Filomat 31:13 (2017), 3965–3974
https://doi.org/10.2298/FIL1713965H

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Conditions under which Left Derivations are Zero

Amin Hosseini[a]

[a]*Department of Mathematics, Kashmar Higher Education Institute, Kashmar, Iran*

Abstract. In this study, we show that every continuous Jordan left derivation on a (commutative or non-commutative) prime UMV-Banach algebra with the identity element $\mathbf{1}$ is identically zero. Moreover, we prove that every continuous left derivation on a unital finite dimensional Banach algebra, under certain conditions, is identically zero. As another result in this regard, it is proved that if $\mathfrak{R}$ is a 2-torsion free semiprime ring such that $ann\{[y,z] \mid y,z \in \mathfrak{R}\} = \{0\}$, then every Jordan left derivation $\mathfrak{L} : \mathfrak{R} \to \mathfrak{R}$ is identically zero. In addition, we provide several other results in this regard.

1. Introduction and Preliminaries

Throughout the paper, $\mathfrak{R}$ denotes an associative ring. Before everything else, let us recall some basic definitions and set the notations which we use in the sequel. A ring $\mathfrak{R}$ is called unital if there exists an element $\mathbf{1} \in \mathfrak{R}$ such that $x\mathbf{1} = \mathbf{1}x = x$ holds for all $x \in \mathfrak{R}$. A ring $\mathfrak{R}$ is said to be a domain if $\mathfrak{R} \neq \{0\}$ and $x = 0$ or $y = 0$, whenever $xy = 0$ in $\mathfrak{R}$. A ring $\mathfrak{R}$ is called prime if for $x, y \in \mathfrak{R}$, $x\mathfrak{R}y = \{0\}$ implies $x = 0$ or $y = 0$, and is semiprime in case $x\mathfrak{R}x = \{0\}$ implies $x = 0$. Let $\mathcal{S}$ be a subset of a ring $\mathfrak{R}$. The left annihilator of $\mathcal{S}$ is $lann(\mathcal{S}) := \{x \in \mathfrak{R} \mid x\mathcal{S} = \{0\}\}$. Similarly, the right annihilator of $\mathcal{S}$ is $rann(\mathcal{S}) := \{x \in \mathfrak{R} \mid \mathcal{S}x = \{0\}\}$. The annihilator of $\mathcal{S}$ is defined as $ann(\mathcal{S}) := lann(\mathcal{S}) \cap rann(\mathcal{S})$. A ring $\mathfrak{R}$ is called simple if $\mathfrak{R}^2 \neq \{0\}$ and $\{0\}$ and $\mathfrak{R}$ are the only ideals in $\mathfrak{R}$. Recall that the center of a ring $\mathfrak{R}$ is $Z(\mathfrak{R}) := \{x \in \mathfrak{R} \mid xy = yx \ for \ all \ y \in \mathfrak{R}\}$. The above-mentioned definitions and notations are also considered for algebras.

Let $\mathcal{A}$ be an associative algebra. A non-zero linear functional φ on an algebra $\mathcal{A}$ is called a *character* if $\varphi(ab) = \varphi(a)\varphi(b)$ for every $a, b \in \mathcal{A}$. Throughout this article, $\Phi_{\mathcal{A}}$ denotes the set of all characters on $\mathcal{A}$. As usual, the set of all primitive ideals is denoted by $\Pi(\mathcal{A})$. The Jacobson radical of an algebra $\mathcal{A}$ is defined to be the intersection of the primitive ideals of $\mathcal{A}$; it is denoted by $rad(\mathcal{A})$. In deed, $rad(\mathcal{A}) = \bigcap_{\mathcal{P}\in\Pi(\mathcal{A})} \mathcal{P}$. An algebra $\mathcal{A}$ is semisimple if $rad(\mathcal{A}) = \{0\}$. If $\mathcal{A}$ is a $*$-algebra, then $S(\mathcal{A})$ denotes the set of all self-adjoint elements of $\mathcal{A}$ (i.e., $S(\mathcal{A}) := \{s \in \mathcal{A} \,|\, s^* = s\}$) and $P(\mathcal{A})$ denotes the set of all projections in $\mathcal{A}$ (i.e., $P(\mathcal{A}) := \{p \in \mathcal{A} \,|\, p^2 = p,\ p^* = p\}$). The set of those elements in $\mathcal{A}$ which can be represented as finite real-linear combinations of mutually orthogonal projections is denoted by $O_{\mathcal{A}}$. Of course, $P(\mathcal{A}) \subseteq O(\mathcal{A}) \subseteq S(\mathcal{A})$. In the case of a von Neumann algebra $\mathcal{A}$, the set $O(\mathcal{A})$ is norm dense in $S(\mathcal{A})$. More generally, this is true for AW^*-algebras. Recall that the spectrum of an arbitrary element a of an algebra $\mathcal{A}$ is $\mathfrak{S}(a) := \{\lambda \in \mathbb{C} \mid \lambda\mathbf{1} - a \ is \ not \ invertible \ in \ \mathcal{A}\}$, where $\mathbf{1}$ stands for the identity element of $\mathcal{A}$. The above-mentioned definitions and concepts can all be found in [6, 15, 16, 20, 22].

2010 *Mathematics Subject Classification*. Primary 47B47; Secondary 13N15, 17B40
Keywords. derivation, left derivation, Jordan left derivation.
Received: 29 December 2015; Accepted: 05 February 2017
Communicated by Dragan S. Djordjević
Email address: hosseini.amin82@gmail.com (Amin Hosseini)

A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called a derivation if $d(ab) = d(a)b + ad(b)$ holds for all pairs $a, b \in \mathcal{A}$ and is called a Jordan derivation in case $d(a^2) = d(a)a + ad(a)$ is fulfilled for all $a \in \mathcal{A}$. A left derivation on $\mathcal{A}$ is a linear mapping $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ if $\mathfrak{L}(ab) = a\mathfrak{L}(b) + b\mathfrak{L}(a)$ holds for all pairs $a, b \in \mathcal{A}$ and is called a Jordan left derivation if $\mathfrak{L}(a^2) = 2a\mathfrak{L}(a)$ is fulfilled for all $a \in \mathcal{A}$. Recently, a number of authors ([1, 8, 13, 21, 23]) have studied left derivations and various generalized notions of them in the context of pure algebra, extensively. As a pioneering work, Brešar and Vukman [5] proved that every left derivation on a semiprime ring $\mathfrak{R}$ is a derivation which maps $\mathfrak{R}$ into its center. Furthermore, they also showed that if $\mathcal{A}$ is a Banach algebra, then every continuous left derivation $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ maps $\mathcal{A}$ into its radical. The question under which conditions left derivations and derivations are zero on a given Banach algebra have attracted much attention of authors (for instance, see [1, 5, 8–11, 13, 18, 21, 23]). In this paper, we also concentrate on this topic. This research has been motivated by the works [5, 9, 19, 23]. First, we present a definition as follows. An element a of a unital Banach algebra $\mathcal{A}$ has the uniformly mean value property (UMV-property, briefly) if for every closed interval $[\alpha, \beta] \subseteq \mathbb{R}$ there exists a real number $c_{\alpha,\beta} \in (\alpha, \beta)$ such that $e^{\beta a} - e^{\alpha a} = (\beta - \alpha)ae^{c_{\alpha,\beta}a}$. A unital Banach algebra $\mathcal{A}$ is called UMV-Banach algebra if every element of $\mathcal{A}$ has the UMV-property. As a result in the current paper, we prove that every continuous left derivation on a unital, prime UMV-Banach algebra is identically zero. Clearly, the same result is true for continuous left derivations on a unital UMV-Banach algebra which also is a domain. In this work, we try to make clear the status of continuous left derivations on unital finite dimensional Banach algebras as follows. Let n be a positive integer and let $\mathcal{A}$ be an n-dimensional unital Banach algebra with the basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$. Suppose that for every integer k, $1 \leq k \leq n$, an ideal $\mathcal{I}_k$ generated by $\mathcal{B} - \{b_k\}$ is a proper subset of $\mathcal{A}$. Then every continuous left derivation on $\mathcal{A}$ is identically zero. Furthermore, it is proved that if $\mathfrak{R}$ is a 2-torsion free semiprime ring such that $ann\{[y, z] \mid y, z \in \mathfrak{R}\} = \{0\}$, then every Jordan left derivation $\mathfrak{L} : \mathfrak{R} \to \mathfrak{R}$ is identically zero. As another result in this regard, we show that every continuous Jordan left derivation on a normed $*$-algebra $\mathcal{A}$ satisfying $\overline{O(\mathcal{A})} = S(\mathcal{A})$ is identically zero. In 2008, J. Vukman proved that every Jordan left derivation on a semisimple Banach algebra is zero (see [23], Theorem 4). We believe he could prove this theorem easier. In this article, we establish a simpler proof of that theorem.

2. Main Results

We begin with the following definition which has been presented in [9].

Definition 2.1. *Let $\mathcal{A}$ be a unital Banach algebra. An element a of $\mathcal{A}$ has the uniformly mean value property (UMV-property, briefly) if for every closed interval $[\alpha, \beta] \subseteq \mathbb{R}$ there exists an element $c_{\alpha,\beta} \in (\alpha, \beta)$ such that $e^{\beta a} - e^{\alpha a} = (\beta - \alpha)ae^{ac_{\alpha,\beta}}$. A unital Banach algebra $\mathcal{A}$ is called* UMV *if every element of $\mathcal{A}$ has the UMV-property.*

Let a be an idempotent element of a unital Banach algebra $\mathcal{A}$, i.e. $a^2 = a$. We have

$$\begin{aligned} e^{ta} &= \sum_{n=0}^{\infty} \frac{t^n a^n}{n!} = \mathbf{1} + \sum_{n=1}^{\infty} \frac{t^n a}{n!} \\ &= \mathbf{1} + \sum_{n=0}^{\infty} \frac{t^n a}{n!} - a \\ &= e^t a - a + \mathbf{1} \end{aligned}$$

for all $t \in \mathbb{R}$. Hence,

$$e^{\beta a} - e^{\alpha a} = e^{\beta} a - a + 1 - (e^{\alpha} a - a + \mathbf{1}) = (e^{\beta} - e^{\alpha})a. \tag{1}$$

According to the classical mean value theorem for the function $f(t) = e^t$ on $[\alpha, \beta]$, there exists an element $c_{\alpha,\beta} \in (\alpha, \beta)$ such that $e^{\beta} - e^{\alpha} = (\beta - \alpha)e^{c_{\alpha,\beta}}$. This equality along with (1) imply that, $e^{\beta a} - e^{\alpha a} = (\beta - \alpha)e^{c_{\alpha,\beta}}a$. Now, we show that $e^{c_{\alpha,\beta}}a = ae^{c_{\alpha,\beta}a}$. We have

$$ae^{c_{\alpha,\beta}a} = a(e^{c_{\alpha,\beta}}a - a + \mathbf{1}) = e^{c_{\alpha,\beta}}a^2 - a^2 + a = e^{c_{\alpha,\beta}}a - a + a = e^{c_{\alpha,\beta}}a.$$

Thus, $e^{\beta a} - e^{\alpha a} = (\beta - \alpha)ae^{c_{\alpha,\beta}a}$. This means that a has the UMV-property.

In the following theorem, $\mathcal{A}$ denotes a unital Banach algebra.

Theorem 2.2. *Let $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ be a left derivation and let $b \in \mathcal{A}$ has the UMV-property. Assume that $f(b)\mathfrak{L}(b) = 0$ forces $f(b) = 0$ or $\mathfrak{L}(b) = 0$ for some function f. Moreover, suppose that $\mathfrak{L}(e^{c_{0,1}b}) = c_{0,1}e^{c_{0,1}b}\mathfrak{L}(b)$, where $c_{0,1} \in (0,1) \subseteq \mathbb{R}$ is obtained from the UMV-property of b and further $\mathfrak{L}(e^b) = e^b\mathfrak{L}(b)$. Then $\mathfrak{L}(b) = 0$.*

Proof. If $b = 0$, then there is nothing to be proved. Let b be a non-zero element of $\mathcal{A}$ having the UMV-property. Hence, there exists an element $c_{0,1} = c$ of (0,1) such that $e^b - \mathbf{1} = be^{cb}$. Using the latest equality along with the aforementioned assumptions that $\mathfrak{L}(e^{cb}) = ce^{cb}\mathfrak{L}(b)$ and $\mathfrak{L}(e^b) = e^b\mathfrak{L}(b)$, we deduce that $0 = e^b\mathfrak{L}(b) - \mathfrak{L}(\mathbf{1}) - b\mathfrak{L}(e^{cb}) - e^{cb}\mathfrak{L}(b) = e^b\mathfrak{L}(b) - cbe^{cb}\mathfrak{L}(b) - e^{cb}\mathfrak{L}(b)$. Indeed, we have $(e^b - cbe^{cb} - e^{cb})\mathfrak{L}(b) = 0$. This equation along with the hypothesis that $f(b)\mathfrak{L}(b) = 0$ forces $f(b) = 0$ or $\mathfrak{L}(b) = 0$, imply that $\mathfrak{L}(b) = 0$ or $e^b - cbe^{cb} - e^{cb} = 0$. If $\mathfrak{L}(b) = 0$, then our goal is achieved. If not, suppose that

$$e^b - cbe^{cb} - e^{cb} = 0. \tag{2}$$

Therefore,

$$\begin{aligned} 0 &= \mathfrak{L}(e^b - cbe^{cb} - e^{cb}) = e^b\mathfrak{L}(b) - c\Big(b\mathfrak{L}(e^{cb}) + e^{cb}\mathfrak{L}(b)\Big) - ce^{cb}\mathfrak{L}(b) \\ &= \Big(e^b - c^2be^{cb} - 2ce^{cb}\Big)\mathfrak{L}(b). \end{aligned}$$

Reusing the above supposition, we obtain that $e^b - c^2be^{cb} - 2ce^{cb} = 0$ or $\mathfrak{L}(b) = 0$. If $\mathfrak{L}(b) = 0$, then we get the required result. If not, $e^b - c^2be^{cb} - 2ce^{cb} = 0$. So, we have

$$e^b = ce^{cb}(cb + 2). \tag{3}$$

Comparing (2) and (3), we find that $ce^{cb}(cb + 2) = e^{cb}(cb + 1)$. From this and using the fact that e^{cb} is an invertible element of $\mathcal{A}$, we arrive at $b = \frac{1-2c}{c(c-1)}\mathbf{1}$. It implies that $\mathfrak{L}(b) = 0$ and our assertion is achieved. □

An immediate corollary of Theorem 2.2 reads as follows.

Corollary 2.3. *Every continuous left derivation on a unital UMV-Banach algebra which is also a domain is identically zero.*

Proof. Let $\mathcal{A}$ be a unital UMV-Banach algebra which is also a domain and let $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ be a continuous left derivation. Evidently, $\mathfrak{L}(e^a) = e^a\mathfrak{L}(a)$ holds for all $a \in \mathcal{A}$. Now, Theorem 2.2 is exactly what we need to complete the proof. □

By using an argument similar to the proof of Theorem 2.2, we show that every Jordan left derivation on a commutative or non-commutative prime Banach algebra, under certain conditions, is identically zero. Recall that an algebra $\mathcal{A}$ is prime if $a\mathcal{A}b = \{0\}$ implies that $a = 0$ or $b = 0$

Theorem 2.4. *Let $\mathcal{A}$ be a (commutative or non-commutative) prime Banach algebra with the identity element $\mathbf{1}$, $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ be a Jordan left derivation, and let $b \in \mathcal{A}$ has the UMV-property. Suppose that $\mathfrak{L}(e^{c_{0,1}b}) = c_{0,1}e^{c_{0,1}b}\mathfrak{L}(b)$, where $c_{0,1} \in (0,1) \subseteq \mathbb{R}$ is obtained from the UMV-property of b and further $\mathfrak{L}(e^b) = e^b\mathfrak{L}(b)$. In this case $\mathfrak{L}(b) = 0$.*

Proof. It follows from Theorem 2 of [23] that $\mathfrak{L}$ is a derivation mapping $\mathcal{A}$ into $Z(\mathcal{A})$. If $b = 0$, then there is nothing to be proved. Let b be a non-zero element of $\mathcal{A}$ having the UMV-property. Hence, there exists an element $c_{0,1} = c$ of (0,1) such that $e^b - \mathbf{1} = be^{cb}$. Using the latest equality along with the aforementioned assumptions that $\mathfrak{L}(e^{cb}) = ce^{cb}\mathfrak{L}(b)$ and $\mathfrak{L}(e^b) = e^b\mathfrak{L}(b)$, we deduce that $0 = e^b\mathfrak{L}(b) - \mathfrak{L}(\mathbf{1}) - b\mathfrak{L}(e^{cb}) - e^{cb}\mathfrak{L}(b) = e^b\mathfrak{L}(b) - cbe^{cb}\mathfrak{L}(b) - e^{cb}\mathfrak{L}(b)$. Indeed, we have $(e^b - cbe^{cb} - e^{cb})\mathfrak{L}(b) = 0$. From this and using the fact that $\mathfrak{L}(\mathcal{A}) \subseteq Z(\mathcal{A})$, we obtain that $0 = (e^b - cbe^{cb} - e^{cb})\mathfrak{L}(b)a = (e^b - cbe^{cb} - e^{cb})a\mathfrak{L}(b)$ for all $a \in \mathcal{A}$. The primeness of $\mathcal{A}$ forces that $\mathfrak{L}(b) = 0$ or $e^b - cbe^{cb} - e^{cb} = 0$. If $\mathfrak{L}(b) = 0$, then our goal is achieved. If not, suppose that

$$e^b - cbe^{cb} - e^{cb} = 0. \tag{4}$$

Therefore,

$$0 = \mathfrak{L}(e^b - cbe^{cb} - e^{cb}) = e^b\mathfrak{L}(b) - c\Big(b\mathfrak{L}(e^{cb}) + e^{cb}\mathfrak{L}(b)\Big) - ce^{cb}\mathfrak{L}(b)$$
$$= \Big(e^b - c^2be^{cb} - 2ce^{cb}\Big)\mathfrak{L}(b).$$

Since $\mathfrak{L}(\mathcal{A}) \subseteq Z(\mathcal{A})$ and $\mathcal{A}$ is prime, $e^b - c^2be^{cb} - 2ce^{cb} = 0$ or $\mathfrak{L}(b) = 0$. If $\mathfrak{L}(b) = 0$, then we get the required result. If not, $e^b - c^2be^{cb} - 2ce^{cb} = 0$. So, we have

$$e^b = ce^{cb}(cb + 2). \tag{5}$$

Comparing (4) and (5), we obtain that $ce^{cb}(cb + 2) = e^{cb}(cb + 1)$. From this and using the fact that e^{cb} is an invertible element of $\mathcal{A}$, we arrive at $b = \frac{1-2c}{c(c-1)}\mathbf{1}$. It implies that $\mathfrak{L}(b) = 0$. This completes the proof of our theorem. □

An immediate conclusion is:

Corollary 2.5. *Every continuous Jordan left derivation on a unital, prime UMV-Banach algebra is identically zero.*

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{P}$ be a proper closed ideal of finite codimension in $\mathcal{A}$ such that $a \in \mathcal{P}$ or $b \in \mathcal{P}$ whenever $ab \in \mathcal{P}$. If $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ is a continuous left derivation, then $\mathfrak{L}(\mathcal{A}) \subseteq \mathcal{P}$.*

Proof. According to page 42 of [6], $\mathcal{P}$ is a prime ideal in $\mathcal{A}$. It is clear that the quotient algebra $\frac{\mathcal{A}}{\mathcal{P}}$ is a domain. It follows from Corollary 1.4.38 of [6] that $\mathcal{P}$ is a primitive ideal of $\mathcal{A}$ and the proof of Theorem 2.1 of [5] implies that $\mathfrak{L}(\mathcal{P}) \subseteq \mathcal{P}$. Since $\mathfrak{L}(\mathcal{P}) \subseteq \mathcal{P}$, the linear mapping $\Lambda : \frac{\mathcal{A}}{\mathcal{P}} \to \frac{\mathcal{A}}{\mathcal{P}}$ defined by $\Lambda(a + \mathcal{P}) = \mathfrak{L}(a) + \mathcal{P}$ ($a \in \mathcal{A}$) is a well-defined left derivation. It follows from Proposition 1.3.56 of [6] that $\frac{\mathcal{A}}{\mathcal{P}} = \mathbb{C}\mathbf{1}$, and we deduce that Λ is identically zero. Consequently, $\mathfrak{L}(\mathcal{A}) \subseteq \mathcal{P}$. □

Here, we focus on the image of Jordan left derivatives to show that every Jordan left derivation, under certain circumstances, on a prime algebra is zero.

Theorem 2.7. *Let $\mathcal{A}$ be a unital, prime algebra, and let $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ be a Jordan left derivation. If the rank of $\mathfrak{L}$ is at most one, i.e.* $\dim(\mathfrak{L}(\mathcal{A})) \leq 1$, *then $\mathfrak{L}$ is identically zero.*

Proof. It follows from Theorem 2 of [23] that $\mathfrak{L}$ is a derivation mapping $\mathcal{A}$ into $Z(\mathcal{A})$. If $\dim(\mathfrak{L}(\mathcal{A})) = 0$, then there is nothing to be proved. Suppose that $\dim(\mathfrak{L}(\mathcal{A})) = 1$. So, we can consider a non-zero element x of $\mathcal{A}$ and a functional $\Omega : \mathcal{A} \to \mathbb{C}$ such that $\mathfrak{L}(a) = \Omega(a)x$ for all $a \in \mathcal{A}$. We are going to show that $\mathfrak{L}$ is identically zero. To obtain a contradiction, assume that there exists an element $a_0 \in \mathcal{A}$ such that $\mathfrak{L}(a_0) \neq 0$. So, $\Omega(a_0) \neq 0$, too. Assume that $\mathfrak{L}(x) = 0$. So, $\Omega(x)x = 0$ and it implies that $\Omega(x) = 0$. We have $\Omega(a_0^2)x = \mathfrak{L}(a_0^2) = 2a_0\mathfrak{L}(a_0) = 2\Omega(a_0)a_0x$. Therefore,

$$0 = \Omega(a_0^2)\mathfrak{L}(x) = \mathfrak{L}(\Omega(a_0^2)x) = \mathfrak{L}(2\Omega(a_0)a_0x)$$
$$= 2\Omega(a_0)\Big(x\mathfrak{L}(a_0) + a_0\mathfrak{L}(x)\Big)$$
$$= 2\Omega(a_0)x\mathfrak{L}(a_0).$$

It means that $2\Omega(a_0)x\mathfrak{L}(a_0) = 0$ and so, $x\mathfrak{L}(a_0) = 0$. From this and using the fact that $\mathfrak{L}(\mathcal{A}) \subseteq Z(\mathcal{A})$, we obtain that $0 = x\mathfrak{L}(a_0)a = xa\mathfrak{L}(a_0)$ for all $a \in \mathcal{A}$. The primeness of $\mathcal{A}$ forces $x = 0$ or $\mathfrak{L}(a_0) = 0$, a contradiction. Now, suppose that $\mathfrak{L}(x) \neq 0$. Clearly, $\Omega(x) \neq 0$, too. Note that

$$\Omega(x^2)x = \mathfrak{L}(x^2) = 2x\mathfrak{L}(x) = 2\Omega(x)x^2.$$

Hence, we have

$$0 = \mathfrak{L}\Big(\Omega(x^2)x - 2\Omega(x)x^2\Big) = \Omega(x^2)\mathfrak{L}(x) - 4\Omega(x)x\mathfrak{L}(x)$$
$$= \Big(\Omega(x^2)\mathbf{1} - 4\Omega(x)x\Big)\mathfrak{L}(x).$$

From the former equation and using the fact $\mathfrak{L}(\mathcal{A}) \subseteq Z(\mathcal{A})$, we have $0 = \Big(\Omega(x^2)\mathbf{1} - 4\Omega(x)x\Big)\mathfrak{L}(x)a = \Big(\Omega(x^2)\mathbf{1} - 4\Omega(x)x\Big)a\mathfrak{L}(x)$ for all $a \in \mathcal{A}$. The primeness of $\mathcal{A}$ implies that $\mathfrak{L}(x) = 0$, a contradiction, or $\Omega(x^2)\mathbf{1} - 4\Omega(x)x = 0$. Thus, $0 = \mathfrak{L}\Big(\Omega(x^2)\mathbf{1} - 4\Omega(x)x\Big) = 0 - 4\Omega(x)\mathfrak{L}(x)$ and since $\Omega(x) \neq 0$, it is concluded that $\mathfrak{L}(x) = 0$. But this is a contradiction of the supposition that $\mathfrak{L}(x) \neq 0$. We see that both cases $\mathfrak{L}(x) = 0$ and $\mathfrak{L}(x) \neq 0$ lead to a contradiction. This contradiction shows that there is no element a_0 of $\mathcal{A}$ such that $\mathfrak{L}(a_0) \neq 0$. Thereby, $\mathfrak{L}$ is identically zero. □

Corollary 2.8. *Let $\mathcal{A}$ be a unital, prime algebra, and let $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ be a non-zero Jordan left derivation. Then, $\dim(\mathfrak{L}(\mathcal{A})) \geq 2$.*

Applying Theorem 2.7, we show that continuous left derivations on unital finite-dimensional Banach algebras, under certain conditions, are zero. Let n be a positive integer, and let $\mathcal{A}$ be an n-dimensional unital Banach algebra with the basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$. We prove the following theorem.

Theorem 2.9. *Suppose that for every integer k, $1 \leq k \leq n$, an ideal $\mathcal{I}_k$ generated by $\mathcal{B} - \{b_k\}$ is a proper subset of $\mathcal{A}$. Then every continuous left derivation on $\mathcal{A}$ is identically zero.*

Proof. It is easy to see that every $\mathcal{I}_k$, $1 \leq k \leq n$, is a maximal ideal of $\mathcal{A}$. Suppose that $\mathcal{I}_k$ is not a maximal ideal of $\mathcal{A}$ for some k, $1 \leq k \leq n$. Then there exists a maximal ideal $\mathcal{M}_k$ of $\mathcal{A}$ such that $\mathcal{I}_k \subset \mathcal{M}_k \subset \mathcal{A}$. But then $n - 1 = \dim(\mathcal{I}_k) < \dim(\mathcal{M}_k) < n$, a contradiction. Hence, every $\mathcal{I}_k$, $1 \leq k \leq n$, must be a maximal ideal of $\mathcal{A}$. It follows from Proposition 1.4.34 and Theorem 2.2.28 in [6] that $\mathcal{I}_k$, $1 \leq k \leq n$, are closed primitive ideals of $\mathcal{A}$. Moreover, according to Proposition 1.4.34 of [6], $\mathcal{I}_k$, $1 \leq k \leq n$, are also prime ideals of $\mathcal{A}$. Thus, the quotient algebra $\frac{\mathcal{A}}{\mathcal{I}_k}$ is a prime algebra. Let $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ be a continuous left derivation. In view of Theorem 2.1 of [5], we obtain $\mathfrak{L}(\mathcal{I}_k) \subseteq \mathcal{I}_k$, $1 \leq k \leq n$. Thus, the mapping $\Lambda : \frac{\mathcal{A}}{\mathcal{I}_k} \to \frac{\mathcal{A}}{\mathcal{I}_k}$ defined by $\Lambda(a + \mathcal{I}_k) = \mathfrak{L}(a) + \mathcal{I}_k$ is a left derivation. Since $\dim(\frac{\mathcal{A}}{\mathcal{I}_k}) = 1$ for every $1 \leq k \leq n$, it follows from Theorem 2.7 that the left derivation $\Lambda : \frac{\mathcal{A}}{\mathcal{I}_k} \to \frac{\mathcal{A}}{\mathcal{I}_k}$ is identically zero. It means that $\mathfrak{L}(\mathcal{A}) \subseteq \mathcal{I}_k$, for every $k \in \{1, 2, ..., n\}$. Hence, $\mathfrak{L}(\mathcal{A}) \subseteq \bigcap_{k=1}^{n} \mathcal{I}_k$. Assume towards a contradiction that there exists an element a_0 of $\mathcal{A}$ such that $\mathfrak{L}(a_0) \neq 0$. Since $\mathcal{B} = \{b_1, b_2, ..., b_n\}$ is a basis for $\mathcal{A}$, there exist the complex numbers α_{i_j}, and the elements b_{i_j} of $\mathcal{B}$ such that

$$\mathfrak{L}(a_0) = \sum_{j=1}^{m} \alpha_{i_j} b_{i_j} = \alpha_{i_1} b_{i_1} + \alpha_{i_2} b_{i_2} + ... + \alpha_{i_m} b_{i_m}, \quad (m \leq n).$$

Since $\mathfrak{L}(\mathcal{A}) \subseteq \mathcal{I}_k$ for every $k \in \{1, 2, ..., n\}$, we may assume that $\mathfrak{L}(\mathcal{A}) \subseteq \mathcal{I}_{i_1}$. Therefore, we have

$$\mathfrak{L}(a_0) = \alpha_{i_1} b_{i_1} + \alpha_{i_2} b_{i_2} + ... + \alpha_{i_m} b_{i_m} \in \mathcal{I}_{i_1}.$$

The previous equation shows that $b_{i_1} \in \mathcal{I}_{i_1}$, which it is a contradiction. This contradiction proves the claim that $\mathfrak{L}$ is identically zero on $\mathcal{A}$. □

Remark 2.10. *Let $\mathcal{A}$ be a semisimple Banach algebra with the identity element $\mathbf{1}$, and let $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ be a linear map satisfying $\mathfrak{L}(ab) = a\mathfrak{L}(b) - b\mathfrak{L}(a)$ for all $a, b \in \mathcal{A}$. We claim that $\mathfrak{L}$ is identically zero. Clearly, $\mathfrak{L}(\mathbf{1}) = 0$. For every invertible element $x \in \mathcal{A}$, we have $\mathfrak{L}(x) = x^2\mathfrak{L}(x^{-1})$. It follows from Theorem 5 of [23] that $\mathfrak{L}(a) = a\mathfrak{L}(\mathbf{1}) = 0$ for all $a \in \mathcal{A}$. It means that $\mathfrak{L}$ is zero.*

In the following theorem we show that there are no nonzero continuous Jordan left derivations on normed $*$-algebras with $\overline{O(\mathcal{A})} = S(\mathcal{A})$.

Theorem 2.11. *Every continuous Jordan left derivation on a normed $*$-algebra $\mathcal{A}$ satisfying $\overline{O(\mathcal{A})} = S(\mathcal{A})$ is identically zero.*

Proof. Let $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ be a continuous Jordan left derivation. We have to prove that $\mathfrak{L}(s) = 0$ for all $s \in S(\mathcal{A})$. Namely, for every $a \in \mathcal{A}$, there exist $s_1, s_2 \in S(\mathcal{A})$ such that $a = s_1 + is_2$, where i denotes the imaginary unit. Thus, $\mathfrak{L}(a) = \mathfrak{L}(s_1 + is_2) = \mathfrak{L}(s_1) + i\mathfrak{L}(s_2) = 0$. So, let $p \in \mathcal{A}$ be an arbitrary projection. We have $\mathfrak{L}(p) = \mathfrak{L}(p^2) = 2p\mathfrak{L}(p)$. This yields that $p\mathfrak{L}(p) = 2p\mathfrak{L}(p)$ and, thus, $p\mathfrak{L}(p) = 0$. Therefore, we conclude that $\mathfrak{L}(p) = 0$ for all projection $p \in P(\mathcal{A})$. Let x be an arbitrary element of $O(\mathcal{A})$. Hence, $x = \sum_{j=1}^{m} r_j p_j$, where $p_1, p_2, \ldots, p_m$ are mutually orthogonal projections in $\mathcal{A}$ and $r_1, r_2, \ldots, r_m$ are real numbers. We have $\mathfrak{L}(x) = \mathfrak{L}(\sum_{j=1}^{m} r_j p_j) = \sum_{j=1}^{m} r_j \mathfrak{L}(p_j) = 0$. Since $\overline{O(\mathcal{A})} = S(\mathcal{A})$, $\mathfrak{L}(s) = 0$ for every $s \in S(\mathcal{A})$, as desired. □

It is evident that if $\mathcal{A}$ is a unital Banach algebra and $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ is a continuous left derivation, then $\mathfrak{L}(e^a) = e^a \mathfrak{L}(a)$ holds for all $a \in \mathcal{A}$. We may think that this equation is valid only if $\mathfrak{L}$ is continuous. In this work, we establish an example to show that the equation $\mathfrak{L}(e^a) = e^a \mathfrak{L}(a)$ can be fulfilled for some discontinuous (equivalently, unbounded) left derivations. The following problem has been raised in [9]. Here, we answer it.

Problem 2.12. *Let $d : \mathcal{A} \to \mathcal{A}$ be a derivation satisfying $d(e^a) = e^a d(a)$ for all $a \in \mathcal{A}$. Is d a continuous operator?*

We give a negative answer to the above question. Indeed, we define a discontinuous derivation (left derivation) D on a given Banach algebra $\mathfrak{B}$ satisfying $D(e^b) = e^b D(b)$ for all $b \in \mathfrak{B}$. Let $\mathcal{A}$ be a Banach algebra. Consider $\mathfrak{B} = \mathbb{C} \bigoplus \mathcal{A}$ as an algebra with pointwise addition, scalar multiplication and the product $(\alpha, a).(\beta, b) = (\alpha\beta, \alpha b + \beta a)$ for all $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$. The algebra $\mathfrak{B}$ with the norm $\|(\alpha, a)\| = |\alpha| + \|a\|$ is a Banach algebra. Clearly, $\mathfrak{B}$ is a unital commutative Banach algebra (see [12]). Suppose that $T : \mathcal{A} \to \mathcal{A}$ is an unbounded linear map. Define $D : \mathfrak{B} \to \mathfrak{B}$ by $D(\alpha, a) = (0, T(a))$. It is evident that D is an unbounded linear map. Furthermore, we have

$$\begin{aligned} D((\alpha, a)(\beta, b)) &= D(\alpha\beta, \alpha b + \beta a) \\ &= (0, \alpha T(b) + \beta T(a)) \\ &= (\alpha, a)(0, T(b)) + (\beta, b)(0, T(a)) \\ &= (\alpha, a)D(\beta, b) + D(\alpha, a)(\beta, b) \\ &= (\alpha, a)D(\beta, b) + (\beta, b)D(\alpha, a) \end{aligned}$$

Since $\mathfrak{B}$ is a commutative algebra, D is both a left derivation and a derivation on $\mathfrak{B}$. Note that $e^{(\alpha,a)} = \sum_{n=0}^{\infty} \frac{(\alpha^n, n\alpha^{n-1}a)}{n!} = (e^\alpha, e^\alpha a)$. Thus, $D(e^{(\alpha,a)}) = D(e^\alpha, e^\alpha a) = (0, T(e^\alpha a)) = (0, e^\alpha T(a))$ for all $a \in \mathcal{A}$, $\alpha \in \mathbb{C}$. On the other hand, $e^{(\alpha,a)} D(\alpha, a) = (e^\alpha, e^\alpha a)(0, T(a)) = (0, e^\alpha T(a))$. Therefore, $D(e^{(\alpha,a)}) = e^{(\alpha,a)} D(\alpha, a)$ while D is an unbounded derivation (left derivation) on $\mathfrak{B}$.

The following theorem has been proved by Vukman [23]. Below, we prove it using a simpler proof.

Theorem 2.13. *Let $\mathcal{A}$ be a semisimple Banach algebra and let $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ be a Jordan left derivation. Then $\mathfrak{L}$ is identically zero.*

Proof. We know that every semisimple algebra is also semiprime. It follows from Theorem 2 of [23] that $\mathfrak{L}$ is a derivation mapping $\mathcal{A}$ into $Z(\mathcal{A})$. We therefore have $\mathfrak{L}(ab) = \mathfrak{L}(a)b + a\mathfrak{L}(b) = a\mathfrak{L}(b) + b\mathfrak{L}(a)$ and it means that $\mathfrak{L}$ is a left derivation, as well. Since $\mathfrak{L}$ is a derivation, Remark 4.3 of [14] implies that $\mathfrak{L}$ is continuous. Therefore, $\mathfrak{L}$ is a continuous left derivation. At this moment, Theorem 2.1 of [5] implies that $\mathfrak{L}(\mathcal{A}) \subseteq rad(\mathcal{A}) = \{0\}$. Thereby, our goal is achieved. □

Applying the above-mentioned argument, we can achieve the following theorem.

Theorem 2.14. *Let $\mathcal{A}$ be a Banach algebra, $\mathfrak{L} : \mathcal{A} \to \mathcal{A}$ be a Jordan left derivation, and let $\mathcal{P}$ be a primitive ideal of $\mathcal{A}$. If $\mathfrak{L}(\mathcal{P}) \subseteq \mathcal{P}$, then $\mathfrak{L}(\mathcal{A}) \subseteq \mathcal{P}$.*

Proof. Straightforward. □

By getting idea from [1], we define an l-two variable left derivation (resp. Jordan l-two variable left derivation) as follows.

Definition 2.15. *A biadditive mapping* $\Lambda : \Re \times \Re \rightarrow \Re$ *is called an l-two variable left derivation (resp. Jordan l-two variable left derivation) if* $\Lambda(xy, z) = x\Lambda(y, z) + y\Lambda(x, z)$ *(resp.* $\Lambda(x^2, y) = 2x\Lambda(x, y)$*) holds for all* $x, y, z \in \Re$.

For example, if $\mathfrak{L} : \Re \rightarrow \Re$ is a left derivation, then $\Lambda : \Re \times \Re \rightarrow \Re$ defined by $\Lambda(x, y) = \mathfrak{L}(x)y$ is an l-two variable left derivation. Because $\Lambda(xy, z) = \mathfrak{L}(xy)z = x\mathfrak{L}(y)z + y\mathfrak{L}(x)z = x\Lambda(y, z) + y\Lambda(x, z)$ holds for all $x, y, z \in \Re$.

Lemma 2.16. *Let* $\Re$ *be a 2-torsion free semiprime ring, and let* $\Lambda : \Re \times \Re \rightarrow \Re$ *be a Jordan l-two variable left derivation. In this case* $\Lambda(\Re \times \Re) \subseteq Z(\Re)$.

Proof. For an arbitrary fixed element $y \in \Re$, we define $\mathfrak{L}_y : \Re \rightarrow \Re$ by $\mathfrak{L}_y(x) = \Lambda(x, y)$. Clearly, $\mathfrak{L}_y(x^2) = \Lambda(x^2, y) = 2x\Lambda(x, y) = 2x\mathfrak{L}_y(x)$ for all $x \in \Re$. It means that $\mathfrak{L}_y$ is a Jordan left derivation on $\Re$. By Theorem 2 of [23], $\mathfrak{L}_y$ is a derivation mapping $\Re$ into $Z(\Re)$. Hence, $\Lambda(x, y) = \mathfrak{L}_y(x) \in Z(\Re)$ for all $x \in \Re$. Since we are assuming that y is an arbitrary element of $\Re$, $\Lambda(x, y) \in Z(\Re)$ for all $x, y \in \Re$. This proves the lemma completely. □

Theorem 2.17. *Let* $\Re$ *be a 2-torsion free semiprime ring, and let* $\mathfrak{L} : \Re \rightarrow \Re$ *be a Jordan left derivation. Then,* $\mathfrak{L}(x)[y, z] = 0$ *for all* $x, y, z \in \Re$.

Proof. If $\Re$ is commutative, then there is nothing to be proved. Now, suppose that $\Re$ is a non-commutative ring. We know that the biadditive map $\Lambda : \Re \times \Re \rightarrow \Re$ defined by $\Lambda(x, y) = \mathfrak{L}(x)y$ is a Jordan l-two variable left derivation. Note that $\mathfrak{L}(\Re) \subseteq Z(\Re)$ (see Theorem 2 of [23]). Application of Lemma 2.16 yields that $\mathfrak{L}(x)y = \Lambda(x, y) \in Z(\Re)$ for all $x, y \in \Re$. Therefore, we have $0 = [\mathfrak{L}(x)y, z] = \mathfrak{L}(x)[y, z] + [\mathfrak{L}(x), z]y = \mathfrak{L}(x)[y, z]$ for all $x, y, z \in \Re$. It means that $\mathfrak{L}(x)[y, z] = 0$ for all $x, y, z \in \Re$. Since $\mathfrak{L}(\Re) \subseteq Z(\Re)$, it is observed that $\mathfrak{L}(\Re) \subseteq ann\{[y, z] \mid y, z \in \Re\}$. □

In the next corollary, we show that every Jordan left derivation on a non-commutative semiprime ring, under certain conditions, is identically zero.

Corollary 2.18. *Let* $\Re$ *be a 2-torsion free semiprime ring such that* $ann\{[y, z] \mid y, z \in \Re\} = \{0\}$. *Then every Jordan left derivation* $\mathfrak{L} : \Re \rightarrow \Re$ *is identically zero.*

Proof. This is an immediate consequence of Theorem 2.17. □

It is clear that if $ann\{[y, z] \mid y, z \in \Re\} = \{0\}$, then $\Re$ is a non-commutative ring. Let $\Re$ be a semiprime ring and $a \in lann\{[y, z] \mid y, z \in \Re\}$. It follows from Lemma 1.3 of [24] that $a \in Z(\Re)$. It means that $lann\{[y, z] \mid y, z \in \Re\} \subseteq Z(\Re)$. Similarly, we can see that $rann\{[y, z] \mid y, z \in \Re\} \subseteq Z(\Re)$, too. Therefore, $lann\{[y, z] \mid y, z \in \Re\} \bigcup rann\{[y, z] \mid y, z \in \Re\} \subseteq Z(\Re)$.

Theorem 2.19. *Let* $\mathcal{A}$ *be a unital, prime algebra, and let* $\mathfrak{L} : \mathcal{A} \rightarrow \mathcal{A}$ *be a Jordan left derivation. If* $\dim \Big(ann\{[a, b] \mid a, b \in \mathcal{A}\}\Big) \leq 1$, *then* $\mathfrak{L}$ *is identically zero.*

Proof. Applying Theorems 2.7 and 2.17, we achieve our goal. □

In the following theorem, we investigate Jordan left derivations on simple rings.

Theorem 2.20. *Let* $\Re$ *be a non-commutative 2-torsion free simple ring, and let* $\mathfrak{L} : \Re \rightarrow \Re$ *be a Jordan left derivation. In this case* $\mathfrak{L} = 0$.

Proof. It is obvious that $\Re$ is semiprime. According to Theorem 2.17, $\mathfrak{L}(x)[y, z] = 0$ for all $x, y, z \in \Re$. It follows from Lemma 1.3 of [24] that for every $x \in \Re$ there exists an ideal $\mathcal{I}_x$ of $\Re$ such that $\mathfrak{L}(x) \in \mathcal{I}_x \subseteq Z(\Re)$. Since $\Re$ is a simple ring, either $\mathcal{I}_x = \{0\}$ or $\mathcal{I}_x = \Re$. If $\mathcal{I}_x = \Re$, then we have $\Re = \mathcal{I}_x \subseteq Z(\Re)$ and so $\Re$ is commutative, a contradiction. Hence $\mathcal{I}_x = \{0\}$, and since $\mathfrak{L}(x) \in \mathcal{I}_x$, $\mathfrak{L}(x) = 0$. Since x is an arbitrary element of $\Re$, our assertion is proved. □

M. Brešar and J. Vukman [[5], Corollary 1.3] proved that every Jordan left derivation on a non-commutative 2-torsion free and 3-torsion free prime ring is identically zero. Here, we prove the same result without using the assumption that the ring is 3-torsion free.

Theorem 2.21. *Let $\mathfrak{R}$ be a non-commutative 2-torsion free prime ring, and let $\mathfrak{L}:\mathfrak{R}\to\mathfrak{R}$ be a Jordan left derivation. In this case, $\mathfrak{L}$ is zero.*

Proof. According to Theorem 2.17, $\mathfrak{L}(x)[y,z]=0$ for all $x,y,z\in\mathfrak{R}$. It follows from Theorem 2 of [23] that $\mathfrak{L}(\mathfrak{R})\subseteq Z(\mathfrak{R})$. Therefore, we have $0=r\mathfrak{L}(x)[y,z]=\mathfrak{L}(x)r[y,z]$ for all $x,y,z,r\in\mathfrak{R}$. The primeness of $\mathfrak{R}$ forces that $\mathfrak{L}(x)=0$ or $[y,z]=0$. Since $\mathfrak{R}$ is non-commutative and also x is an arbitrary element of $\mathfrak{R}$, $\mathfrak{L}=0$ is achieved. □

Theorem 2.22. *Let $\mathfrak{R}$ be a 2-torsion free, unital, simprime ring and let $\mathfrak{L}:\mathfrak{R}\to\mathfrak{R}$ be a Jordan left derivation. If there exists an element x_0 of $\mathfrak{R}$ such that $\mathfrak{L}(x_0)$ is invertible, then $\mathfrak{R}$ is commutative.*

Proof. It follows from Theorem 2.17 that $\mathfrak{L}(x)[y,z]=0$ for all $x,y,z\in\mathfrak{R}$. Thus, $\mathfrak{L}(x_0)[y,z]=0$ for all $y,z\in\mathfrak{R}$. This equation along with the assumption that $\mathfrak{L}(x_0)$ is invertible imply that $[y,z]=0$ for all $y,z\in\mathfrak{R}$, and consequently, $\mathfrak{R}$ is commutative. □

In Proposition 2.24, we show that every Jordan derivation on a (**commutative or non-commutative**) 2-torsion free prime ring is identically zero. To get such results, most authors assume that a ring or an algebra to be non-commutative. But in the following proposition, we do not use this assumption. We need the following lemma to establish Proposition 2.24.

Lemma 2.23. [[17], Lemma 2.2] *Let $\mathfrak{R}$ be a 2-torsion free prime ring and $\mathcal{I}$ be a non-zero Jordan ideal of $\mathfrak{R}$. If d is a derivation of $\mathfrak{R}$ such that $d(x^2)=0$ for all $x\in\mathcal{I}$, then $d=0$.*

Proposition 2.24. *Let $\mathfrak{R}$ be a 2-torsion free prime ring, $\mathcal{I}$ be a non-zero Jordan ideal of $\mathfrak{R}$, and let $d:\mathfrak{R}\to\mathfrak{R}$ be a Jordan derivation such that $d(a^2)\in Z(\mathfrak{R})$ for all $a\in\mathcal{I}$. If $ad(a)a=0$ for all $a\in\mathcal{I}$, then d is identically zero.*

Proof. It follows from Theorem 1 of [4] that d is a derivation. In this proof, $[a,b]$ and $<a,b>$ denote $ab-ba$ and $ab+ba$, respectively. First, note that
i) $[a,b]+<a,b>=2ab$,
ii) $<ab,c>=a<b,c>-[a,c]b$,
iii) $<a,bc>=<a,b>c-b[a,c]$
iv) $<a+b,c>=<a,c>+<b,c>$ and $<a,b+c>=<a,b>+<a,c>$.
By using the above symbols, we see that $d(a^2)=d(a)a+ad(a)=<a,d(a)>$. Let $\mathcal{I}$ be the above-mentioned Jordan ideal. Since $[d(a^2),a]=0$ for all $a\in\mathcal{I}$, we have $[d(a),a^2]=0$, i.e. $d(a)a^2=a^2d(a)$ for all $a\in\mathcal{I}$. In the next step, we show that $(d(a^2))^2a=0$ for all $a\in\mathcal{I}$. By using the equality (ii) and the assumptions that $ad(a)a=0$ and $d(a^2)\in Z(\mathfrak{R})$ for all $a\in\mathcal{I}$, we have

$$\begin{aligned}(d(a^2))^2 =<a,d(a)>^2 &=<<a,d(a)>a,d(a)>+[<a,d(a)>,d(a)]a\\ &=<<a,d(a)>a,d(a)>=<ad(a)a+d(a)a^2,d(a)>\\ &=<d(a)a^2,d(a)>=d(a)a^2d(a)+d(a)d(a)a^2\\ &=d(a)a^2d(a)+d(a)a^2d(a)\\ &=2d(a)a^2d(a).\end{aligned}$$

Therefore, we have

$$<a,d(a)>^2 a=(d(a^2))^2a=2d(a)a^2d(a)a=2d(a)aad(a)a=0,\ for\ all\ a\in\mathcal{I}.$$

Evidently, $d(a^2)\in Z(\mathfrak{R})$ causes that $(d(a^2))^2\in Z(\mathfrak{R})$ as well. Thus $(d(a^2))^3=<a,d(a)>^3=<<a,d(a)>^2 a,d(a)>+[<a,d(a)>^2,d(a)]a=0$ for all $a\in\mathcal{I}$. Hence, $(d(a^2))^4=<a,d(a)>^4=0$ for all $a\in\mathcal{I}$ and so

$< a, d(a) >^2 x < a, d(a) >^2 = < a, d(a) >^4 x = 0$ for all $x \in \Re$. The primeness of $\Re$ forces $< a, d(a) >^2 = 0$ for all $a \in \mathcal{I}$. Similarly, since $< a, d(a) > \in Z(\Re)$, we have $< a, d(a) > x < a, d(a) > = < a, d(a) >^2 x = 0$ for all $x \in \Re$. Reusing the primeness of $\Re$ implies that $< a, d(a) > = 0$, i.e. $d(a^2) = 0$ for all $a \in \mathcal{I}$. Here, Lemma 2.23 completes the proof. □

Corollary 2.25. *Let $\Re$ be a (commutative or non-commutative) 2-torsion free prime ring, $\mathcal{I}$ be a non-zero Jordan ideal of $\Re$, and let $\mathfrak{L} : \Re \to \Re$ be a Jordan left derivation such that $a\mathfrak{L}(a)a = 0$ for all $a \in \mathcal{I}$. Then, $\mathfrak{L}$ is identically zero.*

Proof. It follows from Theorem 2 of [23] that $\mathfrak{L}$ is a derivation which maps $\Re$ into $Z(\Re)$. Therefore, all the assumptions of Proposition 2.24 are fulfilled and consequently, our objective is achieved. □

We feel that in Corollary 2.25, the assumption $a\mathfrak{L}(a)a = 0$ for all $a \in \mathcal{I}$ can be removed. But we are unable to prove the result without this requirement.

A discussion on the presented conjecture in [9]:
After reviewing examples concerning UMV-property, it was seen that the spectrum of such elements is contained in the real numbers set (see Conjecture 2.12 in [9]). For example, let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$ be an idempotent element. We know that a has the UMV-property and clearly, $\mathfrak{S}(a) = \{0, 1\}$. Let $\mathcal{A}$ be a commutative unital Banach algebra. It follows from Theorem 1.3.4 of [16] that $\mathfrak{S}(a) = \{\varphi(a) \,|\, \varphi \in \Phi_{\mathcal{A}}\}$. Moreover, we know that every character is continuous (see Theorem 3.1.3 of [7]). Suppose that $a \in \mathcal{A}$ has the UMV-property. It means that for every closed interval $[\alpha, \beta] \subseteq \mathbb{R}$ there exists an element $c_{\alpha,\beta} \in (\alpha, \beta)$ such that $e^{\beta a} - e^{\alpha a} = (\beta - \alpha)ae^{ac_{\alpha,\beta}}$. If φ is an arbitrary character on $\mathcal{A}$, then we have

$$\varphi(e^{\beta a} - e^{\alpha a}) = \varphi((\beta - \alpha)ae^{ac_{\alpha,\beta}}) = (\beta - \alpha)\varphi(a)\varphi(e^{ac_{\alpha,\beta}}).$$

Since φ is a continuous linear mapping, we obtain that

$$e^{\beta\varphi(a)} - e^{\alpha\varphi(a)} = (\beta - \alpha)\varphi(a)e^{c_{\alpha,\beta}\varphi(a)}.$$

Having considered $z = \varphi(a) \in \mathbb{C}$, the above-mentioned equality turns into

$$e^{\beta z} - e^{\alpha z} = (\beta - \alpha)ze^{c_{\alpha,\beta}z}. \tag{6}$$

It is clear that $z = 0$ is a result of equation (6). We know that function $f(x) = e^x$ is continuous on the closed interval $[\alpha, \beta]$ and also is differentiable on the open interval (α, β). So, by the classical mean value theorem, we obtain that $f(\beta) - f(\alpha) = (\beta - \alpha)f'(c_{\alpha,\beta})$ for come $c_{\alpha,\beta} \in (\alpha, \beta)$. It means that

$$e^{\beta} - e^{\alpha} = (\beta - \alpha)e^{c_{\alpha,\beta}}.$$

Therefore, $z = 1$ is another result of equation (6). Using MATLAB software to solve equation (6), we see that this software acquires $z = 0, 1$ as the results of this equation.

Based on the above discussion, we see that if $\mathcal{A}$ is a unital, commutative Banach algebra and $a \in \mathcal{A}$ has the UMV-property, then $\mathfrak{S}(a) \subseteq \mathbb{R}$.

Acknowledgements The author is greatly indebted to the referee for his/her valuable suggestions and careful reading of the paper. Moreover, we would like to thank Dr. Fatemeh Ghomanjani for her help to solve equation (6) by using MATLAB software.

References

[1] S. Ali, *On Generalized Left Derivations in Rings and Banach Algebras*, Aequat. Math. **81** (2011), 209–226.
[2] M. Ashraf and N. Rehman, *On Lie ideals and Jordan left derivatins of prime rings*, Arch. Math. (Brno). **36** (2000), 201-206.

[3] M. Ashraf, S. Ali, N. Rehman and M. R. Mozumder, *On generalized left (α, β)-derivations in rings* , Bull. Iranian. Math. Soc. **38** (2012), 893–905.
[4] M. Bresar, Jordan derivations on semiprime rings, *Proc. Amer. Math. Soc.* **140** 4 (1988), 1003–1006.
[5] M. Brešar and J. Vukman, *On Left Derivations and Related Mappings,* Proc. Amer. Math. Soc. **110** (1990), 7–16.
[6] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monographs, New Series, 24, Oxford University Press, 2000.
[7] H. G. Dales, Pietro Aiena, Jorg Eschmeier, Kjeld Laursen and George A. Willis, *Introduction to Banach Algebras, Operators and Harmonic Analysis,* Cambridge University Press, 2003.
[8] A. Hosseini, *A characterization of weak n_m-Jordan (α, β)-derivations by generalized centralizers*, Rend. Circ. Math. Palermo. **64** (2015), 221–227.
[9] A. Hosseini, *A characterization of derivations on uniformly mean value Banach algebras*, Turkish J. Math. **40** (2016), 1058–1070.
[10] A. Hosseini, Some conditions under which derivations are zero on Banach $*$-algebras, Acta Univ. Sapientiae, Mathematica, **9** (2017), 176–184.
[11] A. Hosseini, *Commutativity of Banach Algebras Characterized by Primitive Ideals and Spectra*, Filomat, **31** (2017), 2053-2060.
[12] A. Hosseini, M. Hassani, A. Niknam and S. Hejazian, *Some results on σ-Derivations*, Ann. Funct. Anal. **2** (2011), 75-84.
[13] Y. S. Jung, *On left derivations and derivations of Banach algebras*, Bull. Korean Math. Soc. **35** (1998), 659–667.
[14] B. E. Johnson and A. M. Sinclair,*Continuity of derivations and a problem of Kaplansky*, Amer. J. Math. **90** (1968), 10681073.
[15] I. Kaplansky, *Projections in Banach Algebras*, Ann. of Math. 53 (2) (1951), 235-249.
[16] G. J. Murphy, C^*-Algebras and Operator Theory. Boston Academic Press, (1990).
[17] L. Oukhtite, A. Mamounib, C, Beddani, Derivations and Jordan ideals in prime rings, J. Taibah Univ. Sci. **8** (2014), 364369.
[18] E. C. Posner, Derivations in prime rings, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.
[19] S. Pedersen, *Anticommuting derivations*, Proc. Amer. Math. Soc. **127** (1999), 1103–1108.
[20] S. Sakai, *Operator Algebras in Dynamical Systems,* Cambridge University Press, Cambridge, 1991.
[21] N. Shahzad, *Jordan and Left Derivations on Locally C^*-algebras*, Southeast Asian Bull. Math. **31** (2007), 1183–1190.
[22] M. Takesaki, *Theory of Operator Algebras I,* Springer-Verlag Berlin Heidelberg New York, 2001.
[23] J. Vukman, *On Left Jordan Derivations of Rings and Banach Algebras,* Aequat. Math. **75** (2008), 260–266.
[24] B. Zalar, *On centralizers of semiprime rings*, Comment. Math. Univ. Carolin, 32 (4) (1991), 609-614.