigma,\ z\in D} \left|f(z) - \alpha\left(1 + \phi(z)\right)\right| = O\left(\mu^1(\left|\varsigma - \gamma_0\right|)\right),\quad \varsigma \in (\partial D\backslash P),\ \varsigma \to \gamma_0, \tag{1.8}$$

(ii) for all $\forall\gamma \in A_f - \{\gamma_0\}$

$$\limsup_{z\to\varsigma,\ z\in D} \left|f(z) - \alpha\left(1 + \phi(z)\right)\right| = O\left(\mu^2(\left|\varsigma - \gamma\right|)\right),\quad \varsigma \in (\partial D\backslash P),\ \varsigma \to \gamma. \tag{1.9}$$

Then $f(z) = \alpha\left(1 + \phi(z)\right)$.

Proof. Let

$$\Phi(z)=\frac{f(z)-\alpha}{\alpha}.$$

From (1.8) and (1.9), we obtain for a given $\gamma_0\in A_f$

$$\begin{aligned}\limsup_{z\to\varsigma,\ z\in D}\left|\Phi(z)-\phi(z)\right| &= \limsup_{z\to\varsigma,\ z\in D}\left|\frac{f(z)-\alpha}{\alpha}-\phi(z)\right|\\ &= \limsup_{z\to\varsigma,\ z\in D}\left|\frac{f(z)-\alpha(1+\phi(z))}{\alpha}\right|\\ &= O\left(\mu^1(\left|\varsigma-\gamma_0\right|)\right),\ \ \varsigma\in(\partial D\backslash P)\,,\ \varsigma\to\gamma_0,\end{aligned}$$

$$\limsup_{z\to\varsigma,\ z\in D}\left|\Phi(z)-\phi(z)\right|=O\left(\mu^1(\left|\varsigma-\gamma_0\right|)\right),\ \ \varsigma\in(\partial D\backslash P)\,,\ \varsigma\to\gamma_0, \tag{1.10}$$

and for all $\forall\gamma\in A_f-\{\gamma_0\}$

$$\begin{aligned}\limsup_{z\to\varsigma,\ z\in D}\left|\Phi(z)-\phi(z)\right| &= \limsup_{z\to\varsigma,\ z\in D}\left|\frac{f(z)-\alpha(1+\phi(z))}{\alpha}\right|\\ &= O\left(\mu^2(\left|\varsigma-\gamma\right|)\right),\quad \varsigma\in(\partial D\backslash P)\,,\ \varsigma\to\gamma,\end{aligned}$$

$$\limsup_{z\to\varsigma,\ z\in D}\left|\Phi(z)-\phi(z)\right|=O\left(\mu^2(\left|\varsigma-\gamma\right|)\right),\quad \varsigma\in(\partial D\backslash P)\,,\ \varsigma\to\gamma. \tag{1.11}$$

Without loss of generality, we may assume that $\tau=1$ and that $\gamma_0=1$. Due to (1.10), there exist numbers $c_3>0$, $\eta_0\in(0,1)$ such that

$$\limsup_{z\to\varsigma,\ z\in D}\left|\Phi(z)-\phi(z)\right|\le c_3\mu^1(|\varsigma-1|),\quad \forall\varsigma\in(\partial D\backslash P)\,,\ |\varsigma-1|<\eta_0.$$

Let us denote p and c_4 as follows;

$$p=\sup_{\substack{|z-1|=\delta_0\\ z\in D}}\left|\Phi(z)-\phi(z)\right|,$$

$$c_4=\max\left\{\frac{p}{\mu^1(\delta_0)},c_3\right\}.$$

Clearly,

$$\limsup_{z\to\varsigma,\ z\in D}\left|\Phi(z)-\phi(z)\right|\le c_4\mu^1(|\varsigma-1|)$$

inequality is satisfied at every boundary points of the set of $D\cap\mathbb{U}(1,\eta_0)$. Therefore, it is seen that the same inequality from Theorem 3 ([5]) is satisfied at the set of $D\cap\mathbb{U}(1,\eta_0)$. That is,

$$\left|\Phi(z)-\phi(z)\right|\le c_4\mu^1(|z-1|),\ \ \forall z\in D\cap\mathbb{U}(1,\eta_0)\,. \tag{1.12}$$

From $\mu_0>3$ follows that there are some positive constants ε and $\sigma<\min(\eta_0,1)$ such that inequality (1.4) is satisfied. Combining (1.4) and (1.12), we obtain

$$\left|\Phi(z)-\phi(z)\right|\le c_4\,(|z-1|)^{3+\varepsilon}\,,\ \ \forall z\in D\cap\mathbb{U}(1,\sigma)\,. \tag{1.13}$$

From the hypothesis, $\mu_0^2 > 2$, $\mu_0^2 \in \mathcal{M}$,

$$\mu_0^2 = \lim_{x \to 0} \frac{\log \mu^2(x)}{\log x} > 2,$$

$$\frac{\log \mu^2(x)}{\log x} \geq 2 + \varepsilon, \ \ \forall x \in (0, \sigma), \ \sigma < \min(\eta_0, 1),$$

$$\log \mu^2(x) \leq (2 + \varepsilon) \log x, \ \ \forall x \in (0, \sigma)$$

and finally we take

$$\mu^2(x) \leq x^{2+\varepsilon}, \ \ \forall x \in (0, \sigma). \tag{1.14}$$

Analogously, for any point $\gamma \in A_f - \{1\}$, from (1.14) and (1.11) we have

$$\left|\Phi(z) - \phi(z)\right| \leq c_5 \left(|z-1|\right)^{2+\varepsilon}, \ \ \forall z \in D \cap \mathbb{U}(\gamma, \sigma_1) \tag{1.15}$$

with some constant c_5 and σ_1.

We introduce the harmonic function

$$\Theta(z) = Re\left(\frac{1+\Phi(z)}{1-\Phi(z)}\right) - Re\left(\frac{1+\phi(z)}{1-\phi(z)}\right).$$

Since an inner function ϕ is a holomorphic function throughout $\overline{D}$ and that $\left|\phi\right| = 1$ on ∂D, we have that the second term of $\Theta(z)$ is zero on $\partial D - \left\{A_f\right\}$. The first term of $\Theta(z)$ is nonnegative. Consequently, when taking limits to any boundary point in $(\partial D \backslash P) - \left\{A_f\right\}$, one always obtains a nonnegative value.

Now, let's examine behavior of the function $\Theta(z)$ at points of set A_f.

Let $\Psi(z) = \Phi(z) - \phi(z)$. Under the simple calculations, we obtain

$$\Theta(z) = Re\left(\frac{2\Psi(z)}{(1-\Phi(z))\left(1-\phi(z)\right)}\right).$$

Now, let's take any point $\gamma \in A_f - \{1\}$. According to (1.15), the numerator of the last fraction approaches zero as $O\left(\left|z-\gamma\right|^{2+\varepsilon}\right)$. From Remark1, the denominator can decrease no more quickly than $O\left(\left|z-\gamma\right|^{2}\right)$. Thus, $\Theta(z)$ must have a lim inf at γ.

From (1.13), the numerator of the last fraction approaches zero as $O\left(\left|z-\gamma\right|^{3+\varepsilon}\right)$. According to Remark 1.6, the denominator can decrease no more quickly than $O\left(\left|z-\gamma\right|^{2}\right)$. As in the proof of Theorem 2.1, we obtain $\Theta(z)$ is $O\left(\left|z-\gamma\right|^{1+\varepsilon}\right)$ in some neighborhood of 1. Thus, from the Phragmen-Lindelöf principle ([6], p.232) we obtain either $\Theta(z) > 0$, $\forall z \in D$ or $\Theta \equiv 0$. If Θ is not constant, it takes minimum at the point $z = 1$ and is $O\left(\left|z-\gamma\right|^{1+\varepsilon}\right)$ there, as well. This contradicts with Hopf's lemma statement. Consequently, $\Theta \equiv 0$. This $\Phi(z) = \phi(z)$ and $f(z) = \alpha\left(1+\phi(z)\right)$ □

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References

[1] D.M. Burns and S.G. Krantz, *Rigidity of holomorphic mappings and a new Schwarz Lemma at the boundary*, J. Amer. Math. Soc. **7** (1994), 661-676 .

[2] Dov Chelts, *A generalized Schwarz lemma at the boundary, Proceedings of the American Math. Society*, **129** (2001), No.11, 3275–3278 .

[3] D. Gilbarg and N.S. Trudinger, *Eliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin, 1983.

[4] G.M. Golusin, *Geometric Theory of Functions of Complex Variable,* [in Russian], 2nd edn., Moscow 1966.

[5] P. M. Tamrazov, *Holomorphic functions and mappings in the contour-solid problem*, Dokl. Akad. Nauk SSSR **279** (1984), No.1, 38-43 .

[6] W.K. Hayman and P.B. Kennedy, *Subharmonic Function*, Academic Press, London, 1976.

[7] T. L. Kriete and B. D. MacCluer, *A rigidity theorem for composition operators on certain Bergman spaces*, Mich. Math. J. **42** (1995), 379-386.

[8] M. Brelot, *Lectures on Potential Theory*, Tata Institute of Funamental Research, Bombay, 1967.

[9] M. Mateljević, *The Lower Bound for the Modulus of the Derivatives and Jacobian of Harmonic Injective Mappings*, Filomat **29**:2 (2015), 221-244.

[10] M. Mateljević, *Distortion of harmonic functions and harmonic quasiconformal quasi-isometry*, Revue Roum. Math. Pures Appl. Vol. **51** (2006) 56, 711-722.

[11] M. Mateljević, *Ahlfors-Schwarz lemma and curvature*, Kragujevac J. Math. **25** (2003), 155-164.

[12] M. Mateljević, *Note on Rigidity of Holomorphic Mappings & Schwarz and Jack Lemma (in preparation)*, ResearchGate, Sep 2015.

[13] M. Mateljević, *What are the most recent versions of The Schwarz Lemma at the Boundary?*, the question posed on ResearchGate May 19, 2015 [accessed Aug 3, 2016].

[14] T. A. Azeroğlu, B. N. Örnek, *A generalized Schwartz lemma at the boundary*, Complex Analysis and Potential theory, World Sci. Publ. Hackensack, NJ, (2007), 125-130.

[15] B. Nafi Örnek and T. Akyel, *Representation with majorant of the Schwarz lemma at the boundary*, Publ. Inst. Math., Nouv. Sér. 2015, In press.

[16] M. Elin, F. Jacobzon, M. Levenshtein, D. Shoikhet, *The Schwarz lemma: Rigidity and Dynamics*, Harmonic and Complex Analysis and its Applications. Springer International Publishing, (2014), 135-230.