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On the Extended Appell-Lauricella Hypergeometric Functions and their Applications

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Abstract. The main object of this paper is to present a systematic introduction to the theory and applications of the extended Appell-Lauricella hypergeometric functions defined by means of the extended beta function and extended Dirichlet's beta integral. Their connections with the Laguerre polynomials, the ordinary Lauricella functions and the Srivastava-Daoust generalized Lauricella functions are established for some specific paramters. Furthermore, by applying the various methods and known formulas (such as fractional integral technique; some results of the Lagrange polynomials), we also derive some elegant generating functions for these new functions.

1. Introduction

Multivariable hypergeometric functions (such as the famous Appell, Lauricella and Kampé de Fériet functions, etc.) and their various generalizations appear in many branches of mathematics and its applications. Many authors have contributed works on this subject; we mention a few: [8], [11], [27] and [28]. In recent years, several authors have considered some interesting extensions of the Appell and Lauricella functions (see, for example, [22], [24] and [30]). Motivated by their works, we introduce a class of new extensions of the Lauricella functions and find their connection with other celebrated special functions.

The following extension of beta function, introduced in [17], plays a key role in the construction of our new functions.

Definition 1.1. *The extended beta function $\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y)$ with $\Re(b)>0$ is defined by*

$$\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;-\frac{b}{t^{\rho}(1-t)^{\lambda}}\right)\mathrm{d}t, \tag{1}$$

where $\rho\geq 0$, $\lambda\geq 0$, $\min\{\Re(\alpha),\Re(\beta)\}>0$, $\Re(x)>-\Re(\rho\alpha)$, $\Re(y)>-\Re(\lambda\alpha)$ and ${}_1F_1(\alpha;\beta;z)$ is the confluent hypergeometric function [28, Section 1.3].

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For the case when $\alpha = \beta$ and $\rho = \lambda = 1$, we always write

$$B_b(x,y) \equiv \mathcal{B}_{b;1,1}^{(\alpha,\alpha)}(x,y) \quad (x, y \in \mathbb{C}).$$

When $b = 0$, (1) reduces to the ordinary beta function $B(x,y)$ $(\min\{\Re(x), \Re(y)\} > 0)$.

Throughout the paper, r is assumed to be a positive integer. Boldface letters with subscript r denote vectors of dimension r; for instances $\mathbf{m}_r := (m_1, \cdots, m_r) \in \mathbb{N}_0^r$ and $\mathbf{x}_r := (x_1, \cdots, x_r) \in \mathbb{C}^r$. The length of vector $\mathbf{m}_r$ is given by $|\mathbf{m}_r| := m_1 + \cdots + m_r$. More generally, if its subscript starts at i $(1 \leq i \leq r)$, then we write $|\mathbf{x}_{i,r}| := x_i + \cdots + x_r$ for vector $\mathbf{x}_{i,r} = (x_i, \cdots, x_r)$. The inner product of two r-dimensional vectors $\mathbf{u}_r$ and $\mathbf{v}_r$ is defined by $\langle \mathbf{u}_r, \mathbf{v}_r \rangle := u_1 v_1 + \cdots u_r v_r$. When there is no danger of confusion, multiple series are written in simplified notation:

$$\sum_{\mathbf{m}_r=0}^{\infty} \text{ means } \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty}$$

and

$$\sum_{k=0}^{|\mathbf{m}_r|} \text{ means } \sum_{k=0}^{m_1+\cdots+m_r}.$$

Moreover, we always write

$$\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(x+n, y+m) := \frac{\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(x+n, y+m)}{B(x,y)} \quad (n, m \in \mathbb{N}_0), \tag{2}$$

where parameters α, β, ρ, λ and b are assumed to satisfy the restrictions stated in Definition 1.1 unless otherwise specified.

With the help of Definition 1.1 and notation (2), the *extended Appell-Lauricella hypergeometric functions* defined (by single or multiple integrals) in Section 3 can be expressed properly as the following series:

$$\begin{aligned} F_A^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b] &\equiv F_A^{(r:\alpha,\beta;\rho,\lambda)}[a, \mathbf{b}_r; \mathbf{c}_r; \mathbf{x}_r; b] \\ &= \sum_{\mathbf{m}_r=0}^{\infty} (a)_{|\mathbf{m}_r|} \prod_{i=1}^{r} \widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(b_i + m_i, c_i - b_i) \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!}, \\ &\quad (\Re(c_i) > \Re(b_i) > 0,\ i = 1, \cdots, r;\ |x_1| + \cdots + |x_r| < 1) \end{aligned} \tag{3}$$

$$\begin{aligned} F_B^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b] &\equiv F_B^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{a}_r, \mathbf{b}_r; c; \mathbf{x}_r; b] \\ &= \sum_{\mathbf{m}_r=0}^{\infty} (b_1)_{m_1} \cdots (b_r)_{m_r} \prod_{i=1}^{r} \widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(a_i + m_i, |\mathbf{a}_{i+1,r+1}| + |\mathbf{m}_{i+1,r+1}|) \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!}, \\ &\quad (a_{r+1} = c - |\mathbf{a}_r|;\ m_{r+1} = 0;\ \Re(a_i) > 0, i = 1, \cdots, r,\ \Re(c - |\mathbf{a}_r|) > 0;\ \max\{|x_1|, \cdots, |x_r|\} < 1) \end{aligned} \tag{4}$$

$$\begin{aligned} {}_1F_D^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b] &\equiv {}_1F_D^{(r:\alpha,\beta;\rho,\lambda)}[a, \mathbf{b}_r; c; \mathbf{x}_r; b] \\ &= \sum_{\mathbf{m}_r=0}^{\infty} (b_1)_{m_1} \cdots (b_r)_{m_r} \widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(a + |\mathbf{m}_r|, c - a) \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!}, \\ &\quad (\Re(c) > \Re(a) > 0;\ \max\{|x_1|, \cdots, |x_r|\} < 1) \end{aligned} \tag{5}$$

$$\begin{aligned} {}_2F_D^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b] &\equiv {}_2F_D^{(r:\alpha,\beta;\rho,\lambda)}[a, \mathbf{b}_r; c; \mathbf{x}_r; b] \\ &= \sum_{\mathbf{m}_r=0}^{\infty} (a)_{|\mathbf{m}_r|} \prod_{i=1}^{r} \widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(b_i + m_i, |\mathbf{b}_{i+1,r+1}| + |\mathbf{m}_{i+1,r+1}|) \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!}, \end{aligned} \tag{6}$$

$$(b_{r+1} = c - |\mathbf{b}_r|;\ m_{r+1} = 0;\ \Re(b_i) > 0, i = 1, \cdots, r,\ \Re(c - |\mathbf{b}_r|) > 0;\ \max\{|x_1|, \cdots, |x_r|\} < 1)$$

where $(v)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(v)_n := \frac{\Gamma(v+n)}{\Gamma(v)} = \begin{cases} 1 & (n = 0; v \in \mathbb{C} \setminus \{0\}) \\ v(v+1)\cdots(v+n-1) & (n \in \mathbb{N}; v \in \mathbb{C}). \end{cases}$$

It is noted that, by setting $b = 0$ in (3)–(6), we obtain the Lauricella hypergeometric functions $F_A^{(r)}$, $F_B^{(r)}$ and $F_D^{(r)}$ (see, for example, [27, p. 33]). In addition, if we set $b = 0$ and $r = 2$ in (3)–(6), they may reduce to the Appell hypergeometric functions F_2, F_3 and F_1, respectively (see [27, pp. 22–23]). They also have some natural connections with the extended generalized hypergeometric functions [17, p. 632, Eq. (6)]. For instance, in view of the elementary series identity [28, p. 61, Eq. (9)]

$$\sum_{\mathbf{m}_r=0}^{\infty} f(|\mathbf{m}_r|) \frac{(b_1)_{m_1}}{m_1!} \cdots \frac{(b_r)_{m_r}}{m_r!} x^{|\mathbf{m}_r|} = \sum_{k=0}^{\infty} f(k) (|\mathbf{b}_r|)_k \frac{x^k}{k!},$$

we can easily find the reduction formula

$${}_1F_D^{(r:\alpha,\beta;\rho,\lambda)}[a, \mathbf{b}_r; c; x, \cdots, x; b] = {}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}|\mathbf{b}_r|, a \\ c\end{matrix}; x; b\right], \tag{7}$$

where ${}_2F_1^{(\alpha,\beta;\rho,\lambda)}[x;b]$ is the extended Gauss hypergeometric function defined by [17, p. 632, Eq. (5)]

$${}_2F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}A, B \\ C\end{matrix}; x; b\right] = \sum_{n=0}^{\infty} (A)_n \, \widehat{\mathscr{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(B+n, C-B) \frac{x^n}{n!} \quad (\Re(C) > \Re(B) > 0;\ |x| < 1). \tag{8}$$

The confluent form of (8) is given by

$${}_1F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}B \\ C\end{matrix}; x; b\right] = \sum_{n=0}^{\infty} \widehat{\mathscr{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(B+n, C-B) \frac{x^n}{n!} \quad (\Re(C) > \Re(B) > 0;\ |x| < \infty), \tag{9}$$

which will be used in Section 4.

When $b \geq 0$, $\alpha = \beta$ and $\rho = \lambda = 1$, we use the following notations:

$$F_A^{(r)}[\mathbf{x}_r; b] \equiv F_A^{(r:\alpha,\alpha;1,1)}[\mathbf{x}_r; b], \qquad F_B^{(r)}[\mathbf{x}_r; b] \equiv F_B^{(r:\alpha,\alpha;1,1)}[\mathbf{x}_r; b],$$

$${}_1F_D^{(r)}[\mathbf{x}_r; b] \equiv {}_1F_D^{(r:\alpha,\alpha;1,1)}[\mathbf{x}_r; b], \qquad {}_2F_D^{(r)}[\mathbf{x}_r; b] \equiv {}_2F_D^{(r:\alpha,\alpha;1,1)}[\mathbf{x}_r; b].$$

and

$${}_2F_1\left[\begin{matrix}A, B \\ C\end{matrix}; x; b\right] = {}_2F_1^{(\alpha,\alpha;1,1)}\left[\begin{matrix}A, B \\ C\end{matrix}; x; b\right].$$

These special cases attract great interest because they are relatively easy to handle, and the relationship between such extensions and their original forms are clear.

The paper is organized as follows. In Section 2, we introduce a new extension of the Dirichlet integral. In Section 3, we first state the formal definitions of the extended Appell-Lauricella hypergeometric functions by using the extended beta function (1) and the extended Dirichlet integral (see Theorem 2.1). Then, by applying some results of the Laguerre polynomials, we expand $F_A^{(r)}[\mathbf{x}_r; b]$ and ${}_1F_D^{(r)}[\mathbf{x}_r; b]$ in terms of the Laguerre polynomials and the ordinary Lauricella functions $F_A^{(r)}$ and $F_D^{(r)}$. We also prove in this section that $F_B^{(r)}[\mathbf{x}_r; b]$ and ${}_2F_D^{(r)}[\mathbf{x}_r; b]$ can be expanded by using the Laguerre polynomials and Srivastava-Daoust generalized Lauricella functions. In Section 4, we establish several generating functions for $F_A^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b]$. A generating function for $F_B^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b]$ is contained in Section 5. Generating functions for ${}_1F_D^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b]$ and ${}_2F_D^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b]$ and their connections with the Lagrange polynomials and extended fractional integral operators are considered in Section 6.

2. An Extension of Dirichlet's Integral

The well-known Dirichlet integral is given by (see [1, p. 62, Eq. (4)]; see also [13, p. 434, Eq. (2.1)])

$$\int_{E_{k-1}} u_1^{b_1-1}\cdots u_{k-1}^{b_{k-1}-1}\left(1-\sum_{i=1}^{k-1}u_i\right)^{b_k-1}\mathrm{d}u_1\cdots\mathrm{d}u_{k-1}=\frac{\Gamma(b_1)\cdots\Gamma(b_k)}{\Gamma(b_1+\cdots+b_k)}, \tag{10}$$

where $\Re(b_i)>0$ $(i=1,\cdots,k)$, $k\geq 2$, and

$$E_{k-1}:=\left\{(u_1,\cdots,u_{k-1}):u_1\geq 0,\cdots,u_{k-1}\geq 0,\ \sum_{i=1}^{k-1}u_i\leq 1\right\} \tag{11}$$

is the standard simplex in $\mathbb{R}^{k-1}$.

The complex measure $\mu_{\mathbf{b}_k}$ defined on simplex E_{k-1} by

$$\mathrm{d}\mu_{\mathbf{b}_k}(u_{k-1}):=\frac{1}{\mathbf{B}(\mathbf{b}_k)}u_1^{b_1-1}\cdots u_{k-1}^{b_{k-1}-1}\left(1-\sum_{i=1}^{k-1}u_i\right)^{b_k-1}\mathrm{d}u_1\cdots\mathrm{d}u_{k-1} \tag{12}$$

is called a Dirichlet measure, which is introduced by Carlson (see [1, p. 64]). The normalizing constant $\mathrm{B}(\mathbf{b}_k)$ is given by

$$\mathbf{B}(\mathbf{b}_k)\equiv\mathbf{B}(b_1,\cdots,b_k)=\frac{\Gamma(b_1)\cdots\Gamma(b_k)}{\Gamma(b_1+\cdots+b_k)}.$$

By (10), it is easy to see that

$$\mu_{\mathbf{b}_k}(E_{k-1})=\int_{E_{k-1}}\mathrm{d}\mu_{\mathbf{b}_k}(u_{k-1})=1.$$

The total variation measure $|\mu_{\mathbf{b}_k}|$ is given by

$$\mathrm{d}|\mu_{\mathbf{b}_k}|(u_{k-1})=\frac{\mathbf{B}(\Re(\mathbf{b}_k))}{|\mathbf{B}(\mathbf{b}_k)|}\mathrm{d}\mu_{\Re(\mathbf{b}_k)}(u_{k-1}),\quad |\mu_{\mathbf{b}_k}|(E_{k-1})=\frac{\mathbf{B}(\Re(\mathbf{b}_k))}{|\mathbf{B}(\mathbf{b}_k)|}<\infty.$$

When $k=2$, we shall write (12) as

$$\mathrm{d}\mu(u_1):=\frac{u_1^{b_1-1}(1-u_1)^{b_2-1}}{B(b_1,b_2)}\mathrm{d}u_1, \tag{13}$$

where $B(b_1,b_2)$ is the ordinary beta function.

Theorem 2.1. *Let E_{k-1} be the standard simplex in $\mathbb{R}^{k-1}$ defined by* (11). *Then*

$$\begin{aligned}\int_{E_{k-1}} u_1^{b_1-1}\cdots u_{k-1}^{b_{k-1}-1}\left(1-\sum_{i=1}^{k-1}u_i\right)^{b_k-1}\prod_{i=1}^{k-1}{}_1F_1\left(\alpha;\beta;-\frac{\omega_i^{\rho+\lambda}b}{u_i^{\rho}(\omega_i-u_i)^{\lambda}}\right)\mathrm{d}u_1\cdots\mathrm{d}u_{k-1}\\ =\prod_{j=1}^{k-1}\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}\left(b_j,b_{j+1}+\cdots+b_k\right),\end{aligned} \tag{14}$$

where $\rho,\lambda\geq 0$, $\min\{\Re(b),\Re(\alpha),\Re(\beta)\}>0$,

$$\Re(b_1)>-\Re(\rho\alpha),\ \Re(b_i)>-\frac{\Re(\lambda\alpha)}{k-1}\ \ (i=2,\cdots,k),$$

and

$$\omega_1:=1,\ \omega_2:=(1-u_1),\ \omega_3:=(1-u_1-u_2),\ \cdots,\ \omega_{k-1}:=(1-u_1-\cdots-u_{k-2}).$$

Proof. Denote the integral in (14) by $I_k(b_1,\cdots,b_k)$. We can express $I_k(b_1,\cdots,b_k)$ as

$$I_k(b_1,\cdots,b_k) = \mathbf{B}(\mathbf{b}_k)\int_{E_{k-1}}\prod_{i=1}^{k-1}{}_1F_1\left(\alpha;\beta;-\frac{\omega_i^{\rho+\lambda}b}{u_i^{\rho}(\omega_i-u_i)^{\lambda}}\right)\mathrm{d}\mu_{\mathbf{b}_k}(u_{k-1}), \tag{15}$$

where $\mu_{\mathbf{b}_k}(u_{k-1})$ is the Dirichlet measure defined by (12). From the properties of the Dirichlet measure $\mu_{\mathbf{b}_k}(u_{k-1})$ and confluent hypergeometric function ${}_1F_1$, it is clear that the integral in (14) exists.

For $k=2$, we have

$$\begin{aligned} I_2(b_1,b_2) &= \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(b_1+b_2)}\int_0^1 {}_1F_1\left(\alpha;\beta;-\frac{\omega_1^{\rho+\lambda}b}{u_1^{\rho}(\omega_1-u_1)^{\lambda}}\right)\mathrm{d}\mu(u_1)\\ &= \int_0^1 u_1^{b_1-1}(1-u_1)^{b_2-1}\,{}_1F_1\left(\alpha;\beta;-\frac{b}{u_1^{\rho}(1-u_1)^{\lambda}}\right)\mathrm{d}u_1\\ &= \mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(b_1,b_2),\quad (\omega_1=1) \end{aligned} \tag{16}$$

which is just the extended beta function (1). If we set $k=3$, then

$$I_3(b_1,b_2,b_3) = \mathbf{B}(\mathbf{b}_3)\int_{E_2}{}_1F_1\left(\alpha;\beta;-\frac{\omega_1^{\rho+\lambda}b}{u_1^{\rho}(\omega_1-u_1)^{\lambda}}\right){}_1F_1\left(\alpha;\beta;-\frac{\omega_2^{\rho+\lambda}b}{u_2^{\rho}(\omega_2-u_2)^{\lambda}}\right)\mathrm{d}\mu_{\mathbf{b}_3}(u_2). \tag{17}$$

Integrating over u_2 and noting that $\omega_2 = 1-u_1$ with $\omega_1=1$, we have

$$\begin{aligned} I_3(b_1,b_2,b_3) &= \mathbf{B}(\mathbf{b}_3)\int_{E_2}{}_1F_1\left(\alpha;\beta;-\frac{b}{u_1^{\rho}(1-u_1)^{\lambda}}\right){}_1F_1\left(\alpha;\beta;-\frac{\omega_2^{\rho+\lambda}b}{u_2^{\rho}(\omega_2-u_2)^{\lambda}}\right)\mathrm{d}\mu_{\mathbf{b}_3}(u_2)\\ &= \int_0^1 u_1^{b_1-1}{}_1F_1\left(\alpha;\beta;-\frac{b}{u_1^{\rho}(1-u_1)^{\lambda}}\right)\mathrm{d}u_1\\ &\qquad\cdot\int_0^{\omega_2} u_2^{b_2-1}(\omega_2-u_2)^{b_3-1}\,{}_1F_1\left(\alpha;\beta;-\frac{\omega_2^{\rho+\lambda}b}{u_2^{\rho}(\omega_2-u_2)^{\lambda}}\right)\mathrm{d}u_2. \end{aligned}$$

Setting $u_2 = \omega_2 v$, we get

$$\begin{aligned} I_3(b_1,b_2,b_3) &= \int_0^1 u_1^{b_1-1}\omega_2^{b_2+b_3-1}{}_1F_1\left(\alpha;\beta;-\frac{b}{u_1^{\rho}(1-u_1)^{\lambda}}\right)\mathrm{d}u_1\\ &\qquad\cdot\int_0^1 v^{b_2-1}(1-v)^{b_3-1}\,{}_1F_1\left(\alpha;\beta;-\frac{b}{v^{\rho}(1-v)^{\lambda}}\right)\mathrm{d}v\\ &= \mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(b_2,b_3)\int_0^1 u_1^{b_1-1}(1-u_1)^{b_2+b_3-1}\,{}_1F_1\left(\alpha;\beta;-\frac{b}{u_1^{\rho}(1-u_1)^{\lambda}}\right)\mathrm{d}u_1\\ &= \mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(b_2,b_3)\,\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(b_1,b_2+b_3). \end{aligned} \tag{18}$$

In above evaluation of $I_3(b_1,b_2,b_3)$, we have used Fubini's theorem (see [1, p. 294, Theorem B.2]). Actually, there exists a positive number $\mathfrak{M}$ such that

$$\int_0^1\int_0^1\left|{}_1F_1\left(\alpha;\beta;-\frac{b}{u_1^{\rho}(1-u_1)^{\lambda}}\right){}_1F_1\left(\alpha;\beta;-\frac{b}{v^{\rho}(1-v)^{\lambda}}\right)\right|\mathrm{d}\left|\mu\right|(u_1)\,\mathrm{d}\left|\mu\right|(v)$$

$$\begin{aligned}&\leq \mathfrak{M}\int_0^1 \mathrm{d}\left|\mu\right|(u_1)\int_0^1 \mathrm{d}\left|\mu\right|(v)\\&= \mathfrak{M}\frac{B\left(\Re\left(b_1\right),\Re\left(b_2+b_3\right)\right)}{\left|B\left(b_1,b_2+b_3\right)\right|}\frac{B\left(\Re\left(b_2\right),\Re\left(b_3\right)\right)}{\left|B\left(b_2,b_3\right)\right|}<\infty,\end{aligned}\tag{19}$$

where $\min\{\Re\left(b_1\right),\Re\left(b_2\right),\Re\left(b_3\right)\}>0$,

$$\left|\mu\right|(u_1)=\frac{u_1^{\Re(b_1)-1}\left(1-u_1\right)^{\Re(b_2)+(b_3)-1}}{\left|B\left(b_1,b_2+b_3\right)\right|}\mathrm{d}u_1 \quad \text{and} \quad \left|\mu\right|(v)=\frac{v^{\Re(b_2)-1}\left(1-v\right)^{\Re(b_3)-1}}{\left|B\left(b_2,b_3\right)\right|}\mathrm{d}v,$$

which means the use of Fubini's theorem in evaluating $I_3\left(b_1,b_2,b_3\right)$ is valid. According to the method used in [17, p. 633, Theorem 2.1], we know that the range of parameters $\Re\left(b_i\right)$ $(i=1,2,3)$ can be easily extended.

In general, we have

$$\begin{aligned}I_k\left(b_1,\cdots,b_k\right)&=\int_{E_{k-2}}u_1^{b_1-1}\cdots u_{k-2}^{b_{k-2}-1}\prod_{i=1}^{k-2}{}_1F_1\left(\alpha;\beta;-\frac{\omega_i^{\rho+\lambda}b}{u_i^{\rho}\left(\omega_i-u_i\right)^{\lambda}}\right)\mathrm{d}u_1\cdots\mathrm{d}u_{k-2}\\&\quad\cdot\int_0^{\omega_{k-1}}u_{k-1}^{b_{k-1}-1}\left(\omega_{k-1}-u_{k-1}\right)^{b_k-1}{}_1F_1\left(\alpha;\beta;-\frac{\omega_{k-1}^{\rho+\lambda}b}{u_{k-1}^{\rho}\left(\omega_{k-1}-u_{k-1}\right)^{\lambda}}\right)\mathrm{d}u_{k-1}\\&=\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}\left(b_{k-1},b_k\right)\int_{E_{k-2}}u_1^{b_1-1}\cdots u_{k-2}^{b_{k-2}-1}\omega_{k-1}^{b_{k-1}+b_k-1}\prod_{i=1}^{k-2}{}_1F_1\left(\alpha;\beta;-\frac{\omega_i^{\rho+\lambda}b}{u_i^{\rho}\left(\omega_i-u_i\right)^{\lambda}}\right)\mathrm{d}u_1\cdots\mathrm{d}u_{k-2}\\&=\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}\left(b_{k-1},b_k\right)I_{k-1}\left(b_1,\cdots,b_{k-2},b_{k-1}+b_k\right)\\&=\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}\left(b_{k-1},b_k\right)\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}\left(b_{k-2},b_{k-1}+b_k\right)I_{k-2}\left(b_1,\cdots,b_{k-3},b_{k-2}+b_{k-1}+b_k\right)\\&\quad\cdots\cdots\\&=\prod_{j=1}^{k-1}\mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}\left(b_j,b_{j+1}+\cdots+b_k\right).\end{aligned}\tag{20}$$

The validity of evaluating integral $I_k\left(b_1,\cdots,b_k\right)$ as an iterated integral can be verified in the same manner as we have done to $I_3\left(b_1,b_2,b_3\right)$, and then Fubini's theorem gives the final result.

Finally, the ranges for parameters b_i $(i=1,\cdots,k)$ are decided by the extended beta function involving the most parameters. The requirement $\Re\left(b_1\right)>-\Re\left(\rho\alpha\right)$ is natural. We also see that

$$\Re\left(b_i\right)>-\frac{\Re\left(\lambda\alpha\right)}{k-1}\quad(i=1,\cdots,k)\quad\Longrightarrow\quad\Re\left(b_2+\cdots+b_k\right)>-\Re\left(\lambda\alpha\right).$$

Hence, each extended beta function appearing in the final expression is proper. □

Remark 2.2. *If we set $b=0$ in* (14), *it reduces to the Dirichlet integral* (10).

3. Extended Appell-Lauricella Hypergeometric Functions

In this section, we introduce four new multivariable hypergeometric functions by means of their Euler type integrals.

Definition 3.1. *The extended Appell-Lauricella hypergeometric functions of r variables can be defined by the following four Euler type integrals:*

- *The extension of $F_A^{(r)}$ denoted by $F_A^{(r:\alpha,\beta;\rho,\lambda)}$ is defined by*

$$F_A^{(r:\alpha,\beta;\rho,\lambda)}[a,\mathbf{b}_r;\mathbf{c}_r;\mathbf{x}_r;b]=\prod_{i=1}^{r}\frac{1}{B(b_i,c_i-b_i)}$$
$$\cdot\int_0^1\cdots\int_0^1\prod_{i=1}^{r}u_i^{b_i-1}(1-u_i)^{c_i-b_i-1}\,{}_1F_1\left(\alpha;\beta;-\frac{b}{u_i^{\rho}(1-u_i)^{\lambda}}\right)(1-\langle\mathbf{x}_r,\mathbf{u}_r\rangle)^{-a}\,\mathrm{d}u_1\cdots\mathrm{d}u_r. \tag{21}$$

$(\Re(c_i)>\Re(b_i)>0, i=1,\cdots,r)$

- *The extension of $F_B^{(r)}$ denoted by $F_B^{(r:\alpha,\beta;\rho,\lambda)}$ is defined by*

$$F_B^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{a}_r,\mathbf{b}_r;c;\mathbf{x}_r;b]=\frac{\Gamma(c)}{\Gamma(a_1)\cdots\Gamma(a_r)\Gamma(c-|\mathbf{a}_r|)}$$
$$\cdot\int_{E_r}\prod_{i=1}^{r}u_i^{a_i-1}(1-u_ix_i)^{-b_i}\,{}_1F_1\left(\alpha;\beta;-\frac{\omega_i^{\rho+\lambda}b}{u_i^{\rho}(\omega_i-u_i)^{\lambda}}\right)(1-|\mathbf{u}_r|)^{a_{r+1}-1}\,\mathrm{d}u_1\cdots\mathrm{d}u_r. \tag{22}$$

$$(a_{r+1}=c-|\mathbf{a}_r|;\ \Re(a_i)>0, i=1,\cdots,r;\ \Re(c-|\mathbf{a}_r|)>0)$$

- *The function $F_D^{(r)}$ has two reasonable extensions, they are*

$${}_1F_D^{(r:\alpha,\beta;\rho,\lambda)}[a,\mathbf{b}_r;c;\mathbf{x}_r;b]=\frac{1}{B(a,c-a)}$$
$$\cdot\int_0^1 u^{a-1}(1-u)^{c-a-1}\,{}_1F_1\left(\alpha;\beta;-\frac{b}{u^{\rho}(1-u)^{\lambda}}\right)\prod_{i=1}^{r}(1-x_iu)^{-b_i}\,\mathrm{d}u \tag{23}$$

$(\Re(c)>\Re(a)>0)$

and

$${}_2F_D^{(r:\alpha,\beta;\rho,\lambda)}[a,\mathbf{b}_r;c;\mathbf{x}_r;b]=\frac{\Gamma(c)}{\Gamma(b_1)\cdots\Gamma(b_r)\Gamma(c-|\mathbf{b}_r|)}$$
$$\cdot\int_{E_r}\prod_{i=1}^{r}u_i^{b_i-1}{}_1F_1\left(\alpha;\beta;-\frac{\omega_i^{\rho+\lambda}b}{u_i^{\rho}(\omega_i-u_i)^{\lambda}}\right)(1-|\mathbf{u}_r|)^{b_{r+1}-1}(1-\langle\mathbf{u}_r,\mathbf{x}_r\rangle)^{-a}\,\mathrm{d}u_1\cdots\mathrm{d}u_r. \tag{24}$$

$$(b_{r+1}=c-|\mathbf{b}_r|;\ \Re(b_i)>0, i=1,\cdots,r;\ \Re(c-|\mathbf{b}_r|)>0)$$

Remark 3.2.

a. *In the case $b=0$, the above extended Appell-Lauricella hypergeometric functions reduces immediately to the Euler type integral representations of the original Lauricella functions $F_A^{(r)}$, $F_B^{(r)}$ and $F_D^{(r)}$, respectively (see [11, pp. 48–49, Section 2.3]).*

b. *For $\alpha=\beta$ and $\rho=\lambda=1$ the extended hypergeometric functions in* (21), (22) *and* (24)*, becomes into the known results due to Şahin (see [24, p. 1142]). Here, it is important to mentioning that Şahin also point out that the third kind of Lauricella's hypergeometric function $F_C^{(r)}$ can not be extended in this manner, since its coefficient can not be expressed as a product of beta functions.*

c. *By suitably expanding the integrands in above integrals, we can easily find their series representations* (3)–(6) *by termwise integration. The series expressions* (3)–(6) *(may be not heuristic) can also be considered as the definitions of the extended Appell-Lauricella hypergeometric functions, since they are easier to be manipulated especially in deriving some identities.*

3.1. Expansions for $F_A^{(r)}[\mathbf{x}_r;b]$ and ${}_1F_D^{(r)}[\mathbf{x}_r;b]$

The (generalized) Laguerre polynomials $L_n^{(\alpha)}(x)$, $\alpha\in\mathbb{C}$, $x\geq 0$, and $n=0,1,2,\cdots$ can be defined by the generating function [28, p. 84, Eq. (14)]:

$$(1-z)^{-\alpha-1}\exp\left(-\frac{xz}{1-z}\right)=\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)\,z^n,\quad |z|<1. \tag{25}$$

For $\alpha=0$, we simply write $L_n(x)=L_n^{(0)}(x)$.

Theorem 3.3. *For the extended Appell-Lauricella hypergeometric function $F_A^{(r)}[\mathbf{x}_r;b]$, the following expression holds true:*

$$\begin{aligned}F_A^{(r)}[a,\mathbf{b}_r;\mathbf{c}_r;\mathbf{x}_r;b]=e^{-2rb}\sum_{\mathbf{m}_{2r}=0}^{\infty}\sum_{k=0}^{|\mathbf{m}_{2r}|}\gamma_k L_k^{(\alpha)}(q)\prod_{i=1}^{r}\frac{(b_i)_{m_{2i-1}+1}(c_i-b_i)_{m_{2i}+1}}{(c_i)_{m_{2i-1}+m_{2i}+2}}\\ \cdot F_A^{(r)}[a,b_1+m_1+1,\cdots,b_r+m_{2r-1}+1;c_1+m_1+m_2+2,\cdots,c_r+m_{2r-1}+m_{2r}+2;\mathbf{x}_r],\end{aligned} \tag{26}$$

where $q>0$, $|x_1|+\cdots+|x_r|<1$ and

$$\gamma_k=F_A^{(2r+1)}\Bigg[\alpha+1,-\mathbf{m}_{2r},k;\underbrace{1,\cdots,1}_{2r},\alpha+1;\underbrace{\frac{b}{q},\cdots,\frac{b}{q}}_{2r},1\Bigg],$$

it being assumed that each member of the assertion (26) *exists.*

Proof. Setting $\alpha=\beta$ and $\rho=\lambda=1$ in the integral representation (21), we get

$$F_A^{(r)}[a,\mathbf{b}_r;\mathbf{c}_r;\mathbf{x}_r;b]=\prod_{i=1}^{r}\frac{1}{B(b_i,c_i-b_i)}\int_0^1\cdots\int_0^1\prod_{i=1}^{r}u_i^{b_i-1}(1-u_i)^{c_i-b_i-1}e^{-\frac{b}{u_i(1-u_i)}}(1-\langle\mathbf{x}_r,\mathbf{u}_r\rangle)^{-a}\,\mathrm{d}u_1\cdots\mathrm{d}u_r. \tag{27}$$

The key part of the proof is to expand $\prod_{i=1}^{r}e^{-\frac{b}{u_i(1-u_i)}}$ in a suitable way. By applying the familiar expansion [2, p. 238, Eq. (5.155)]

$$e^{-\frac{b}{t(1-t)}}=e^{-2b}\sum_{m,n=0}^{\infty}L_m(b)\,L_n(b)\,t^{m+1}(1-t)^{n+1}, \tag{28}$$

and writing (technically) $b=tq$ (>0) in it, we can get

$$\begin{aligned}\prod_{i=1}^{r}e^{-\frac{tq}{u_i(1-u_i)}}&=e^{-2tq}\sum_{m_1,m_2=0}^{\infty}L_{m_1}(tq)\,L_{m_2}(tq)\,u_1^{m_1+1}(1-u_1)^{m_2+1}\\ &\qquad\cdots e^{-2tq}\sum_{m_{2r-1},m_{2r}=0}^{\infty}L_{m_{2r-1}}(tq)\,L_{m_{2r}}(tq)\,u_r^{m_{2r-1}+1}(1-u_r)^{m_{2r}+1}\\ &=e^{-2rtq}\sum_{\mathbf{m}_{2r}=0}^{\infty}\prod_{j=1}^{2r}L_{m_j}(tq)\prod_{i=1}^{r}u_i^{m_{2i-1}+1}(1-u_i)^{m_{2i}+1}.\end{aligned} \tag{29}$$

It is known that for the products of various Laguerre polynomials we have the following elegant result due to Erdélyi (see [7, p. 156, Eq. (5)]; see also [27, p. 48, Eq. (2)]):

$$\prod_{j=1}^{r}L_{m_j}^{(a_j)}\left(z_jx\right)=\sum_{k=0}^{|\mathbf{m}_r|}\varphi_k L_k^{(\alpha)}(x), \tag{30}$$

where the coefficients φ_k $(k \geq 0)$ are given by

$$\varphi_k = \binom{m_1 + a_1}{m_1} \cdots \binom{m_r + a_r}{m_r} F_A^{(r+1)} \left[\alpha + 1, -\mathbf{m}_r, -k; \mathbf{a}_r + \mathbf{1}, \alpha + 1; \mathbf{z}_r, 1\right]. \tag{31}$$

With the help of the expansion formula (30), we can express (29) as

$$\prod_{i=1}^{r} e^{-\frac{tq}{u_i(1-u_i)}} = e^{-2rtq} \sum_{\mathbf{m}_{2r}=0}^{\infty} \sum_{k=0}^{|\mathbf{m}_{2r}|} \gamma_k L_k^{(\alpha)}(q) \prod_{i=1}^{r} u_i^{m_{2i-1}+1} (1-u_i)^{m_{2i}+1}, \tag{32}$$

where

$$\gamma_k = F_A^{(2r+1)} \Big[\alpha + 1, -\mathbf{m}_{2r}, -k; \underbrace{1, \cdots, 1}_{2r}, \alpha + 1; \underbrace{t, \cdots, t}_{2r}, 1\Big].$$

Now, we can set $t = b/q$ to find that

$$\prod_{i=1}^{r} e^{-\frac{b}{u_i(1-u_i)}} = e^{-2rb} \sum_{\mathbf{m}_{2r}=0}^{\infty} \sum_{k=0}^{|\mathbf{m}_{2r}|} \gamma_k L_k^{(\alpha)}(q) \prod_{i=1}^{r} u_i^{m_{2i-1}+1} (1-u_i)^{m_{2i}+1}, \tag{33}$$

where

$$\gamma_k = F_A^{(2r+1)} \left[\alpha + 1, -\mathbf{m}_{2r}, -k; \underbrace{1, \cdots, 1}_{2r}, \alpha + 1; \underbrace{\frac{b}{q}, \cdots, \frac{b}{q}}_{2r}, 1\right].$$

Finally, making use of (33), the integral (27) can be evaluated as

$$\begin{aligned} F_A^{(r)}[a, \mathbf{b}_r; \mathbf{c}_r; \mathbf{x}_r; b] &= e^{-2rb} \prod_{i=1}^{r} \frac{1}{B(b_i, c_i - b_i)} \sum_{\mathbf{m}_{2r}=0}^{\infty} \sum_{k=0}^{|\mathbf{m}_{2r}|} \gamma_k L_k^{(\alpha)}(q) \\ &\quad \cdot \int_0^1 \cdots \int_0^1 \prod_{i=1}^{r} u_i^{b_i + m_{2i-1}} (1-u_i)^{c_i + m_{2i} - b_i} (1 - \langle \mathbf{x}_r, \mathbf{u}_r \rangle)^{-a} \, \mathrm{d}u_1 \cdots \mathrm{d}u_r \\ &= e^{-2rb} \prod_{i=1}^{r} \frac{1}{B(b_i, c_i - b_i)} \sum_{\mathbf{m}_{2r}=0}^{\infty} \sum_{k=0}^{|\mathbf{m}_{2r}|} \gamma_k L_k^{(\alpha)}(q) \prod_{i=1}^{r} B(b_i + m_{2i-1} + 1, c_i - b_i + m_{2i} + 1) \\ &\quad \cdot F_A^{(r)}[a, b_1 + m_1 + 1, \cdots, b_r + m_{2r-1} + 1; c_1 + m_1 + m_2 + 2, \cdots, c_r + m_{2r-1} + m_{2r} + 2; \mathbf{x}_r]. \end{aligned} \tag{34}$$

This completes the proof. □

Theorem 3.4. *For the extended Appell-Lauricella hypergeometric function ${}_1F_D^{(r)}[\mathbf{x}_r; b]$, the following expression holds true:*

$$ {}_1F_D^{(r)}[a, \mathbf{b}_r; c; \mathbf{x}_r; b] = e^{-2b} \sum_{m,n=0}^{\infty} \sum_{k=0}^{m+n} \gamma_k L_k^{(\alpha)}(q) \frac{(a)_{m+1} (c-a)_{n+1}}{(c)_{m+n+2}} F_D^{(r)}[a + m + 1, \mathbf{b}_r; c + m + n + 2; \mathbf{x}_r], \tag{35}$$

where $q > 0$, $\max\{|x_1|, \cdots, |x_r|\} < 1$ *and*

$$\gamma_k = F_A^{(3)} \left[\alpha + 1, -m, -n, -k; 1, 1, \alpha + 1; \frac{b}{q}, \frac{b}{q}, 1\right],$$

it being assumed that each member of the assertion (35) *exists.*

Proof. By setting $\alpha=\beta$, $\rho=\lambda=1$ and $b=tq$ (>0) in the integral representation (23), we get

$$ {}_1F_D^{(r)}\left[a,\mathbf{b}_r;c;\mathbf{x}_r;tq\right]=\frac{1}{B(a,c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}e^{-\frac{tq}{u(1-u)}}\prod_{i=1}^{r}(1-x_iu)^{-b_i}\,\mathrm{d}u. \tag{36}$$

Now, the same argument gives

$$\begin{aligned} {}_1F_D^{(r)}\left[a,\mathbf{b}_r;c;\mathbf{x}_r;tq\right]&=e^{-2tq}\sum_{m,n=0}^{\infty}L_m(tq)\,L_n(tq)\int_0^1\frac{u^{a+m}(1-u)^{c+n-a}}{B(a,c-a)}\prod_{i=1}^{r}(1-x_iu)^{-b_i}\,\mathrm{d}u\\ &=\frac{e^{-2tq}}{B(a,c-a)}\sum_{m,n=0}^{\infty}L_m(tq)\,L_n(tq)\,B(a+m+1,c-a+n+1)\\ &\quad\cdot F_D^{(r)}\left[a+m+1,\mathbf{b}_r;c+m+n+2;\mathbf{x}_r\right]\\ &=\frac{e^{-2tq}}{B(a,c-a)}\sum_{m,n=0}^{\infty}\sum_{k=0}^{m+n}\gamma_kL_k^{(\alpha)}(q)\,B(a+m+1,c-a+n+1)\\ &\quad\cdot F_D^{(r)}\left[a+m+1,\mathbf{b}_r;c+m+n+2;\mathbf{x}_r\right], \end{aligned}\tag{37}$$

where

$$\gamma_k=F_A^{(3)}\left[\alpha+1,-m,-n,-k;1,1,\alpha+1;t,t,1\right]. \tag{38}$$

The result follows directly by letting $t=b/q$ in (37) and (38). □

Remark 3.5. *Setting $x_1=\cdots=x_r=x$ in (35) and using the reduction formula (7), we can find that*

$${}_2F_1\left[\begin{matrix}|\mathbf{b}_r|,a\\ c\end{matrix};x;b\right]=e^{-2b}\sum_{m,n=0}^{\infty}\sum_{k=0}^{m+n}\gamma_kL_k^{(\alpha)}(q)\frac{(a)_{m+1}(c-a)_{n+1}}{(c)_{m+n+2}}\,{}_2F_1\left[\begin{matrix}|\mathbf{b}_r|,a+m+1\\ c+m+n+2\end{matrix};x\right],$$

which is equivalent to the known result [20, p. 18, Section 3].

3.2. Expansions for $F_B^{(r)}[\mathbf{x}_r;b]$ and ${}_2F_D^{(r)}[\mathbf{x}_r;b]$

The Srivastava-Daoust generalized Lauricella function is defined by [27, p. 37]

$$\begin{aligned} F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}&=F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\left[\begin{matrix}[(a):\theta^{(1)},\cdots,\theta^{(r)}]:[\left(b^{(1)}\right):\phi^{(1)}];\cdots;[\left(b^{(r)}\right):\phi^{(r)}];\\ \\ [(c):\psi^{(1)},\cdots,\psi^{(r)}]:[\left(d^{(1)}\right):\delta^{(1)}];\cdots;[\left(d^{(r)}\right):\delta^{(r)}];\end{matrix}\ \mathbf{x}_r\right]\\ &=\sum_{\mathbf{m}_r=0}^{\infty}\Omega(\mathbf{m}_r)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!}, \end{aligned}\tag{39}$$

where, for convenience,

$$\Omega(\mathbf{m}_r)=\frac{\prod_{j=1}^{A}\left(a_j\right)_{m_1\theta_j^{(1)}+\cdots+m_r\theta_j^{(r)}}\prod_{j=1}^{B^{(1)}}\left(b_j^{(1)}\right)_{m_1\phi_j^{(1)}}\cdots\prod_{j=1}^{B^{(r)}}\left(b_j^{(r)}\right)_{m_r\phi_j^{(r)}}}{\prod_{j=1}^{C}\left(c_j\right)_{m_1\psi_j^{(1)}+\cdots+m_r\psi_j^{(r)}}\prod_{j=1}^{D^{(1)}}\left(d_j^{(1)}\right)_{m_1\delta_j^{(1)}}\cdots\prod_{j=1}^{D^{(r)}}\left(d_j^{(r)}\right)_{m_r\delta_j^{(r)}}},$$

the coefficients

$$\begin{cases}\theta_j^{(k)}, j=1,\cdots,A; & \phi_j^{(k)}, j=1,\cdots,B^{(k)};\\ \psi_j^{(k)}, j=1,\cdots,C; & \delta_j^{(k)}, j=1,\cdots,D^{(k)}; \quad \forall k\in\{1,\cdots,r\}\end{cases}$$

are real and positive, and (a) abbreviates the array of A parameters $a_1,\cdots,a_A$, $\left(b^{(k)}\right)$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}$, $j=1,\cdots,B^{(k)}$; $\forall k\in\{1,\cdots,r\}$, with similar interpretations for (c) and $\left(d^{(k)}\right)$, $k=1,\cdots,r$; *et cetera*. For the precise conditions under which the multiple series (39) and its special cases converge absolutely, see [25, pp. 157–158, Section 5].

First, we need the following lemma.

Lemma 3.6. *If* $b\geq 0$ *and* $\min\{\Re(x),\Re(y)\}>0$, *then we have*

$$e^{-2b}\sum_{k,l=0}^{\infty}|B(x+k+1,y+l+1)||L_k(b)||L_l(b)|<e^{-b+\frac{\pi}{2}|\Im(x+y)|}B(\Re(x),\Re(y)) \tag{40}$$

for $\Im(x+y)\neq 0$, *and*

$$e^{-2b}\sum_{k,l=0}^{\infty}|B(x+k+1,y+l+1)||L_k(b)||L_l(b)|\leq e^{-b}B(\Re(x),\Re(y)) \tag{41}$$

for $\Im(x+y)=0$.

Proof. For the Laguerre polynomial $L_k(b)$, we have the following useful inequality [21, p. 450, Eq. (18.14.8)]

$$|L_k(b)|\leq e^{b/2}\quad (b\geq 0;\ k\in\mathbb{N}_0),$$

and for gamma function we have $|\Gamma(x)|\leq\Gamma(\Re(x))$ $(\Re(x)>0)$ and [1, p. 51, Eq. (7)]

$$|\Gamma(x)|\geq\Gamma(\Re(x))(\operatorname{sech}\pi\Im(x))^{1/2}>\Gamma(\Re(x))e^{-\frac{\pi}{2}|\Im(x)|}\quad \left(\Re(x)\geq\frac{1}{2};\ \Im(x)\neq 0\right).$$

By using these inequalities, we get:

$$\begin{aligned}
&e^{-2b}\sum_{k,l=0}^{\infty}|B(x+k+1,y+l+1)||L_k(b)||L_l(b)|\\
&\leq e^{-b}\sum_{k,l=0}^{\infty}\left|\frac{\Gamma(x+k+1)\Gamma(y+l+1)}{\Gamma(x+y+k+l+2)}\right|<e^{-b}\sum_{k,l=0}^{\infty}\frac{\Gamma(\Re(x)+k+1)\Gamma(\Re(y)+l+1)}{e^{-\frac{\pi}{2}|\Im(x+y)|}\Gamma(\Re(x+y)+k+l+2)}\quad (\Im(x+y)\neq 0)\\
&=e^{-b+\frac{\pi}{2}|\Im(x+y)|}\frac{\Gamma(\Re(x)+1)\Gamma(\Re(y)+1)}{\Gamma(\Re(x+y)+2)}\sum_{k,l=0}^{\infty}\frac{(\Re(x)+1)_k(\Re(y)+1)_l}{(\Re(x+y)+2)_{k+l}}\frac{(1)_k}{k!}\frac{(1)_l}{l!}\\
&=e^{-b+\frac{\pi}{2}|\Im(x+y)|}B(\Re(x)+1,\Re(y)+1)F_3[\Re(x)+1,\Re(y)+1,1,1;\Re(x+y)+2;1,1]<\infty,
\end{aligned}\tag{42}$$

where F_3 is the Appell function defined by [27, p. 23, Eq. (4)]

$$F_3[a,a',b,b';c;x,y]=\sum_{k,l=0}^{\infty}\frac{(a)_k(a')_l(b)_k(b')_l}{(c)_{k+l}}\frac{x^k}{k!}\frac{y^l}{l!}\quad (\max\{|x|,|y|\}<1).$$

The conditions under which the Appell function F_3 converges absolutely for $|x|=|y|=1$ can be found, for example, in [23] and [14]. By making use of the integral representation [27, p. 279, Eq. (17)]

$$F_3[a,a',b,b';a+a';x,y]=\frac{1}{B(a,a')}\int_0^1 u^{a-1}(1-u)^{a'-1}(1-ux)^{-b}[1-(1-u)y]^{-b'}\,du,$$

$(\max\{\Re(a), \Re(a')\} > 0;\ |x| < 1, |y| < 1)$

we can find that

$$B(\Re(x)+1, \Re(y)+1)\, F_3[\Re(x)+1, \Re(y)+1, 1, 1; \Re(x+y)+2; 1, 1] = B(\Re(x), \Re(y)). \tag{43}$$

Substituting (43) into (42) we get the inequality (40). The inequality (41) can be found similarly. □

Theorem 3.7. *For the extended Appell-Lauricella hypergeometric function ${}_2F_D^{(r)}[\mathbf{x}_r; b]$, the following expression holds true:*

$$\begin{aligned}
{}_2F_D^{(r)}[a, \mathbf{b}_r; c; \mathbf{x}_r; b] &= \frac{e^{-2rb}}{\mathbf{B}(\mathbf{b}_{r+1})} \sum_{\mathbf{l}_{2r}=0}^{\infty} \sum_{k=0}^{|\mathbf{l}_{2r}|} \gamma_k L_k^{(\alpha)}(q) \prod_{i=1}^{r} B(b_i + l_{2i-1} + 1, |\mathbf{b}_{i+1,r+1}| + l_{2i} + 1) \\
&\cdot F^{r:1;\cdots;1}_{r:0;\cdots;0}\left[\begin{array}{l}
[a; \overbrace{1,\cdots,1}^{r}], [|\mathbf{b}_{2,r+1}| + l_2 + 1 : 0, \overbrace{1,\cdots,1}^{r-1}], \cdots, [|\mathbf{b}_{r,r+1}| + l_{2r} + 1 : \overbrace{0,\cdots,0}^{r-1}, 1] : \\
[|\mathbf{b}_{1,r+1}| + l_1 + l_2 + 2 : \underbrace{1,\cdots,1}_{r}], \cdots, [|\mathbf{b}_{r,r+1}| + l_{2r-1} + l_{2r} + 1 : \underbrace{0,\cdots,0}_{r-1}, 1] : \\
\qquad [b_1 + l_1 + 1 : 1]; \cdots; [b_r + l_{2r-1} + 1 : 1]; \\
\qquad \text{———}; \cdots; \text{———};
\end{array} \mathbf{x}_r\right],
\end{aligned} \tag{44}$$

where $b_{r+1} = c - |\mathbf{b}_r|$, $m_{r+1} = 0$, $q > 0$, $\max\{|x_1|, \cdots, |x_r|\} < 1$ *and*

$$\gamma_k = F_A^{(2r+1)}\left[\alpha + 1, -\mathbf{l}_{2r}, k; \underbrace{1,\cdots,1}_{2r}, \alpha + 1; \underbrace{\frac{b}{q},\cdots,\frac{b}{q}}_{2r}, 1\right],$$

it being assumed that each member of assertion (44) *exists.*

Proof. For the extended beta function $B_b(x, y)$, we have the following expansion [2, p. 238, Theorem 5.13]:

$$B_b(x, y) = e^{-2b} \sum_{k,l=0}^{\infty} B(x + k + 1, y + l + 1)\, L_k(b)\, L_l(b) \qquad (b \geq 0;\ \ \min\{\Re(x), \Re(y)\} > 0). \tag{45}$$

Then, from the series expression (6), we have

$$\begin{aligned}
{}_2F_D^{(r)}[\mathbf{x}_r; b] &= \frac{1}{\mathbf{B}(\mathbf{b}_{r+1})} \sum_{\mathbf{m}_r=0}^{\infty} (a)_{|\mathbf{m}_r|} \prod_{i=1}^{r} B_b(b_i + m_i, |\mathbf{b}_{i+1,r+1}| + |\mathbf{m}_{i+1,r+1}|) \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!} \\
&= \frac{e^{-2rb}}{\mathbf{B}(\mathbf{b}_{r+1})} \sum_{\mathbf{m}_r=0}^{\infty} (a)_{|\mathbf{m}_r|} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!} \sum_{l_1,l_2=0}^{\infty} B(b_1 + m_1 + l_1 + 1, |\mathbf{b}_{2,r+1}| + |\mathbf{m}_{2,r+1}| + l_2 + 1)\, L_{l_1}(b)\, L_{l_2}(b) \\
&\qquad \cdots \sum_{l_{2r-1},l_{2r}=0}^{\infty} B(b_r + m_r + l_{2r-1} + 1, |\mathbf{b}_{r+1,r+1}| + |\mathbf{m}_{r+1,r+1}| + l_{2r} + 1)\, L_{l_{2r-1}}(b)\, L_{l_{2r}}(b) \\
&= \frac{e^{-2rb}}{\mathbf{B}(\mathbf{b}_{r+1})} \sum_{\mathbf{m}_r=0}^{\infty} (a)_{|\mathbf{m}_r|} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!} \\
&\qquad \cdot \sum_{\mathbf{l}_{2r}=0}^{\infty} \prod_{j=1}^{2r} L_{l_j}(b) \prod_{i=1}^{r} B(b_i + m_i + l_{2i-1} + 1, |\mathbf{b}_{i+1,r+1}| + |\mathbf{m}_{i+1,r+1}| + l_{2i} + 1).
\end{aligned} \tag{46}$$

By using Lemma 3.6, we can easily derive the fact that the multiple series (46) is absolutely convergent for $\max\{|x_1|,\cdots,|x_r|\}<1$. In fact, we only need to prove the convergence of the following series

$$\Psi(\mathbf{x}_r):=e^{-2rb}\sum_{\mathbf{m}_r=0}^{\infty}(|a|)_{|\mathbf{m}_r|}\frac{|x_1|^{m_1}}{m_1!}\cdots\frac{|x_r|^{m_r}}{m_r!}$$
$$\cdot\sum_{\mathbf{l}_{2r}=0}^{\infty}\prod_{j=1}^{2r}|L_{l_j}(b)|\prod_{i=1}^{r}\left|B\left(b_i+m_i+l_{2i-1}+1,|\mathbf{b}_{i+1,r+1}|+|\mathbf{m}_{i+1,r+1}|+l_{2i}+1\right)\right|. \tag{47}$$

Without loss of generality we may assume that $\Im(|\mathbf{b}_{i,r+1}|)\neq 0$, $i=1,\cdots,r$. The use of (40) gives

$$\Psi(\mathbf{x}_r)<e^{-rb}\prod_{i=1}^{r}e^{\frac{\pi}{2}|\Im(|\mathbf{b}_{i,r+1}|)|}B\left(\Re(b_i),\Re(|\mathbf{b}_{i+1,r+1}|)\right)$$
$$\cdot\sum_{\mathbf{m}_r=0}^{\infty}(|a|)_{|\mathbf{m}_r|}\prod_{i=1}^{r}\frac{(\Re(b_i))_{m_i}\left(\Re(|\mathbf{b}_{i+1,r+1}|)\right)_{|\mathbf{m}_{i+1,r+1}|}}{\left(\Re(|\mathbf{b}_{i,r+1}|)\right)_{|\mathbf{m}_{i,r+1}|}}\frac{|x_1|^{m_1}}{m_1!}\cdots\frac{|x_r|^{m_r}}{m_r!}$$
$$=e^{-rb}\prod_{i=1}^{r}e^{\frac{\pi}{2}|\Im(|\mathbf{b}_{i,r+1}|)|}B\left(\Re(b_i),\Re(|\mathbf{b}_{i+1,r+1}|)\right)$$
$$\cdot\sum_{\mathbf{m}_r=0}^{\infty}\frac{(|a|)_{|\mathbf{m}_r|}}{\left(\Re(|\mathbf{b}_{1,r+1}|)\right)_{|\mathbf{m}_r|}}\frac{\prod_{i=1}^{r-1}\left(\Re(|\mathbf{b}_{i+1,r+1}|)\right)_{|\mathbf{m}_{i+1,r+1}|}}{\prod_{i=2}^{r}\left(\Re(|\mathbf{b}_{i,r+1}|)\right)_{|\mathbf{m}_{i,r+1}|}}\frac{(\Re(b_1))_{m_1}}{m_1!}\cdots\frac{(\Re(b_r))_{m_r}}{m_r!}|x_1|^{m_1}\cdots|x_r|^{m_r}$$
$$=e^{-rb}\prod_{i=1}^{r}e^{\frac{\pi}{2}|\Im(|\mathbf{b}_{i,r+1}|)|}B\left(\Re(b_i),\Re(|\mathbf{b}_{i+1,r+1}|)\right)$$
$$\cdot F^{r:1;\cdots;1}_{r:0;\cdots;0}\left[\begin{array}{l}[|a|;\overbrace{1,\cdots,1}^{r}],[\Re(|\mathbf{b}_{2,r+1}|):0,\overbrace{1,\cdots,1}^{r-1}],\cdots,[\Re(|\mathbf{b}_{r,r+1}|):\overbrace{0,\cdots,0}^{r-1},1]:\\ \qquad[\Re(|\mathbf{b}_{1,r+1}|):\underbrace{1,\cdots,1}_{r}],\cdots,[\Re(|\mathbf{b}_{r,r+1}|):\underbrace{0,\cdots,0}_{r-1},1]:\end{array}\right.$$
$$\left.\begin{array}{l}[\Re(b_1):1];\cdots;[\Re(b_r):1];\\ \text{———};\cdots;\text{———};\end{array}|x_1|,\cdots,|x_r|\right]<\infty\quad\text{for}\quad\max\{|x_1|,\cdots,|x_r|\}<1. \tag{48}$$

Thus, we can interchange the order of summations and find that

$$_2F_D^{(r)}[\mathbf{x}_r;b]=\frac{e^{-2rb}}{\mathbf{B}(\mathbf{b}_{r+1})}\sum_{\mathbf{l}_{2r}=0}^{\infty}\prod_{j=1}^{2r}L_{l_j}(b)$$
$$\cdot\sum_{\mathbf{m}_r=0}^{\infty}(a)_{|\mathbf{m}_r|}\prod_{i=1}^{r}B\left(b_i+m_i+l_{2i-1}+1,|\mathbf{b}_{i+1,r+1}|+|\mathbf{m}_{i+1,r+1}|+l_{2i}+1\right)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!}$$
$$=\frac{e^{-2rb}}{\mathbf{B}(\mathbf{b}_{r+1})}\sum_{\mathbf{l}_{2r}=0}^{\infty}\prod_{j=1}^{2r}L_{l_j}(b)\prod_{i=1}^{r}B\left(b_i+l_{2i-1}+1,|\mathbf{b}_{i+1,r+1}|+l_{2i}+1\right)$$
$$\cdot\sum_{\mathbf{m}_r=0}^{\infty}(a)_{|\mathbf{m}_r|}\prod_{i=1}^{r}\frac{(b_i+l_{2i-1}+1)_{m_i}\left(|\mathbf{b}_{i+1,r+1}|+l_{2i}+1\right)_{|\mathbf{m}_{i+1,r+1}|}}{\left(|\mathbf{b}_{i,r+1}|+l_{2i-1}+l_{2i}+2\right)_{|\mathbf{m}_{i,r+1}|}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!}. \tag{49}$$

The final result (44) can be obtained by interpreting the inner series in (49) as the Srivastava-Daoust generalized Lauricella function and using formula (30). □

We can prove the following result for $F_B^{(r)}[\mathbf{x}_r;b]$ in a similar manner, so we omit its proof.

Theorem 3.8. *For the extended Appell-Lauricella hypergeometric function $F_B^{(r)}[\mathbf{x}_r;b]$, the following result holds true:*

$$F_B^{(r)}[\mathbf{a}_r,\mathbf{b}_r;c;\mathbf{x}_r;b]=\frac{e^{-2rb}}{\mathbf{B}(\mathbf{a}_{r+1})}\sum_{\mathbf{l}_{2r}=0}^{\infty}\sum_{k=0}^{|\mathbf{l}_{2r}|}\gamma_k L_k^{(\alpha)}(q)\prod_{i=1}^{r}B(a_i+l_{2i-1}+1,|\mathbf{a}_{i+1,r+1}|+l_{2i}+1)$$
$$\cdot F^{r-1:2;\cdots;2}_{r:0;\cdots;0}\left[\begin{array}{l}[|\mathbf{a}_{2,r+1}|+l_2+1:0,\overbrace{1,\cdots,1}^{r-1}],\cdots,[|\mathbf{a}_{r,r+1}|+l_{2r}+1:\overbrace{0,\cdots,0}^{r-1},1]:\\ [|\mathbf{a}_{1,r+1}|+l_1+l_2+2:\underbrace{1,\cdots,1}_{r}],\cdots,[|\mathbf{a}_{r,r+1}|+l_{2r-1}+l_{2r}+1:\underbrace{0,\cdots,0}_{r-1},1]:\end{array}\right.$$
$$\left.\begin{array}{l}[b_1:1],[a_1+l_1+1:1];\cdots;[b_r:1],[a_r+l_{2r-1}+1:1];\\ \text{———};\cdots;\text{———};\end{array}\mathbf{x}_r\right], \tag{50}$$

where $a_{r+1}=c-|\mathbf{a}_r|$, $m_{r+1}=0$, $q>0$, $\max\{|x_1|,\cdots,|x_r|\}<1$ and

$$\gamma_k=F_A^{(2r+1)}\left[\alpha+1,-\mathbf{l}_{2r},k;\underbrace{1,\cdots,1}_{2r},\alpha+1;\underbrace{\frac{b}{q},\cdots,\frac{b}{q}}_{2r},1\right],$$

it being assumed that each member of the assertion (50) *exists.*

4. Generating Functions for $F_A^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r;b]$

In this section, we derive several generating functions for the extended Appell-Lauricella hypergeometric function $F_A^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r;b]$ with the help of the method considered in [6] and [26].

Theorem 4.1. *The following generating function holds true:*

$$\sum_{k=0}^{\infty}\binom{a+k}{k}F_A^{(r:\alpha,\beta;\rho,\lambda)}[-k,\mathbf{b}_r;\mathbf{c}_r;\mathbf{x}_r;b]\,z^k=(1-z)^{-a-1}F_A^{(r:\alpha,\beta;\rho,\lambda)}\left[a+1,\mathbf{b}_r;\mathbf{c}_r;\frac{x_1z}{z-1},\cdots,\frac{x_rz}{z-1};b\right]. \tag{51}$$

In particular, we have

$$\sum_{k=0}^{\infty}F_A^{(r:\alpha,\beta;\rho,\lambda)}[-k,\mathbf{b}_r;\mathbf{c}_r;\mathbf{x}_r;b]\,z^k=e^z\prod_{j=1}^{r}{}_1F_1^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix}b_j\\c_j\end{matrix};-x_jz;b\right], \tag{52}$$

where ${}_1F_1^{(\alpha,\beta;\rho,\lambda)}[z;b]$ is the extended confluent hypergeometric function given by (9).

Proof. Since (52) is just the confluent form of (51), which can be easily derived by replacing z by z/a and letting $a\to\infty$, it would suffice to prove the generating function (51).

Let $\Delta(z)$ deonte the left-hand side of (51) and let

$$\Theta(m_1,\cdots,m_r):=\prod_{i=1}^{r}\widehat{\mathscr{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(b_i+m_i,c_i-b_i)\quad(m_i\in\mathbb{N}_0;i=1,\cdots,r).$$

Then, by substituting the series expression (3) for the function $F_A^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r;b]$ and using the elementary identities

$$(-k)_N = \begin{cases} \dfrac{(-1)^N k!}{(k-N)!} & 0 \leq N \leq k, \\ 0, & N > k \end{cases}$$

and

$$\binom{a+k}{k} = \frac{(a+1)_k}{k!},$$

we find that

$$\begin{aligned} \Delta(z) &= \sum_{k=0}^{\infty} \frac{(a+1)_k}{k!} \sum_{m_1,\cdots,m_r=0}^{\infty} (-k)_{|\mathbf{m}_r|}\, \Theta(m_1,\cdots,m_r) \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!} z^k \\ &= \sum_{k=0}^{\infty} \sum_{m_1,\cdots,m_r=0}^{0\leq m_1+\cdots+m_r\leq k} \frac{(-1)^{|\mathbf{m}_r|}(a+1)_k}{(k-|\mathbf{m}_r|)!} \Theta(m_1,\cdots,m_r) \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!} z^k. \end{aligned} \tag{53}$$

By applying the multiple series identity [28, p. 102, Lemma 4, Eq. (17)]

$$\sum_{k=0}^{\infty} \sum_{m_1,\cdots,m_r=0}^{l_1m_1+\cdots+l_rm_r\leq k} \Phi(m_1,\cdots,m_r;k) = \sum_{k=0}^{\infty} \sum_{m_1,\cdots,m_r=0}^{\infty} \Phi(m_1,\cdots,m_r;k+l_1m_1+\cdots+l_rm_r),$$

we obtain

$$\Delta(z) = \sum_{m_1,\cdots,m_r=0}^{\infty} \Theta(m_1,\cdots,m_r) \frac{(-x_1z)^{m_1}}{m_1!} \cdots \frac{(-x_rz)^{m_r}}{m_r!} \sum_{k=0}^{\infty} (a+1)_{k+|\mathbf{m}_r|} \frac{z^k}{k!}. \tag{54}$$

Now, the use of identity $(\lambda)_{m+n} = (\lambda)_n (\lambda+n)_m$ and the binomial theorem can give us the following result

$$\begin{aligned} \Delta(z) &= (1-z)^{-a-1} \sum_{m_1,\cdots,m_r=0}^{\infty} \Theta(m_1,\cdots,m_r) \frac{(a+1)_{|\mathbf{m}_r|}}{m_1!\cdots m_r!} \left(\frac{x_1z}{z-1}\right)^{m_1} \cdots \left(\frac{x_rz}{z-1}\right)^{m_r} \\ &= (1-z)^{-1-a} F_A^{(r:\alpha,\beta;\rho,\lambda)}\left[a+1, \mathbf{b}_r; \mathbf{c}_r; \frac{x_1z}{z-1}, \cdots, \frac{x_rz}{z-1}; b\right]. \end{aligned} \tag{55}$$

This completes the proof of (51). $\square$

Remark 4.2. *For $b = 0$, the generating function* (51) *and its confluent form* (52) *may reduce to the results derived by Srivastava and Choe [26, p. 60, Eqs. (2.1) and (2.7)].*

Now, let

$$F(\mu;\mathbf{x}_r) := \sum_{\mathbf{j}_r=0}^{\infty} (\mu)_L\, A(\mathbf{j}_r)\, x_1^{j_1} \cdots x_r^{j_r} \tag{56}$$

and

$$G(\mu;\mathbf{y}_s) := \sum_{\mathbf{k}_s=0}^{\infty} (\mu)_M\, B(\mathbf{k}_s)\, y_1^{k_1} \cdots y_s^{k_s}, \tag{57}$$

where, and in what follows,

$$L := l_1 j_1 + \cdots + l_r j_r; \;\; M := m_1 k_1 + \cdots + m_s k_s \quad (l_1, \cdots, l_r \in \mathbb{N}; \mu, m_1, \cdots, m_s \in \mathbb{C})$$

and $\{A(\mathbf{j}_r)\}$ and $\{B(\mathbf{k}_s)\}$ are suitably bounded multiple complex sequences. The modified Hadamard product (or convolution) of F and G, which was first introduced by Chen and Srivastava [6], is defined formally by

$$(F \star G)(\mu; \mathbf{x}_r; \mathbf{y}_s) := \sum_{\mathbf{j}_r=0}^{\infty} \sum_{\mathbf{k}_s=0}^{\infty} (\mu)_{L+M} A(\mathbf{j}_r) B(\mathbf{k}_s) x_1^{j_1} \cdots x_r^{j_r} y_1^{k_1} \cdots y_s^{k_s}.$$

With the help of this concept and some series rearrangement techniques, Chen and Srivastava proved the following *bilateral* generating function [6, p. 2, Theorem 1]:

$$\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} F(-n; \mathbf{x}_r) G(\mu + n; \mathbf{y}_s) t^n$$
$$= (1-t)^{-\mu} (F \star G)\left(\mu; x_1 \left(\frac{t}{t-1}\right)^{l_1}, \cdots, x_r \left(\frac{t}{t-1}\right)^{l_r}, \frac{y_1}{(1-t)^{m_1}}, \cdots, \frac{y_s}{(1-t)^{m_s}}\right) \quad (|t| < 1). \tag{58}$$

Specializing the bounded sequences $\{A(\mathbf{j}_r)\}$ and $\{B(\mathbf{k}_s)\}$ by putting

$$A(\mathbf{j}_r) := \prod_{i=1}^{r} \widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(b_i + j_i, c_i - b_i) \quad \text{and} \quad B(\mathbf{k}_s) := \prod_{i=1}^{s} \widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(d_i + k_i, h_i - d_i),$$

we can deduce from (58) the following bilateral generating functions for $F_A^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b]$.

Theorem 4.3. *The following generating function holds true:*

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} F_A^{(r;\alpha,\beta;\rho,\lambda)}[-n; \mathbf{b}_r; \mathbf{c}_r; \mathbf{x}_r; b] F_A^{(s;\alpha,\beta;\rho,\lambda)}[a+n; \mathbf{d}_s; \mathbf{h}_s; \mathbf{y}_s; b] t^n$$
$$= (1-t)^{-a} F_A^{(r+s;\alpha,\beta;\rho,\lambda)}\left[a; \mathbf{b}_r, \mathbf{d}_s; \mathbf{c}_r, \mathbf{h}_s; \frac{x_1 t}{t-1}, \cdots, \frac{x_r t}{t-1}, \frac{y_1}{1-t}, \cdots, \frac{y_s}{1-t}; b\right]. \tag{59}$$

Remark 4.4. *By setting $y_1 = \cdots = y_s = 0$ and $a = a+1$ in* (59)*, we can get the former result* (51)*. If we put $x_1, \cdots, x_r = 0$ in* (59)*, we obtain the following generating function*

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} F_A^{(s;\alpha,\beta;\rho,\lambda)}[a+n; \mathbf{d}_s; \mathbf{h}_s; \mathbf{y}_s; b] t^n = (1-t)^{-a} F_A^{(s;\alpha,\beta;\rho,\lambda)}\left[a; \mathbf{d}_s; \mathbf{h}_s; \frac{y_1}{1-t}, \cdots, \frac{y_s}{1-t}; b\right].$$

5. Generating Functions for $F_B^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b]$

In this section, we establish a generating function for the extended Appell-Lauricella hypergeometric function $F_B^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b]$. Our theorem mainly depend on the following result due to Exton [12, p. 247, Theorem 1]:

$$(1-t)^{-d} \sum_{\mathbf{m}_r=0}^{\infty} C(\mathbf{m}_r) \frac{(d)_{|\mathbf{m}_r|} (k_1)_{m_1} \cdots (k_r)_{m_r}}{(|\mathbf{k}_r|)_{|\mathbf{m}_r|}} \left(\frac{t}{1-t}\right)^{|\mathbf{m}_r|}$$
$$= \sum_{\mathbf{m}_r=0}^{\infty} \frac{(d)_{|\mathbf{m}_r|} (k_1)_{m_1} \cdots (k_r)_{m_r} t^{|\mathbf{m}_r|}}{(|\mathbf{k}_r|)_{|\mathbf{m}_r|} m_1! \cdots m_r!} \sum_{\mathbf{p}_r=0}^{\infty} C(\mathbf{p}_r) (-1)^{|\mathbf{p}_r|} (-m_1)_{p_1} \cdots (-m_r)_{p_r}, \tag{60}$$

where $C(\mathbf{m}_r)$ is any arbitrary function of $m_1, \cdots, m_r$. The generating function (60) is derived by using Exton's multidimensional generalization of Bailey's transform (see [12]; see also [11, p. 139]).

By suitably specializing the parameters, we obtain the following result.

Theorem 5.1. *The following generating function holds true:*

$$(1-t)^{-|\mathbf{k}_r|} F_B^{(r:\alpha,\beta;\rho,\lambda)}\left[\mathbf{a}_r, \mathbf{k}_r; c; \frac{x_1 t}{t-1}, \cdots, \frac{x_r t}{t-1}; b\right] = \sum_{\mathbf{m}_r=0}^{\infty} \frac{(k_1)_{m_1} \cdots (k_r)_{m_r}}{m_1! \cdots m_r!} t^{|\mathbf{m}_r|} F_B^{(r:\alpha,\beta;\rho,\lambda)}\left[\mathbf{a}_r, -\mathbf{m}_r; c; \mathbf{x}_r; b\right]. \tag{61}$$

Proof. The generating function (61) is easily derived by putting $d = |\mathbf{k}_r|$ and

$$C(\mathbf{m}_r) = \prod_{i=1}^{r} \widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}\left(a_i + m_i, |\mathbf{a}_{i+1,r+1}| + |\mathbf{m}_{i+1,r+1}|\right)(-1)^{|\mathbf{m}_r|} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!} \quad (a_{r+1} = c - |\mathbf{a}_r|,\ m_{r+1} \equiv 0).$$

in (60) and suitably expressing the resulting equation in terms of the function $F_B^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b]$. $\square$

Remark 5.2. *Setting* $b = 0$ *in* (61) *gives the result [12, p. 251, Eq. (A.9)].*

If we specialize the parameters in another way, we may get a generating function for the function ${}_1F_D^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b]$. This result will be given in Section 6.

6. Generating Functions for ${}_1F_D^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b]$ and ${}_2F_D^{(r:\alpha,\beta;\rho,\lambda)}[\mathbf{x}_r; b]$

The familiar (two-variable) polynomials $g_n^{(\alpha,\beta)}(x,y)$ generated by

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y) z^n = (1-xz)^{-\alpha}(1-yz)^{-\beta} \quad \left(|z| < \min\{|x|^{-1}, |y|^{-1}\}\right)$$

are known as the Lagrange polynomials which occur in certain problems in statistics (see [9]; see also [28, pp. 441–442]). The multivariable Lagrange polynomials $g_n^{(\boldsymbol{\alpha}_r)}(\mathbf{x}_r) := g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$, which are popularly known as the *Chan-Chyan-Srivastava polynomials*, are generated by (see [3]; see also [5] and [19])

$$\prod_{j=1}^{r}\left\{\left(1-x_j z\right)^{-\alpha_j}\right\} = \sum_{n=0}^{\infty} g_n^{(\boldsymbol{\alpha}_r)}(\mathbf{x}_r) z^n \quad \left(|z| < \min\{|x_1|^{-1}, \cdots, |x_r|^{-1}\}\right). \tag{62}$$

The explicit representation of the Chan-Chyan-Srivastava polynomial is given by

$$g_n^{(\boldsymbol{\alpha}_r)}(\mathbf{x}_r) = \sum_{k_1+\cdots+k_r=n}^{\infty} (\alpha_1)_{k_1} \cdots (\alpha_r)_{k_r} \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!}, \tag{63}$$

which is equivalent to [16, p. 522, Eq. (17)]

$$g_n^{(\boldsymbol{\alpha}_r)}(\mathbf{x}_r) = \sum_{n_{r-1}=0}^{n} \sum_{n_{r-2}=0}^{n_{r-1}} \cdots \sum_{n_1=0}^{n_2} \frac{(\alpha_1)_{n_1}(\alpha_2)_{n_2-n_1} \cdots (\alpha_r)_{n-n_{r-1}}}{n_1!(n_2-n_1)! \cdots (n-n_{r-1})!} x_1^{n_1} x_2^{n_2-n_1} \cdots x_r^{n-n_{r-1}}. \tag{64}$$

We also present here the following important generating function (see [3, p. 143, Eq. (20)]; see also [5, p. 246, Remark 1])

$$\sum_{n=0}^{\infty} \binom{m+n}{n} g_{m+n}^{(\boldsymbol{\alpha}_r)}(\mathbf{x}_r) z^n = \prod_{j=1}^{r}\left\{\left(1-x_j z\right)^{-\alpha_j}\right\} g_m^{(\boldsymbol{\alpha}_r)}\left(\frac{x_1}{1-x_1 z}, \cdots, \frac{x_r}{1-x_r z}\right). \tag{65}$$

For other recent results concerning these polynomials and their extensions, we refer to [4], [10] and [29].

The following lemma, given in [16, p. 521, Eq. (12) and (13)], will be useful in the sequel.

Lemma 6.1. *The following formulas*

$$\sum_{n_r=0}^{\infty}\sum_{n_{r-1}=0}^{\infty}\cdots\sum_{n_1=0}^{\infty}A\left(n_1,n_2,\cdots,n_r\right)=\sum_{n_r=0}^{\infty}\sum_{n_{r-1}=0}^{n_r}\cdots\sum_{n_1=0}^{n_2}A\left(n_1,n_2-n_1,\cdots,n_r-n_{r-1}\right) \tag{66}$$

and

$$\sum_{n_r=0}^{\infty}\sum_{n_{r-1}=0}^{n_r}\cdots\sum_{n_1=0}^{n_2}A\left(n_1,n_2,\cdots,n_r\right)=\sum_{n_r=0}^{\infty}\sum_{n_{r-1}=0}^{\infty}\cdots\sum_{n_1=0}^{\infty}A\left(n_1,n_1+n_2,\cdots,n_1+\cdots+n_r\right) \tag{67}$$

holds true provided that each of the series involved is absolutely convergent.

Theorem 6.2. *The following bilateral generating function holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty}g_n^{(\gamma_r)}\left(\mathbf{x}_r\right){}_1F_D^{(s:\alpha,\beta;\rho,\lambda)}\left[a,-n,b_2,\cdots,b_s;c;u_1,\cdots,u_s;b\right]z^n\\&=\prod_{j=1}^{r}\left\{\left(1-x_jz\right)^{-\gamma_j}\right\}{}_1F_D^{(s+r-1:\alpha,\beta;\rho,\lambda)}\left[a,\gamma_1,\cdots,\gamma_r,b_2,\cdots,b_s;c;\frac{x_1u_1z}{x_1z-1}\cdots,\frac{x_ru_rz}{x_rz-1},u_2,\cdots,u_s;b\right].\end{aligned} \tag{68}$$

Proof. Let $\Lambda(z)$ denote the left-hand side of (68). Then, by some simple calculations, we get

$$\begin{aligned}\Lambda(z)&:=\sum_{n=0}^{\infty}g_n^{(\gamma_r)}\left(\mathbf{x}_r\right){}_1F_D^{(s:\alpha,\beta;\rho,\lambda)}\left[a,-n,b_2,\cdots,b_s;c;u_1,\cdots,u_s;b\right]z^n\\&=\sum_{n=0}^{\infty}g_n^{(\gamma_r)}\left(\mathbf{x}_r\right)\sum_{m_1,\cdots,m_s=0}^{\infty}\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}\left(a+|\mathbf{m}_s|,c-a\right)(-n)_{m_1}(b_2)_{m_2}\cdots(b_s)_{m_s}\frac{u_1^{m_1}}{m_1!}\cdots\frac{u_s^{m_s}}{m_s!}z^n\\&=\sum_{m_2,\cdots,m_s=0}^{\infty}(b_2)_{m_2}\cdots(b_s)_{m_s}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}\sum_{n=0}^{\infty}\sum_{m_1=0}^{n}g_n^{(\gamma_r)}\left(\mathbf{x}_r\right)\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}\left(a+|\mathbf{m}_s|,c-a\right)(-n)_{m_1}\frac{u_1^{m_1}}{m_1!}z^n\\&=\sum_{m_2,\cdots,m_s=0}^{\infty}(b_2)_{m_2}\cdots(b_s)_{m_s}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}\sum_{n=0}^{\infty}\sum_{m_1=0}^{\infty}g_{n+m_1}^{(\gamma_r)}\left(\mathbf{x}_r\right)\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}\left(a+|\mathbf{m}_s|,c-a\right)(-n-m_1)_{m_1}\frac{u_1^{m_1}}{m_1!}z^{n+m_1}\\&=\sum_{n,m_1,\cdots,m_s=0}^{\infty}\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}\left(a+|\mathbf{m}_s|,c-a\right)(b_2)_{m_2}\cdots(b_s)_{m_s}\frac{u_1^{m_1}}{m_1!}\cdots\frac{u_s^{m_s}}{m_s!}g_{n+m_1}^{(\gamma_r)}\left(\mathbf{x}_r\right)(-1)^{m_1}\frac{(n+m_1)!}{n!}z^{n+m_1}\\&=\sum_{m_1,\cdots,m_s=0}^{\infty}\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}\left(a+|\mathbf{m}_s|,c-a\right)(b_2)_{m_2}\cdots(b_s)_{m_s}(-u_1z)^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}\sum_{n=0}^{\infty}\binom{n+m_1}{n}g_{n+m_1}^{(\gamma_r)}\left(\mathbf{x}_r\right)z^n.\end{aligned} \tag{69}$$

From (65), we have

$$\begin{aligned}\Lambda(z)=\prod_{j=1}^{r}\left\{\left(1-x_jz\right)^{-\gamma_j}\right\}&\sum_{m_1,\cdots,m_s=0}^{\infty}\widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}\left(a+|\mathbf{m}_s|,c-a\right)(b_2)_{m_2}\cdots(b_s)_{m_s}\\&\cdot g_{m_1}^{(\gamma_r)}\left(\frac{x_1}{1-x_1z},\cdots,\frac{x_r}{1-x_rz}\right)(-u_1z)^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}.\end{aligned} \tag{70}$$

Upon substituting (64) into (70), we find that

$$\Lambda(z)=\prod_{j=1}^{r}\left\{\left(1-x_jz\right)^{-\gamma_j}\right\}\sum_{m_2,\cdots,m_s=0}^{\infty}(b_2)_{m_2}\cdots(b_s)_{m_s}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}$$

$$\cdot \sum_{m_1=0}^{\infty} \sum_{n_{r-1}=0}^{m_1} \cdots \sum_{n_1=0}^{n_2} \frac{(\gamma_1)_{n_1} (\gamma_2)_{n_2-n_1} \cdots (\gamma_r)_{n-n_{r-1}}}{n_1! (n_2-n_1)! \cdots (m_1-n_{r-1})!}$$

$$\cdot \widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)} (a+|\mathbf{m}_s|, c-a) (-u_1 z)^{m_1} \left(\frac{x_1}{1-x_1 z}\right)^{n_1} \left(\frac{x_2}{1-x_2 z}\right)^{n_2-n_1} \cdots \left(\frac{x_r}{1-x_r z}\right)^{m_1-n_{r-1}}. \tag{71}$$

By using Lemma 6.1, we get

$$\Lambda(z) = \prod_{j=1}^{r} \left\{ \left(1-x_j z\right)^{-\gamma_j} \right\} \sum_{n_1,\cdots,n_{r-1},m_1,\cdots,m_s=0}^{\infty} \widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)} (a+|\mathbf{m}_s|+|\mathbf{n}_{r-1}|, c-a) (b_2)_{m_2} \cdots (b_s)_{m_s}$$

$$\cdot (\gamma_1)_{n_1} (\gamma_2)_{n_2} \cdots (\gamma_r)_{m_1} \left(\frac{x_1}{1-x_1 z}\right)^{n_1} \cdots \left(\frac{x_r}{1-x_r z}\right)^{m_1} \frac{(-u_1 z)^{m_1+|\mathbf{n}_{r-1}|}}{n_1! \cdots n_{r-1}! m_1!} \frac{u_2^{m_2}}{m_2!} \cdots \frac{u_s^{m_s}}{m_s!}. \tag{72}$$

Now, we interpret this last multiple series by means of (5), and the result follows. □

Remark 6.3. *Setting $b = 0$ in* (68) *yields the result [16, p. 525, Theorem 2.12].*

The next result is derived by performing a suitable fractional integral operator on the generating function (62). The fractional integral operator to be used is defined by [17, p. 647]

$$I_z^{\mu,b} \{f(z)\} := \frac{1}{\Gamma(\mu)} \int_0^z f(t) (z-t)^{\mu-1} {}_1F_1 \left(\alpha; \beta; -\frac{z^{\rho+\lambda} b}{t^{\rho} (z-t)^{\lambda}}\right) \mathrm{d}t, \tag{73}$$

where $\rho \geq 0, \lambda \geq 0$, $\min\{\Re(\alpha), \Re(\beta), \Re(\mu), \Re(b)\} > 0$. It is clear that $I_z^{\mu,b}$ may reduce to the Riemann-Liouville (left-sided) fractional integral operator [15, p. 69] when $b = 0$. The case when $\rho = \lambda = 1$ and $\alpha = \beta$ have been considered in [22]. Similar constructions are also used in [18] and [30].

For this operator, we have [17, p. 647, Eq. (47)]

$$I_z^{\mu,b} \left\{z^{\eta-1}\right\} = \frac{z^{\eta+\mu-1}}{\Gamma(\mu)} \mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)} (\eta, \mu), \quad \Re(\eta) > 0. \tag{74}$$

Lemma 6.4.

$$I_z^{c-a,b} \left\{ z^{a-1} \prod_{j=1}^{r} \left(1-x_j z\right)^{-b_j} \right\} = \frac{\Gamma(a)}{\Gamma(c)} z^{c-1} {}_1F_D^{(r:\alpha,\beta;\rho,\lambda)} [a, \mathbf{b}_r; c; z\mathbf{x}_r]. \tag{75}$$

$(\Re(c) > \Re(a) > 0; \quad \max\{|x_1 z|, \cdots, |x_r z|\} < 1)$

Proof. Direct calculations yield

$$\begin{aligned} I_z^{c-a,b} \left\{ z^{a-1} \prod_{j=1}^{r} \left(1-x_j z\right)^{-b_j} \right\} &= \sum_{\mathbf{m}_r=0}^{\infty} (b_1)_{m_1} \cdots (b_r)_{m_r} I_z^{c-a,b} \left\{ z^{a+|\mathbf{m}_r|-1} \right\} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!} \\ &= \frac{z^{c-1}}{\Gamma(c-a)} \sum_{\mathbf{m}_r=0}^{\infty} (b_1)_{m_1} \cdots (b_r)_{m_r} \mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)} (a+|\mathbf{m}_r|, c-a) \frac{(zx_1)^{m_1}}{m_1!} \cdots \frac{(zx_r)^{m_r}}{m_r!} \\ &= \frac{\Gamma(a)}{\Gamma(c)} z^{c-1} \sum_{\mathbf{m}_r=0}^{\infty} (b_1)_{m_1} \cdots (b_r)_{m_r} \widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)} (a+|\mathbf{m}_r|, c-a) \frac{(zx_1)^{m_1}}{m_1!} \cdots \frac{(zx_r)^{m_r}}{m_r!} \\ &= \frac{\Gamma(a)}{\Gamma(c)} z^{c-1} {}_1F_D^{(r:\alpha,\beta;\rho,\lambda)} [a, \mathbf{b}_r; c; z\mathbf{x}_r], \end{aligned}$$

where we have used property (74) and the interchange of the order of summation and integration can be justified by the absolute convergence of the series involved. □

Theorem 6.5. *The following generating function holds true:*

$$\sum_{n=0}^{\infty} \widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(a+n,c-a)\, g_n^{(\mathbf{b}_r)}(\mathbf{x}_r)\, z^n = {}_1F_D^{(r:\alpha,\beta;\rho,\lambda)}[a,\mathbf{b}_r;c;zx_1,\cdots,zx_r;b]. \tag{76}$$

$$\left(|z| < \min\left\{|x_1|^{-1},\cdots,|x_r|^{-1}\right\}\right)$$

Proof. Upon replacing α_i $(i=1,\cdots,r)$ in (62) by b_i $(i=1,\cdots,r)$, multiplying its both sides by z^{a-1}, and applying the extended fractional integral operator $I_z^{c-a,b}$ to the resulting equation, we get

$$I_z^{c-a,b}\left\{z^{a-1}\prod_{j=1}^{r}\left(1-x_jz\right)^{-b_j}\right\} = I_z^{c-a,b}\left\{\sum_{n=0}^{\infty} g_n^{(\mathbf{b}_r)}(\mathbf{x}_r)\, z^{a+n-1}\right\}. \tag{77}$$

The left-hand side of (77) can be evaluated by using Lemma 6.4, and the right-hand side of (77) can be evaluated as

$$\begin{aligned} I_z^{c-a,b}\left\{\sum_{n=0}^{\infty} g_n^{(\mathbf{b}_r)}(\mathbf{x}_r)\, z^{a+n-1}\right\} &= \sum_{n=0}^{\infty} g_n^{(\mathbf{b}_r)}(\mathbf{x}_r)\, I_z^{c-a,b}\left\{z^{a+n-1}\right\} \\ &= \frac{z^{c-1}}{\Gamma(c-a)}\sum_{n=0}^{\infty} \mathcal{B}_{b;\rho,\lambda}^{(\alpha,\beta)}(a+n,c-a)\, g_n^{(\mathbf{b}_r)}(\mathbf{x}_r)\, z^n \\ &= \frac{\Gamma(a)}{\Gamma(c)} z^{c-1}\sum_{n=0}^{\infty} \widehat{\mathcal{B}}_{b;\rho,\lambda}^{(\alpha,\beta)}(a+n,c-a)\, g_n^{(\mathbf{b}_r)}(\mathbf{x}_r)\, z^n. \end{aligned} \tag{78}$$

Therefore, the generating function (76) follows from (75), (77) and (78). □

Remark 6.6. *By setting $b=0$, the generating function* (76) *reduces to the one derived by* Chan et al *[3, p. 141, Eq. (11)].*

We end this section by giving the following two theorems without proof. In fact, they can be easily derived by applied the methods used in Section 4 and 5.

Theorem 6.7. *The following generating function holds true:*

$$(1-t)^{-|\mathbf{k}_r|}\,{}_1F_D^{(r:\alpha,\beta;\rho,\lambda)}\left[a,\mathbf{k}_r;c;\frac{x_1t}{t-1},\cdots,\frac{x_rt}{t-1};b\right] = \sum_{\mathbf{m}_r=0}^{\infty}\frac{(k_1)_{m_1}\cdots(k_r)_{m_r}}{m_1!\cdots m_r!}t^{|\mathbf{m}_r|}\,{}_1F_D^{(r:\alpha,\beta;\rho,\lambda)}[a,-\mathbf{m}_r;c;\mathbf{x}_r;b]. \tag{79}$$

Theorem 6.8. *The following generating function holds true:*

$$\sum_{k=0}^{\infty}\binom{a+k}{k}{}_2F_D^{(r:\alpha,\beta;\rho,\lambda)}[-k,\mathbf{b}_r;c;\mathbf{x}_r;b]\,z^k = (1-z)^{-a-1}\,{}_2F_D^{(r:\alpha,\beta;\rho,\lambda)}\left[a+1,\mathbf{b}_r;c;\frac{x_1z}{z-1},\cdots,\frac{x_rz}{z-1};b\right]. \tag{80}$$

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