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Yetter-Drinfeld Modules for Weak Hom-Hopf Algebras

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Abstract. The aim of this paper is to define and study Yetter-Drinfeld modules over weak Hom-Hopf algebras. We show that the category ${}_H\mathcal{WYD}^H$ of Yetter-Drinfeld modules with bijective structure maps over weak Hom-Hopf algebras is a rigid category and a braided monoidal category, and obtain a new solution of quantum Hom-Yang-Baxter equation. It turns out that, If H is quasitriangular (respectively, coquasitriangular)weak Hom-Hopf algebras, the category of modules (respectively, comodules) with bijective structure maps over H is a braided monoidal subcategory of the category ${}_H\mathcal{WYD}^H$ of Yetter-Drinfeld modules over weak Hom-Hopf algebras.

1. Introduction

The first examples of Hom-type algebras were related to q-deformations of Witt and Virasoro algebras, which play an important role in Physics, mainly in conformal field theory. The q-deformations of Witt and Virasoro algebras are obtained when the derivation is replaced by a σ-derivation. It was observed in the pioneering works (See [5]-[8]). Motivated by these examples and their generalization, Hartwig, Larsson and Silvestrov introduced the Hom-Lie algebras when they concerned about the q-deformations of Witt and Virasoro algebras in [4]. In a Hom-Lie algebra, the Jacobi identity is replaced by the so called Hom-Jacobi identity via an homomorphism. Hom-associative algebras, the corresponding structure of associative algebras, were introduced by Makhlouf and Silvestrov in [12]. The associativity of the Hom-algebra is twisted by an endomorphism (here we call it the Hom structure map). The generalized notions, Hom-bialgebras, Hom-Hopf algebras were developed in [13], [14]. Caenepeel and Goyvaerts studied in [2] Hom-bialgebras and Hom-Hopf algebras from a categorical view point, and called them monoidal Hom-bialgebras and monoidal Hom-Hopf algebras respectively, which are slightly different from the above Hom-bialgebras and Hom-Hopf algebras. Thus a monoidal Hom-bialgebra is Hom-bialgebra if and only if the Hom-structure map α satisfies $\alpha^2 = id$. Yau introduced Quasitriangular Hom-bialgebras in [17]), which provided a solution of the quantum Hom-Yang-Baxter euqation, a twisted version of the quantum Yang-Baxter equation called the Hom-Yang-Baxter equation in [18]. Zhang and Wang introduced weak Hom-Hopf algebra H, which is generalization of both Hom-Hopf algebras and weak Hopf algebras, and

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discussed the category *Rep*(*H*) (resp. *Corep*(*H*)) of Hom-modules (resp. Hom-comodules) with bijective Hom-structure maps, they proved that if H is a (co)quasitrialgular weak Hom-bialgebra (resp. ribbon weak Hom-Hopf algebra), then *Rep*(*H*) (resp. *Corep*(*H*)) is a braided monoidal category (resp. ribbon category) in [19].

Makhlouf and Panaite defined and studied Yetter-Drinfeld modules over Hom bialgebras, a generalized version of bialgebras obtained by modifying the algebra and coalgebra structures by a homomorphism. Yetter-Drinfeld modules over a Hom bialgebra with bijective structure map provide solutions of the Hom-Yang-Baxter equation in [10]. It is well known that the Yetter-Drinfeld modules category of a (weak) Hopf algebra is a rigid monoidal category, and is braided. Does this result remain true in a weak Hom-Hopf algebra? How the corresponding results appear under the condition that the associativity and coassociativity are twisted by an endomorphism? Is there any relation between this Yetter-Drinfeld modules category and module category or comodule category? This is the motivation of the present article. In order to investigate these questions, we introduce the definition of Yetter-Drinfeld modules over weak Hom-Hopf algebras, which is generalization of both weak Yetter-Drinfeld modules introduced by [3] or [15] and Hom-Yetter-Drinfeld modules introduced by [10] or [11], and consider that when the Yetter-Drinfeld modules category of a weak Hom-Hopf algebra is is a rigid monoidal category, and is braided.

To make sure that the Yetter-Drinfeld modules category of a weak Hom-Hopf algebra H is a monoidal category, we need H is unital and counital, and the Hom structure maps over H are all bijective maps.

The paper is organized as follows. In Section 2, we recall now several concepts and results, fixing thus the terminology to be used in the rest of the paper.

In Section 3, we introduce the definition of Yetter-Drinfeld modules over weak Hom-Hopf algebras and show the category ${}_H\mathcal{WYD}^H$ of Yetter-Drinfeld modules is a monoidal category and a rigid category.

In Section 4, we show that the category ${}_H\mathcal{WYD}^H$ of Yetter-Drinfeld modules is a braided monoidal category and obtain a new solution of quantum Hom-Yang-Baxter equation. It turns out that, if H is a quasitriangular weak Hom-Hopf algebra, the category of left H-modules with bijective structure maps is a braided monoidal subcategory of the category ${}_H\mathcal{WYD}^H$ of Yetter-Drinfeld modules.

In Section 5, we find another braided monoidal category structure on the category ${}_H\mathcal{WYD}^H$ of Yetter-Drinfeld modules , with the property that if H is a coquasitriangular weak Hom-Hopf algebra, then ${}_H\mathcal{WYD}^H$ contains the category of right H-comodules with bijective structure maps as a braided monoidal category.

2. Preliminaries

Throughout the paper, we let $\Bbbk$ be a fixed field and all algebras are supposed to be over $\Bbbk$. For the comultiplication Δ of a vector space C, we use the Sweedler-Heyneman's notation:

$$\Delta(c) = c_1 \otimes c_2,$$

for any $c \in C$. τ means the flip map $\tau(a \otimes b) = b \otimes a$. When we say a "Hom-algebra" or a "Hom-coalgebra", we mean the unital Hom-algebra and counital Hom-coalgebra.

In this section, we will review several definitions and notations related to weak Hom-Hopf algebras and rigid categories.

2.1. Hom-algebras and Hom-coalgebras.

Recall from [12] that a *Hom-associative algebra* is a quadruple (A, μ, η, α_A), in which A is a linear space, $\alpha_A : A \to A$, $\mu : A \otimes A \to A$ and $\eta : \Bbbk \to A$ are linear maps, with notation $\mu(a \otimes b) = ab$ and $\eta(1_\Bbbk) = 1_A$, satisfying the following conditions, for all $a, b, c \in A$:

$$\begin{cases} (1)\ \alpha_A(ab) = \alpha_A(a)\alpha(b); \\ (2)\ \alpha_A(a)(bc) = (ab)\alpha_A(c); \\ (3)\ \alpha_A(1_A) = 1_A; \\ (4)\ 1_A a = a 1_A = \alpha_A(a). \end{cases}$$

A morphism $f : A \to B$ *of Hom-algebras* is a linear map such that $\alpha_B \circ f = f \circ \alpha_A$, $f(1_A) = 1_B$ and $\mu_B \circ (f \otimes f) = f \circ \mu_A$.

Let A be a Hom-algebra. Recall that a *left A-module* is a triple (M, α_M, θ_M), where M is a $\Bbbk$-space, $\alpha_M : M \to M$ and $\theta_M : A \otimes M \to M$ are linear maps with notation $\theta_M(a \otimes m) = a \cdot m$, satisfying the following conditions, for all $a, b \in A, m \in M$:

$$\begin{cases} (1)\ \alpha(a \cdot m) = \alpha(a) \cdot \alpha_M(m); \\ (2)\ \alpha(a) \cdot (b \cdot m) = (ab) \cdot \alpha_M(m); \\ (3)\ 1_A \cdot m = \alpha_M(m). \end{cases}$$

A morphism $f : M \to N$ *of A-modules* is a linear map such that $\alpha_N \circ f = f \circ \alpha_M$ and $\theta_N \circ (id_A \otimes f) = f \circ \theta_M$.

Recall from [14] that a *Hom-coassociative coalgebra* is a quadruple $(C, \Delta, \varepsilon, \alpha_C)$, in which C is a linear space, $\alpha : C \to C, \Delta : C \to C \otimes C$ and $\varepsilon : C \to \Bbbk$ are linear maps, with notation $\Delta(c) = c_1 \otimes c_2$, satisfying the following conditions for all $c \in C$:

$$\begin{cases} (1)\ \Delta(\alpha_C(c)) = \alpha_C(c_1)\alpha(c_2); \\ (2)\ \alpha_C(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \alpha_C(c_2); \\ (3)\ \varepsilon \circ \alpha_C = \varepsilon; \\ (4)\ \varepsilon(c_1)c_2 = c_1\varepsilon(c_2) = \alpha_C(c). \end{cases}$$

A morphism $f : C \to D$ *of Hom-coalgebras* is a linear map such that $\alpha_D \circ f = f \circ \alpha_C$, $\varepsilon_C = \varepsilon_D \circ f$ and $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$.

Let C be a Hom-coalgebra. Recall that a *right C-comodule* is a triple (M, α_M, ρ_M), where M is a $\Bbbk$-space, $\alpha_M : M \to M$ and $\rho_M : M \to M \otimes C$ are linear maps with notation $\rho_M(m) = m_{(0)} \otimes m_{(1)}$, satisfying the following conditions for all $m \in M$:

$$\begin{cases} (1)\ \rho_M(\alpha_M(m)) = \alpha_M(m_{(0)}) \otimes \alpha_C(m_{(1)}); \\ (2)\ \alpha_M(m_{(0)}) \otimes \Delta(m_{(1)}) = \rho_M(m_{(0)}) \otimes \alpha_C(m_{(1)}); \\ (3)\ \varepsilon(m_1)m_0 = \alpha_M(m). \end{cases}$$

A morphism $f : M \to N$ *of C-comodules* is a linear map such that $\alpha_N \circ f = f \circ \alpha_M$ and $\rho_N \circ f = (id_C \otimes f) \circ \rho_M$.

Recall from [19] that a *weak Hom-bialgebra* is a sextuple $H = (H, \alpha_H, \mu, \eta, \Delta, \varepsilon)$ if (H, α_H) is both a Hom-algebra and a Hom-coalgebra, satisfying the following identities for any $a, b, c \in H$:

$$\begin{cases} (1)\ \Delta(ab) = \Delta(a)\Delta(b); \\ (2)\ \varepsilon((ab)c) = \varepsilon(ab_1)\varepsilon(b_2c),\ \ \varepsilon(a(bc)) = \varepsilon(ab_2)\varepsilon(b_1c); \\ (3)\ (\Delta \otimes id_H)\Delta(1_H) = 1_1 \otimes 1_2 1'_1 \otimes 1'_2,\ \ (id_H \otimes \Delta)\Delta(1_H) = 1_1 \otimes 1'_1 1_2 \otimes 1'_2. \end{cases}$$

Recall from [19] that a *Weak Hom-Hopf algebra* is a septuple $(H, \mu, \eta, \Delta, \varepsilon, S, \alpha_H)$, in which (H, α_H) is a weak Hom-bialgebra, if H endowed with a $\Bbbk$-linear map S (the *antipode*), such that for any $h, g \in H$, the following conditions hold:

$$\begin{cases} (1)\ S \circ \alpha_H = \alpha_H \circ S; \\ (2)\ h_1S(h_2) = \varepsilon_t(h),\ \ S(h_1)h_2 = \varepsilon_s(h); \\ (3)\ S(hg) = S(g)S(h),\ \ S(1_H) = 1_H; \\ (4)\ \Delta(S(h)) = S(h_2) \otimes S(h_1),\ \ \varepsilon \circ S = \varepsilon. \end{cases}$$

Let (H, α_H) be a weak Hom-bialgebra. Define linear maps ε_t and ε_s by the formula

$$\varepsilon_t(h) = \varepsilon(1_1h)1_2, \quad \varepsilon_s(h) = 1_1\varepsilon(h1_2),$$

for any $h \in H$, where ε_t, ε_s are called the *target* and *source counital maps*. We adopt the notations $H_t = \varepsilon_t(H)$ and $H_s = \varepsilon_s(H)$ for their images.

Similarly, we define the linear maps $\widehat{\varepsilon_t}$ and $\widehat{\varepsilon_s}$ by the formula

$$\widehat{\varepsilon_t}(h) = \varepsilon(h1_1)1_2, \quad \widehat{\varepsilon_s}(h) = 1_1\varepsilon(1_2h),$$

for any $h \in H$. Their images are denoted by $\widehat{H_t} = \widehat{\varepsilon_t}(H)$ and $\widehat{H_s} = \widehat{\varepsilon_s}(H)$.

2.2. Duality and rigid categories.

Recall from [9] that let $(C, \otimes, I, a, l, r)$ be a monoidal category. $V \in C$, a *left dual* of V is a triple $(V^*, ev_V, coev_V)$, where V^* is an object, $ev_V : V^* \otimes V \to I$ and $coev_V : I \to V \otimes V^*$ are morphisms in C, satisfying

$$r_V \circ (id_V \otimes ev_V) \circ a_{V,V^*,V} \circ (coev_V \otimes id_V) \circ l_V^{-1} = id_V,$$

and

$$l_{V^*} \circ (ev_V \otimes id_{V^*}) \circ a^{-1}_{V^*,V,V^*} \circ (id_{V^*} \otimes coev_V) \circ r^{-1}_{V^*} = id_{V^*}.$$

Similarly, a *right dual* of V is a triple $({}^*V, \widetilde{ev}_V, \widetilde{coev}_V)$, where ${}^*V$ is an object, $\widetilde{ev}_V : V \otimes {}^*V \to I$ and $\widetilde{coev}_V : I \to {}^*V \otimes V$ are morphisms in C, satisfying

$$r_{{}^*V} \circ (id_V \otimes \widetilde{ev}_V) \circ a_{{}^*V,V,{}^*V} \circ (\widetilde{coev}_V \otimes id_{{}^*V}) \circ l^{-1}_{{}^*V} = id_{{}^*V},$$

and

$$l_V \circ (\widetilde{ev}_V \otimes id_V) \circ a^{-1}_{V,{}^*V,V} \circ (id_V \otimes \widetilde{coev}_V) \circ r_V^{-1} = id_V.$$

If each object in C admits a left dual (resp. a right dual, both a left dual and a right dual), then C is called a *left rigid category* (resp. a *right rigid category*, a *rigid category*).

3. Left-right Yetter-Drinfeld Modules over a Weak Hom-Hopf Algebra

Definition 3.1. Let (H, α_H) be a weak Hom-Hopf algebra. A Yetter-Drinfeld module over H is a vector space (M, α_M), such that M is a unital left H-module (with notation $h \otimes m \mapsto h \cdot m$) and a counital right H-comodule (with notation $m \otimes h \mapsto m_{(0)} \otimes m_{(1)}$) with the following compatibility condition:

$$(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = \alpha_H^{-1}(h_{21}) \cdot m_{(0)} \otimes [\alpha^{-2}(h_{22})\alpha_H^{-1}(m_{(1)})]S^{-1}(h_1), \tag{3. 1}$$

for all $h \in H$ and $m \in M$. We denote by ${}_H\mathcal{WYD}^H$ the category of Yetter-Drinfeld modules, morphisms being the H-linear H-colinear maps.

Proposition 3.2. one has that(3.1) is equivalent to the following equations

$$\rho(m) = m_{(0)} \otimes m_{(1)} \in M \otimes_t H \triangleq (1_1 \otimes 1_2) \cdot (M \otimes H) \tag{3. 2}$$

$$\alpha_H(h_1) \cdot m_{(0)} \otimes \alpha_H^2(h_2)\alpha_H(m_{(1)}) = (h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)}\alpha_H^2(h_1) \tag{3. 3}$$

Proof (3.1) $\Longrightarrow$ (3.2), (3.3). We have

$$\begin{aligned}
& m_{(0)} \otimes m_{(1)} \\
= \ & \alpha_H^{-1}(1_{21}) \cdot \alpha_M^{-1}(m_{(0)}) \otimes [\alpha_H^{-2}(1_{22})\alpha_H^{-2}(m_{(1)})]S^{-1}(1_1) \\
= \ & 1'_1 \cdot (1_2 \cdot \alpha_M^{-1}(m_{(0)})) \otimes [\alpha_H^{-2}(1'_2)\alpha_H^{-2}(m_{(1)})]S^{-1}(1_1) \\
= \ & 1'_1 \cdot (1_2 \cdot \alpha_M^{-1}(m_{(0)})) \otimes \alpha_H^{-1}(1'_2)[\alpha_H^{-2}(m_{(1)})S^{-1}(\alpha_H^{-1}(1_1))] \\
= \ & 1'_1 \cdot (1_2 \cdot \alpha_M^{-1}(m_{(0)})) \otimes 1'_2[\alpha_H^{-2}(m_{(1)})S^{-1}(1_1)].
\end{aligned}$$

Then we do a calculation as follows:

$$\begin{aligned}
& (h_2\cdot m)_{(0)}\otimes(h_2\cdot m)_{(1)}\alpha_H^2(h_1)\\
= & \alpha_H^{-1}(h_{221})\cdot m_{(0)}\otimes[(\alpha_H^{-2}(h_{222})\alpha_H^{-1}(m_{(1)}))S^{-1}(h_{21})]\alpha_H^2(h_1)\\
= & h_{21}\cdot m_{(0)}\otimes[(\alpha_H^{-1}(h_{22})\alpha_H^{-1}(m_{(1)}))S^{-1}(h_{12})]\alpha_H(h_{11})\\
= & h_{21}\cdot m_{(0)}\otimes[(h_{22}m_{(1)})]S^{-1}(h_{12})h_{11}\\
= & h_{21}\cdot m_{(0)}\otimes[h_{22}m_{(1)}]S^{-1}(1_1)\varepsilon(h_1 1_2)\\
= & h_{12}\cdot m_{(0)}\otimes[\alpha_H(h_2)m_{(1)}]S^{-1}(1_1)\varepsilon(\alpha_H^{-1}(h_{11})1_2)\\
= & \alpha_H^{-1}(h_{12})1'_2\cdot m_{(0)}\otimes[\alpha_H(h_2)m_{(1)}]S^{-1}(1_1)\varepsilon(\alpha_H^{-2}(h_{11})\alpha_H^{-1}(1'_1)1_2)\\
= & \alpha_H^{-1}(h_{12})1'_2\cdot m_{(0)}\otimes[\alpha_H(h_2)m_{(1)}]S^{-1}(1_1)\varepsilon(\alpha_H^{-1}(h_{11})[\alpha_H^{-1}(1'_1)\alpha_H^{-1}(1_2)])\\
= & \alpha_H^{-1}(h_{12})1'_2\cdot m_{(0)}\otimes[\alpha_H(h_2)m_{(1)}]S^{-1}(1_1)\varepsilon(\alpha_H^{-1}(h_{11})[1'_1 1_2])\\
= & h_1 1_2\cdot m_{(0)}\otimes[h_2 m_{(1)}]S^{-1}(1_1)\\
= & h_1 1'_1 1_2\cdot m_{(0)}\otimes[\alpha_H(h_2)[1'_2\alpha_H^{-1}(m_{(1)})]]S^{-1}(1_1)\\
= & \alpha_H(h_1)[1_2\cdot\alpha_M^{-1}(m_{(0)})]\otimes\alpha_H^2(h_2)[[1_3\alpha_H^{-1}(m_{(1)})]S^{-1}(\alpha_H^{-1}(1_1))]\\
= & \alpha_H(h_1)\cdot m_{(0)}\otimes\alpha_H^2(h_2)\alpha_H(m_{(1)}).
\end{aligned}$$

For (3.2), (3.3) $\Longrightarrow$ (3.1), we have

$$\begin{aligned}
& \alpha_H^{-1}(h_{21})\cdot m_{(0)}\otimes[\alpha_H^{-2}(h_{22})\alpha_H^{-1}(m_{(1)})]S^{-1}(h_1)\\
= & \alpha_M^2(\alpha_H^{-4}(h_{22})\cdot\alpha_M^{-2}(m))_{(0)}\otimes(\alpha_H^{-4}(h_{22})\cdot\alpha_M^{-2}(m))_{(1)}\alpha_H^2(\alpha_H^{-4}(h_{21}))S^{-1}(h_1)\\
= & (\alpha_H^{-1}(h_2)\cdot m)_{(0)}\otimes(\alpha_H^{-2}(h_2)\cdot\alpha_M^{-1}(m))_{(1)}\alpha_H^{-2}(h_{12}))S^{-1}(\alpha_H^{-2}(h_{11}))\\
= & (\alpha_H^{-1}(h_2)\cdot m)_{(0)}\otimes(\alpha_H^{-2}(h_2)\cdot\alpha_M^{-1}(m))_{(1)}1_1\varepsilon(1_2\alpha_H^{-2}(h_1))\\
= & (1'_2\alpha_H^{-2}(h_2)\cdot m)_{(0)}\otimes(1'_2\alpha_H^{-3}(h_2)\cdot\alpha_M^{-1}(m))_{(1)}1_1\varepsilon(1_2 1'_1\alpha_H^{-3}(h_1))\\
= & (1_3\alpha_H^{-2}(h_2)\cdot m)_{(0)}\otimes(1_3\alpha_H^{-3}(h_2)\cdot\alpha_M^{-1}(m))_{(1)}1_1\varepsilon(1_2\alpha_H^{-2}(h_1))\\
= & (1_2\alpha_H^{-1}(h)\cdot m)_{(0)}\otimes(1_2\alpha_H^{-2}(h_2)\cdot\alpha_M^{-1}(m))_{(1)}1_1\\
= & (1_2\cdot(\alpha_H^{-1}(h)\cdot\alpha_H^{-1}(m))_{(0)}\otimes(1_2\cdot(\alpha_H^{-2}(h_2)\cdot\alpha_M^{-2}(m)))_{(1)}1_1\\
= & 1_1\cdot\alpha_M^{-1}(h\cdot m)_{(0)}\otimes 1_2\alpha_H^{-1}(h\cdot m)_{(1)}\\
= & (h\cdot m)_{(0)}\otimes(h\cdot m)_{(1)}.
\end{aligned}$$

Definition 3.3. Let (H,α_H) be a weak Hom-Hopf algebra. Left-right weak-Hom type entwining structure is a triple (A,C,ψ), where (A,α_A) is a Hom-algebra and (C,α_C) is a Hom-coalgebra with a linear map $\psi: A\otimes C\to A\otimes C$ such that $\psi\circ(\alpha_A\otimes\alpha_C)=(\alpha_A\otimes\alpha_C)\circ\psi$ satisfying the following conditions:

$$_{\psi}(ab)\otimes\alpha_C(c^{\psi}) = {}_{\psi}a_{\varphi}b\otimes\alpha_C(c)^{\psi\varphi}, \tag{3. 4}$$

$$\psi(c\otimes 1_A) = \varepsilon(c_1^{\psi})_{\psi}1_A\otimes c_2, \tag{3. 5}$$

$$\alpha_A({}_{\psi}a)\otimes\Delta(c^{\psi}) = \alpha_A(a)_{\varphi\psi})\otimes(c_{(1)}{}^{\psi}\otimes c_{(2)}{}^{\varphi}), \tag{3. 6}$$

$$\varepsilon(c^{\psi})_{\psi}a = \varepsilon(c^{\psi})a({}_{\psi}1_A). \tag{3. 7}$$

Over a weak-Hom type entwining structure (A,C,ψ), a left-right weak-Hom type entwined modules (M,α_M) is both a right C-comodule and a left A-module such that

$$\rho_M(a\cdot m) \quad = \quad \alpha_A({}_{\psi}a)\cdot m_{(0)}\cdot\otimes\alpha_C(m_{(1)}^{\psi}), \tag{3. 8}$$

for all $a\in A$ and $m\in M$. ${}_A\mathcal{WM}(\psi)^C$ will denote the category of left-right weak-Hom type entwined modules and morphisms between them.

Proposition 3.4. Let (H, α_H) be a weak Hom-Hopf algebra. Define $\phi : H \otimes H \to H \otimes H$ given by $\phi(a \otimes c) = \alpha_H^{-1}(a_{21}) \otimes (\alpha_H^{-2}(a_{22})\alpha_H^{-1}(c))S^{-1}(a_1)$ for all $h, g \in H$, and so ${}_H\mathcal{WM}(\psi)^H$ is the category of weak-Hom type entwined modules. In fact, for any $(M, \mu) \in {}_H\mathcal{WM}(\psi)^H$, one has compatible condition

$$\rho_M(a \cdot m) = \alpha_H^{-1}(a_{21}) \cdot m_{(0)} \otimes (\alpha_H^{-2}(a_{22})\alpha_H^{-1}(m_{(1)}))S^{-1}(a_1).$$

Proof. We need to prove that (3.4-3.7) hold. First, it is straightforward to check (3.4) and (3.6). In what follows, we only verify (3.5) and (3.7). In fact, for all $a, b, c \in H$, we have

$$\begin{aligned}
c_2 \otimes \varepsilon(c_1^{\psi})_{\psi}1_A &= c_2 \otimes \varepsilon(\alpha_H^{-2}(1_3)\alpha_H^{-1}(c_1))S^{-1}(1_1))\alpha_H(1_2) \\
&= \widetilde{1}_2\alpha_H^{-1}(c_2) \otimes \varepsilon(1_2''[\widetilde{1}_1\alpha_H^{-2}(c_1))]1_2')\varepsilon(1_1'S^{-1}(1_1))1_21_1'' \\
&= \widetilde{1}_2\alpha_H^{-1}(c_2) \otimes \varepsilon([1_2''\widetilde{1}_1]\alpha_H^{-1}(c_1)1_2')\varepsilon(1_1'S^{-1}(1_1))1_21_1'' \\
&= 1_3''\alpha_H^{-1}(c_2) \otimes \varepsilon([1_2''\alpha_H^{-1}(c_1)]1_2')\varepsilon(1_1'1_2)S(1_1)1_1'' \\
&= 1_3''\alpha_H^{-2}(c_2)1_3' \otimes \varepsilon([1_2''\alpha_H^{-1}(c_1)]1_2')\varepsilon(1_1'1_2)S(1_1)1_1'' \\
&= 1_2''\alpha_H^{-1}(c)1_2' \otimes \varepsilon(1_1'1_2)S(1_1)1_1'' \\
&= 1_2''\alpha_H^{-1}(c)1_2 \otimes S(1_1)1_1'' \\
&= 1_2'\alpha_H^{-1}(c)S^{-1}(1_1) \otimes 1_21_1' \\
&= 1_3\alpha_H^{-1}(c)S^{-1}(1_1) \otimes 1_2 \\
&= c^{\psi} \otimes {}_{\psi}1_A.
\end{aligned}$$

As for (3.7), we compute:

$$\begin{aligned}
\varepsilon(c^{\psi})_{\psi}a &= \varepsilon((\alpha_H^{-2}(a_{22})\alpha_H^{-1}(c))S^{-1}(a_1))\alpha_H(a_{21}) \\
&= \varepsilon((\alpha_H^{-2}(a_{22})\alpha_H^{-1}(c))1_2)\varepsilon(1_1S^{-1}(a_1))\alpha_H(a_{21}) \\
&= \varepsilon(\alpha_H^{-1}(a_{22})(\alpha_H^{-1}(c)1_2))\varepsilon(1_1S^{-1}(a_1))\alpha_H(a_{21}) \\
&= \varepsilon(\alpha_H^{-1}(a_{22})1_1')\varepsilon(1_2'(\alpha_H^{-1}(c)1_2))\varepsilon(1_1S^{-1}(a_1))\alpha_H(a_{21}) \\
&= \varepsilon(\alpha_H^{-1}(a_{22})1_1')\varepsilon(1_2'(\alpha_H^{-1}(c)S^{-1}(1_1)))\varepsilon(a_11_2)\alpha_H(a_{21}) \\
&= \varepsilon(a_21_1')\varepsilon(1_2'(\alpha_H^{-1}(c)S^{-1}(1_1)))\varepsilon(\alpha_H^{-1}(a_{11})1_2)\alpha_H(a_{12}) \\
&= \varepsilon(a_21_1')\varepsilon(1_2'(\alpha_H^{-1}(c)S^{-1}(1_1)))\varepsilon(a_1(1_1''1_2))a_{21}1_2'' \\
&= \varepsilon(\alpha^{-1}(a_2)1_2''1_1')\varepsilon(1_2'(\alpha_H^{-1}(c)S^{-1}(1_1)))a_11_1''1_2 \\
&= \varepsilon(a_2[1_2''1_1'])\varepsilon(1_2'(\alpha_H^{-1}(c)S^{-1}(1_1)))a_11_1''1_2 \\
&= \varepsilon(1_2'(\alpha_H^{-1}(c)S^{-1}(1_1)))\alpha_H(a_1)1_1'1_2 \\
&= \varepsilon(1_2'(\alpha_H^{-1}(c)S^{-1}(1_1)))a_1[1_1'1_2] \\
&= \varepsilon(1_3(\alpha_H^{-1}(c)S^{-1}(1_1)))a_11_2 \\
&= \varepsilon(c^{\psi})a({}_{\psi}1_A).
\end{aligned}$$

Proposition 3.5. Let (H, α_H) be a weak Hom-Hopf algebra, for any $(M, \alpha_M), (N, \alpha_N) \in {}_H\mathcal{WYD}^H$, and define the linear map

$$B_{M,N} : M \otimes N \to N \otimes M, \qquad B_{M,N}(m \otimes n) = n_{(0)} \otimes \alpha_H^{-1}(n_{(1)}) \cdot m.$$

Then, we have $(\alpha_N \otimes \alpha_M) \circ B_{M,N} = B_{M,N} \circ (\alpha_M \otimes \alpha_N)$ and, if $(P, \alpha_P) \in {}_H\mathcal{WYD}^H$, the maps $B_{-,-}$ satisfy the Hom-Yang-Baxter equation:

$$\begin{aligned}
&(\alpha_P \otimes B_{M,N}) \circ (B_{M,P}) \otimes (\alpha_M \otimes B_{N,P}) \\
= \ &(B_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes B_{N,P}) \circ ((B_{M,P} \otimes \alpha_M).
\end{aligned}$$

Proof. The proof is similar to Proposition 3.4 in [10].

lemma 3.6. Let(H,α_H) be a weak Hom-Hopf algebra, then H_s is the unit object in ${}_H\mathcal{WYD}^H$ with the action: for any $h\in H$, $x\in H_s$,

$$h\cdot x=\widehat{\varepsilon_s}(h)(hx),\ \ \rho(x)=x_1\otimes x_2,$$

and $\alpha_{H_s}=\alpha$.

Proof. The proof is similar to [15].

lemma 3.7. Let (H,α_H) be a weak Hom-Hopf algebra, the left and right unit constraints $l_M: H_s\otimes_t M\to M$ and $r_M: M\otimes_t H_s\to M$ and their inverses are given by the formulas

$$l_M(x\otimes m)=S(x)\cdot\alpha_M^{-2}(m),\qquad l_M^{-1}(m)=1_H\otimes\alpha_M(m),$$
$$r_M(m\otimes x)=x\cdot\alpha_M^{-2}(m),\qquad r_M^{-1}(m)=\varepsilon(1_3)\varepsilon_s(1_2)\cdot\alpha_M(m)\otimes 1_1.$$

Proof. It is easy to see that l_M is natural isomorphisms in ${}_H\mathcal{WYD}^H$. We only check that

$$\begin{aligned}
l_M^{-1}l_M(x\otimes_t m) &= l_M^{-1}(S(x)\cdot\alpha_M^{-2}(m))=1_H\otimes_t S(x)\cdot\alpha_M^{-1}(m)\\
&= \varepsilon_t(1_1)\otimes(1_2S(x))\cdot m\\
&= \varepsilon_t(\widehat{\varepsilon_s}(S(x))_1)\otimes\widehat{\varepsilon_s}(x)_2\cdot m\\
&= \varepsilon_t(1_1\widehat{\varepsilon_s}(S(x)))\otimes 1_2\cdot m\\
&= \varepsilon_t(1_1x)\otimes 1_2\cdot m=x\otimes_t m,
\end{aligned}$$

and

$$l_Ml_M^{-1}(m)=l_M(1_H\otimes_t\alpha_M(m))=1_H\cdot\alpha_M^{-2}(\alpha_M(m))=m,$$

which implies l_M^{-1} is the inverse of l_M.

Similarly, we can check that r_M is a natural isomorphism with the inverse r_M^{-1} in ${}_H\mathcal{WYD}^H$.

Theorem 3.8. Let (H,α_H) be a weak Hom-Hopf algebra. Then $({}_H\mathcal{WYD}^H,\otimes_t,H_s)$ is a monoidal category.

Proof. Firstly, for any $(M,\alpha_M),(N,\alpha_N),(P,\alpha_P)\in{}_H\mathcal{WYD}^H$, define an associativity constraint by

$$a_{M,N,P}((m\otimes_t n)\otimes_t p)=\alpha_M^{-1}(m)\otimes_t(n\otimes_t\alpha_P(p)),\ m\in M,\ n\in N,\ p\in P.$$

Obviously that a is natural and satisfies $a_{M,N,P}\circ(\alpha_M\otimes(\alpha_N\otimes\alpha_P))=((\alpha_M\otimes\alpha_N)\otimes\alpha_P)\circ a_{M,N,P}$. For any $h\in H$, since

$$\begin{aligned}
&a_{M,N,P}(h\cdot((m\otimes_t n)\otimes_t p))\\
&=\alpha_M^{-1}(h_{11}\cdot m)\otimes_t(h_{12}\cdot n\otimes_t\alpha_P(h_2\cdot p))\\
&=h_1\cdot\alpha_M^{-1}(m)\otimes_t(h_{21}\cdot n\otimes_t h_{22}\cdot\alpha_P(p))\\
&=h\cdot(\alpha_M^{-1}(m)\otimes_t(n\otimes_t\alpha_P(p)))\\
&=h\cdot(a_{M,N,P}((m\otimes_t n)\otimes_t p)),
\end{aligned}$$

$a_{M,N,P}$ is H-linear.

Next we will check that $a_{M,N,P}$ is H-colinear.

$$\begin{aligned}
&(a_{M,N,P}\otimes id_H)\circ\rho_{(M\hat{\otimes}N)\hat{\otimes}P}((m\otimes n)\otimes p)\\
=\ &(a_{M,N,P}\otimes id_H)((m\otimes n)_{(0)}\otimes p_{(0)}\otimes\alpha_H^{-2}(p_{(1)}(m\otimes n)_{(1)}))\\
=\ &a_{M,N,P}((m_{(0)}\otimes n_{(0)})\otimes p_{(0)})\otimes\alpha_H^{-2}(p_{(1)}\alpha_H^{-2}(n_{(1)}m_{(1)}))\\
=\ &\alpha_M^{-1}(m_{(0)})\otimes(n_{(0)}\otimes\alpha_P(p_{(0)}))\otimes\alpha_H^{-2}(p_{(1)})\alpha_H^{-4}(n_{(1)}m_{(1)}),
\end{aligned}$$

$$
\begin{aligned}
& \rho_{(M\hat{\otimes}N)\hat{\otimes}P} \circ a_{M,N,P}((m \otimes n) \otimes p) \\
= \ & \rho_{(M\hat{\otimes}N)\hat{\otimes}P}(\alpha_M^{-1}(m) \otimes (n \otimes \alpha_P(p))) \\
= \ & \alpha_M^{-1}(m)_{(0)} \otimes (n \otimes \alpha_P(p))_{(0)} \otimes \alpha_H^{-2}((n \otimes \alpha_P(p))_{(1)} \alpha_M^{-1}(m)_{(1)}) \\
= \ & \alpha_M^{-1}(m)_{(0)} \otimes (n_{(0)} \otimes \alpha_P(p)_{(0)}) \otimes \alpha_H^{-2}(\alpha_H^{-2}(\alpha_H(p)_{(1)} n_{(1)}) \alpha_M^{-1}(m)_{(1)}) \\
= \ & \alpha_M^{-1}(m)_{(0)} \otimes (n_{(0)} \otimes \alpha_P(p)_{(0)}) \otimes [\alpha_H^{-3}(p)_{(1)} \alpha_H^{-4}(n_{(1)})] \alpha_M^{-3}(m)_{(1)} \\
= \ & \alpha_M^{-1}(m_{(0)}) \otimes (n_{(0)} \otimes \alpha_P(p_{(0)})) \otimes [\alpha_H^{-2}(p)_{(1)} [\alpha_H^{-4}(n_{(1)}) \alpha_M^{-4}(m)_{(1)}] \\
= \ & \alpha_M^{-1}(m_{(0)}) \otimes (n_{(0)} \otimes \alpha_P(p_{(0)})) \otimes \alpha_H^{-2}(p_{(1)}) \alpha_H^{-4}(n_{(1)} m_{(1)}).
\end{aligned}
$$

And $a_{M,N,P}$ is a bijection because of α_M, α_P are all bijective maps. Thus a is a natural isomorphism in ${}_H\mathcal{WYD}^H$.

Secondly, it is also a direct check to prove that a satisfies the Pentagon Axiom.

At last, we will check the Triangle Axiom. In fact, for any $x \in H_s$, we have

$$
\begin{aligned}
& (r_M \otimes id_N)((m \otimes_t x) \otimes_t n) \\
= \ & x \cdot \alpha_M^{-2}(m) \otimes_t n \\
= \ & 1_1 \cdot (x \cdot \alpha_M^{-2}(m)) \otimes_t 1_2 \cdot n \\
= \ & 1_1 x \cdot \alpha_M^{-1}(m) \otimes_t 1_2 \cdot n \\
= \ & 1_1 \cdot \alpha_M^{-1}(m) \otimes_t 1_2 S(x) \cdot n \\
= \ & 1_1 \cdot \alpha_M^{-1}(m) \otimes_t 1_2 \cdot (S(x) \cdot \alpha_N^{-1}(n)) \\
= \ & \alpha_M^{-1}(m) \otimes S(x) \cdot \alpha_N^{-1}(n) \\
= \ & \alpha_M^{-1}(m) \otimes_t S(x) \cdot \alpha_N^{-2}(\alpha_N(n)) \\
= \ & (id_M \otimes l_N) a_{M,H_t,N}((m \otimes_t x) \otimes_t n).
\end{aligned}
$$

Let (H, α_H) be a weak Hom-Hopf algebra with a bijective antipode S. Consider the full subcategory ${}_H\mathcal{WYD}^H_{f.d.}$ of ${}_H\mathcal{WYD}^H$ whose objects are finite-dimensional. Using the antipode S of H, we can provide $\mathcal{WYD}(H)_{f.d.}$ with a duality.

For any $(M, \alpha_M) \in {}_H\mathcal{WYD}^H_{f.d.}$, we set ${}^*M = Hom(M, k)$, with the action and the coaction of H on M^* given by

$$(h \cdot f)(m) = f(S(\alpha_H^{-1}(h)) \cdot \alpha_M^{-2}(m)) \quad \text{and} \quad f_{(0)}(m) \otimes f_{(1)} = f(\alpha_M^{-2}(m_{(0)})) \otimes S^{-1}(\alpha_H^{-1}(m_{(1)})).$$

Similarly, for any $(M, \alpha_M) \in \mathcal{WYD}(H)_{f.d.}$, we set ${}^*M = Hom(M, k)$, with the action and the coaction of H on M^* given by

$$(h \cdot f)(m) = f(S^{-1}(\alpha_H^{-1}(h)) \cdot \alpha_M^{-2}(m)) \quad \text{and} \quad f_{(0)}(m) \otimes f_{(1)} = f(\alpha_M^{-2}(m_{(0)})) \otimes S(\alpha_H^{-1}(m_{(1)})).$$

Theorem 3.9. The category ${}_H\mathcal{WYD}^H_{f.d.}$ is a rigid category.

Proof. Define the maps

$$coev_M : \ H_t \to M \otimes_t M^*, \quad x \mapsto \sum x \cdot (e_i \otimes_t \alpha_{M^*}(e^i)),$$

where e_i and e^i are bases of M and M^*, respectively, dual to each other, and

$$ev_M : \ M^* \otimes_t M \to H_t, \quad f \otimes_t m \mapsto f(1_1 \cdot m) 1_2.$$

Firstly, we will prove that M^* is indeed an object in ${}_H\mathcal{WYD}^H$, and $\alpha_{^*M}$ is given by

$$\alpha_{^*M}(f)(m) = f(\alpha_M^{-1}(m)), \quad f \in {}^*M, \ m \in M.$$

We have

$$\begin{aligned}(h\cdot f)_{(0)}(m)\otimes(h\cdot f)_{(1)} &= (h\cdot f)(\alpha_M^{-2}(m_{(0)}))\otimes S^{-1}(\alpha_H^{-1}(m_{(1)}))\\ &= f(S(\alpha_H^{-1}(h))\cdot\alpha^{-4}(m_{(0)})))\otimes S^{-1}(\alpha_H^{-1}(m_{(1)})),\end{aligned}$$

$$\begin{aligned}&(\alpha_H^{-1}(h_{21})\cdot f_{(0)})(m)\otimes(\alpha_H^{-2}(h_{22})\alpha_H^{-1}(f_{(1)}))S^{-1}(h_1)\\
=\ & f_{(0)}(S(\alpha_H^{-2}(h_{21}))\cdot\alpha_M^{-2}(m))\otimes(\alpha_H^{-2}(h_{22})\alpha_H^{-1}(f_{(1)}))S^{-1}(h_1)\\
=\ & f(S(\alpha_H^{-4}(h_{21}))\cdot\alpha_M^{-4}(m))_{(0)})\otimes(\alpha_H^{-2}(h_{22})\alpha_H^{-2}(S(\alpha_H^{-1}(h_{21}))\cdot\alpha_H^{-2}(m_{(1)})))S^{-1}(h_1)\\
=\ & f(S(\alpha_H^{-5}(h_{2112}))\cdot\alpha_M^{-4}(m)_{(0)})\otimes(\alpha_H^{-2}(h_{22})\\
& [S^{-1}(\alpha_H^{-4}(h_{212}))(\alpha_H^{-5}(S^{-1}(m_{(1)}))\alpha_H^{-6}(h_{2111}))])S^{-1}(h_1)\\
=\ & f(S(\alpha_H^{-4}(h_{212}))\cdot\alpha_M^{-4}(m)_{(0)}))\otimes([\alpha_H^{-4}(h_{222})S^{-1}(\alpha_H^{-4}(h_{221}))]\\
& (\alpha_H^{-4}(S^{-1}(m_{(1)}))\alpha_H^{-4}(h_{211}))])S^{-1}(h_1)\\
=\ & f(S(\alpha_H^{-4}(h_{212}))\cdot\alpha_M^{-2}(m)_{(0)}))\otimes(1_1\varepsilon(1_2h_{22})\\
& (\alpha_H^{-4}(S^{-1}(m_{(1)}))\alpha_H^{-4}(h_{211}))])S^{-1}(h_1)\\
=\ & f(S(\alpha_H^{-4}(h_{221}))\cdot\alpha_M^{-4}(m)_{(0)}))\otimes(1_1\varepsilon(1_2h_{222})(\alpha_H^{-4}(S^{-1}(m_{(1)}))\alpha_H^{-3}(h_{21}))])S^{-1}(h_1)\\
=\ & f(S(\alpha_H^{-4}(h_{221}))\cdot\alpha_M^{-4}(m)_{(0)}))\otimes(1_1\varepsilon(1_2h_{222})(\alpha_H^{-4}(S^{-1}(m_{(1)}))\alpha_H^{-3}(h_{21}))])S^{-1}(h_1)\\
=\ & f(S(\alpha_H^{-3}(h_{21}))\cdot\alpha_M^{-4}(m)_{(0)}))\otimes(1_1\varepsilon(1_2h_{22})(\alpha_H^{-3}(S^{-1}(m_{(1)}))[\alpha_H^{-2}(h_{12}))S^{-1}(\alpha_H^{-2}(h_{11}))]\\
=\ & f(S(\alpha_H^{-3}(h_{21}))\cdot\alpha_M^{-4}(m)_{(0)}))\otimes(1'_1\varepsilon(1'_2h_{22})(\alpha_H^{-3}(S^{-1}(m_{(1)}))1_1\varepsilon(1_2h_1)\\
=\ & f(S(\alpha_H^{-4}(1''_2h_{12}))\cdot\alpha_M^{-4}(m)_{(0)}))\otimes(1'_1\varepsilon(1'_2h_2)(\alpha_H^{-2}(S^{-1}(m_{(1)}))1_1\varepsilon(1_21''_1h_{11})\\
=\ & f(S(\alpha_H^{-3}(1_2h_1))\cdot\alpha_M^{-4}(m)_{(0)}))\otimes(1'_1\varepsilon(1'_2h_2)(\alpha_H^{-3}(S^{-1}(m_{(1)}))1_1\\
=\ & f(S(\alpha_H^{-4}(1''_1h_1))S(1_2)\cdot\alpha_M^{-4}(m)_{(0)}))\otimes(S^{-1}(1'_2)\varepsilon(1'_11''_2h_2)(\alpha_H^{-3}(S^{-1}(m_{(1)}))1_1\\
=\ & f(S(\alpha_H^{-3}(1_21'_1h))\cdot\alpha_M^{-4}(m)_{(0)}))\otimes(S^{-1}(1'_2)(\alpha_H^{-3}(S^{-1}(m_{(1)}))1_1\\
=\ & f(S(\alpha_H^{-2}(h))S(1_2)\cdot\alpha_M^{-4}(m)_{(0)}))\otimes(S^{-1}(1_3\alpha_H^{-3}(S^{-1}(m_{(1)}))S^{-1}(1_1))\\
=\ & f(S(\alpha_H^{-1}(h))(1_2\cdot\alpha_M^{-5}(m)_{(0)}))\otimes(S^{-1}(1_3\alpha_H^{-3}(S^{-1}(m_{(1)}))S^{-1}(1_1))\\
=\ & f(S(\alpha_H^{-1}(h))\cdot\alpha_M^{-4}(m)_{(0)})\otimes S^{-1}(\alpha_H^{-1}(m_{(1)})).\end{aligned}$$

Which means that

$$(h\cdot f)_{(0)}\otimes(h\cdot f)_{(1)}=(\alpha_H^{-1}(h_{21})\cdot f_{(0)})\otimes(\alpha_H^{-2}(h_{22})\alpha_H^{-1}(f_{(1)}))S^{-1}(h_1).$$

We have known that $H_s\in{}_H\mathcal{WYD}^H$, with left H-module structure $h\cdot z=\widehat{\varepsilon_s}(hz)$ and right H-comodule structure $\rho(x)=x_1\otimes x_2$, for all $x\in H_s$. Next, we will check ev_M and $coev_M$ are morphisms in ${}_H\mathcal{WYD}^H$. For any $h\in H$, $m\in M$, $f\in M^*$, we compute

$$\begin{aligned}ev_M(h\cdot(f\otimes_t m)) &= (h_1\cdot f)(1_1\cdot(h_2\cdot m))1_2\\
&= f(S(1'_1\alpha_H^{-2}(h_1))\cdot((1_1\cdot(1'_2\alpha_H^{-3}(h_2)))\cdot\alpha_M^{-1}(m)))1_2\\
&= f((S(1_1\alpha_H^{-2}(h_1))(1_2\alpha_H^{-2}(h_2)))\cdot\cdot\alpha_M^{-1}(m))1_3\\
&= f(\varepsilon_s(1_1\alpha_H^{-2}(h))\cdot m)1_2\\
&= f(\varepsilon_s(\alpha_H^{-2}(h_1))\cdot v)\varepsilon_t(\alpha_H^{-2}(h_1))\\
&= f(1_1\cdot m)\varepsilon_t(h1_2)=h\cdot(ev_M(f\otimes_t m)),\end{aligned}$$

and on one hand,

$$\begin{aligned}\rho\circ ev_M(f\otimes m) &= \rho(1_2)f(1_1\cdot m)\\
&= f(1_1\cdot m)S(1'_1)\otimes 1'_2S^{-1}(1_2),\end{aligned}$$

on the other hand

$$\begin{aligned}(ev_M \otimes id) \circ \rho(f \otimes m) &= (ev_M \otimes id)(f_{(0)} \otimes m_{(0)}) \otimes \alpha_H^{-2}(m_{(1)} f_{(1)}) \\ &= f_{(0)}(1_1 \cdot m_{(0)})1_2 \otimes \alpha_H^{-2}(m_{(1)} f_{(1)}) \\ &= f(1_2 \cdot \alpha_M^{-2}(m_{(0)(0)}))(1'_2) \otimes \alpha_H^{-2}(m_{(1)}) S^{-1}(1'_1 1_3 \alpha_H^{-3}(m_{(0)(1)}) S^{-1}(1_1)) \\ &= f(1_2 \cdot \alpha_M^{-1}(m)_{(0)})(1'_2) \otimes \alpha_H^{-3}(m_{(1)2}) S^{-1}(1'_1 1_3 \alpha_H^{-3}(m_{(1)1}) S^{-1}(1_1)) \\ &= f(1_2 \cdot \alpha_M^{-1}(m)_{(0)})(1'_2) \otimes S^{-1}(\varepsilon_t(m_{(1)})) S^{-1}(1'_1)) \\ &= f(1_1 \cdot m)1'_2 \otimes S^{-1}(1_2) S^{-1}(1'_1) \\ &= f(1_1 \cdot m) S(1'_1) \otimes 1'_2 S^{-1}(1_2).\end{aligned}$$

Thus ev_M is H-linear and H-colinear. And it is easy to get that $ev_M \circ (\alpha_{M^*} \otimes \alpha_M) = \alpha_H \circ ev_M$, hence $ev_M \in {}_H\mathcal{WYD}^H$.

Next we have

$$\begin{aligned}coev_M(h \cdot x)(m) &= \sum \varepsilon_t(hx) \cdot (e_i \otimes \alpha_{M^*}(e^i))(m) \\ &= \varepsilon_t(hx) \cdot \alpha_M^{-2}(m)),\end{aligned}$$

and

$$\begin{aligned}h \cdot coev_M(x)(m) &= (\alpha_H^{-1}(h_1)x_1) \cdot \alpha_M(e_i) \otimes_t ((\alpha_H^{-1}(h_2)x_2) \cdot \alpha_{M^*}^2(e^i))(m) \\ &= (\alpha_H^{-1}(h_1)x_1) \cdot (S(\alpha_H^{-2}(h_2)\alpha_H^{-1}(x_2)) \cdot \alpha_M^{-3}(m)) \\ &= \varepsilon_t(\alpha_H^{-2}(h)\alpha_H^{-1}(x)) \cdot \alpha_M^{-2}(m) = \varepsilon_t(hx) \cdot \alpha_M^{-2}(m),\end{aligned}$$

hence $coev_V$ is H-linear, it is not hard to check that $coev_V$ is H-colinear and is left to the reader. Obviously that $coev_M \circ \alpha_H = (\alpha_M \otimes \alpha_{M^*}) \circ coev_M$, thus $coev_M \in {}_H\mathcal{WYD}^H$.

Secondly, we consider

$$\begin{aligned}& (r_M \circ (id_M \otimes ev_M) \circ a_{M,M^*,M} \circ (coev_M \otimes id_M) \circ l_M^{-1})(m) \\ = {} & (r_M \circ (id_M \otimes ev_M) \circ a_{M,M^*,M})((1_1 \cdot e_i \otimes_t 1_2 \cdot \alpha_{M^*}(e^i)) \otimes_t \alpha_M(m)) \\ = {} & r_M(e_i \otimes_t 1_2 \cdot \alpha_{M^*}^2(e^i)(1_1 \cdot \alpha_M^2(m))) \\ = {} & \widehat{\varepsilon_s}(1_2) \cdot (1_1 \cdot \alpha_M^2(m)) = m,\end{aligned}$$

and

$$\begin{aligned}& (l_{M^*} \circ (ev_M \otimes id_{M^*}) \circ a_{M^*,M,M^*}^{-1} \circ (id_{M^*} \otimes coev_M) \circ r_{M^*}^{-1})(f)(m) \\ = {} & (l_{M^*} \circ (ev_M \otimes id_{M^*}) \circ a_{M^*,M,M^*}^{-1})(\alpha_{M^*}(f) \otimes_t (\alpha_M(e_i) \otimes_t \alpha_{M^*}^2(e^i)))(m) \\ = {} & l_{V^*}(\alpha_{M^*}^2(f)(1_1 \cdot \alpha_M(e_i))1_2 \otimes_t \alpha_{M^*}(e^i))(m) \\ = {} & f(1_1 \cdot (S(1_2) \cdot \alpha_M^{-2}(m))) = f(m).\end{aligned}$$

Thus ${}_H\mathcal{WYD}^H_{f.d.}$ is a left rigid category.

Similarly, we define the following maps

$$\widetilde{coev}_M : H_t \to {}^*M \otimes_t M, \quad x \mapsto x \cdot (\sum \alpha_{{}^*M}(e^i) \otimes_t e_i),$$

and

$$\widetilde{ev}_M : M \otimes_t {}^*M \to H_t, \quad m \otimes_t f \mapsto f(S^{-1}(1_1) \cdot m)1_2.$$

We can show that $({}^*M, \widetilde{ev}_M, \widetilde{coev}_M)$ is a right dual of M. Thus ${}_H\mathcal{WYD}^H_{f.d.}$ is a right rigid category.

4. A Braided Monoidal Category ${}_H\mathcal{WYD}^H$ I

Proposition 4.1. Let (H,α_H) be a weak Hom-Hopf algebra. For any $(M,\alpha_M),(N,\alpha_N)\in {}_H\mathcal{WYD}^H$, then $M\otimes_t N = 1_1M\otimes 1_2N\in {}_H\mathcal{WYD}^H$ with structures:

$$h\cdot(m\otimes_t n) = h_1\cdot m\otimes_t h_2\cdot n,$$
$$m\otimes_t n\mapsto (m\otimes_t n)_{(0)}\otimes_t (m\otimes_t n)_{(1)} = (m_{(0)}\otimes_t n_{(0)})\otimes\alpha_H^{-2}(n_{(1)}m_{(1)}).$$

Proof. Obviously $M\otimes_t N$ is a left H-module and a right H-comodule. We check now the compatibility condition. We compute:

$$\begin{aligned}
&(h\cdot(m\otimes n))_{(0)}\otimes(h\cdot(m\otimes n))_{(1)}\\
=\;&(h_1\cdot m\otimes h_2\cdot n)_{(0)}\otimes(h_1\cdot m\otimes h_2\cdot n)_{(1)}\\
=\;&((h_1\cdot m)_{(0)}\otimes(h_2\cdot n)_{(0)}\otimes\alpha_H^{-2}((h_2\cdot n)_{(1)}(h_1\cdot m)_{(1)})\\
=\;&(\alpha_H^{-1}(h_{121})\cdot m_{(0)}\otimes\alpha_H^{-1}(h_{221})\cdot n_{(0)})\otimes\alpha_H^{-2}(((\alpha_H^{-2}(h_{222})\alpha_H^{-1}(n_1))S^{-1}(h_{21}))\\
&((\alpha_H^{-2}(h_{122})\alpha_H^{-1}(m_{(1)}))S^{-1}(h_{11})))\\
=\;&(h_{12}\cdot m_{(0)}\otimes\alpha_H^{-1}(h_{212})\cdot n_{(0)})\otimes\alpha_H^{-2}((h_{22}n_1)\\
&((S^{-1}(\alpha_H^{-3}(h_{2112}))\alpha_H^{-3}(h_{2111}))(\alpha_H^{-1}(m_{(1)})S^{-1}(\alpha_H^{-1}(h_{11}))))\\
=\;&(h_{12}\cdot m_{(0)}\otimes\alpha_H^{-1}(h_{212})\cdot n_{(0)})\otimes\alpha_H^{-2}((h_{22}n_1)\\
&(1_2\varepsilon(\alpha_H^{-3}(h_{211})1_1)(\alpha_H^{-1}(m_{(1)})S^{-1}(\alpha_H^{-1}(h_{11}))))\\
=\;&(h_{12}\cdot m_{(0)}\otimes\alpha_H^{-1}(h_{221})\cdot n_{(0)})\otimes\alpha_H^{-2}((h_{222}n_1)\\
&(1_2\varepsilon(\alpha_H^{-2}(h_{21})1_1)(\alpha_H^{-1}(m_{(1)})S^{-1}(\alpha_H^{-1}(h_{11}))))
\end{aligned}$$

$$\begin{aligned}
=\;&(\alpha_H^{-1}(h_{211})\cdot m_{(0)}\otimes\alpha_H^{-1}(h_{221})\cdot n_{(0)})\otimes\alpha_H^{-2}((h_{222}n_1)\\
&(1_2\varepsilon(\alpha_H^{-2}(h_{212})1_1)(\alpha_H^{-1}(m_{(1)})S^{-1}(h_1)))\\
=\;&(\alpha_H^{-2}(h_{211})1'_1\cdot m_{(0)}\otimes\alpha_H^{-1}(h_{221})\cdot n_{(0)})\otimes\alpha_H^{-2}((h_{222}n_1)\\
&(1_2\varepsilon(\alpha_H^{-2}(h_{212})1'_21_1)(\alpha_H^{-1}(m_{(1)})S^{-1}(h_1)))\\
=\;&(\alpha_H^{-2}(h_{211})1'_1\cdot m_{(0)}\otimes\alpha_H^{-1}(h_{221})\cdot n_{(0)})\otimes\alpha_H^{-2}((h_{222}n_1)\\
&(1_2\varepsilon(\alpha_H^{-2}(h_{212})1'_21_1)(\alpha_H^{-1}(m_{(1)})S^{-1}(h_1)))\\
=\;&(\alpha_H^{-2}(h_{211})1'_21_1\cdot m_{(0)}\otimes\alpha_H^{-1}(h_{212})\cdot n_{(0)})\otimes\alpha_H^{-2}((\alpha_H(h_{22})n_1)\\
&(1_2(\alpha_H^{-1}(m_{(1)})S^{-1}(\alpha^{-1}(h_1)1'_1)))\\
=\;&\alpha_H^{-1}(h_{21})\cdot(m\otimes n)_{(0)}\otimes\alpha_H^{-2}(h_{22})\alpha_H^{-1}(m\otimes n)_{(1)}S^{-1}(h_1).
\end{aligned}$$

Hence $M\otimes_t N\in {}_H\mathcal{WYD}^H$.

Proposition 4.2. Let $(M,\alpha_M),(N,\alpha_N)\in {}_H\mathcal{WYD}^H$. Define the map

$$c_{M,N}: M\otimes_t N\to M\otimes_t M,\;\; c_{M,N}(m\otimes n) = \alpha_N^{-1}(n_{(0)})\otimes\alpha_M^{-1}(\alpha_H^{-1}(n_{(1)})\cdot m).$$

Then $c_{M,N}$ is H-linear H-colinear and satisfies the conditions (for $(P,\alpha_P)\in {}_H\mathcal{WYD}^H$)

$$a^{-1}_{P,M,N}\circ c_{M\otimes N,P}\circ a^{-1}_{M,N,P} = (c_{M,P}\otimes id_N)\circ a^{-1}_{M,P,N}\circ(id_M\otimes c_{N,P}), \tag{4. 1}$$

$$a_{N,P,M}\circ c_{M,N\otimes P}\circ a_{M,N,P} = (id_N\otimes c_{M,P})\circ a_{N,M,P}\circ(c_{M,N}\otimes id_P). \tag{4. 2}$$

Proof. First, we prove that $c_{M,N}$ is H-linear, we compute:

$$\begin{aligned}
& c_{M,N}(h\cdot(m\otimes n))\\
= & c_{M,N}(h_1\cdot m\otimes h_2\cdot n)\\
= & \alpha_N^{-1}((h_2\cdot n)_{(0)})\otimes\alpha_M^{-1}(\alpha_H^{-1}((h_2\cdot n)_{(1)}))\cdot(h_1\cdot m))\\
= & \alpha_N^{-1}(\alpha_H^{-1}(h_{221})\cdot n_{(0)})\otimes\alpha_M^{-1}(\alpha_H^{-1}([\alpha_H^{-2}(h_{222})\alpha_H^{-1}(n_{(1)})]S^{-1}(h_{21}))\cdot(h_1\cdot m))\\
= & \alpha_N^{-1}(h_{21}\cdot n_{(0)})\otimes\alpha_M^{-1}(\alpha_H^{-1}([\alpha_H^{-1}(h_{22})\alpha_H^{-1}(n_{(1)})]S^{-1}(h_{12}))\cdot(\alpha_H^{-1}(h_{11})\cdot m))\\
= & \alpha_N^{-1}(h_{21}\cdot n_{(0)})\otimes\alpha_M^{-1}([\alpha_H^{-2}(h_{22})\alpha_H^{-2}(n_{(1)})]S^{-1}(\alpha_H^{-1}(h_{12}))\cdot(\alpha_H^{-1}(h_{11})\cdot m))\\
= & \alpha_N^{-1}(h_{21}\cdot n_{(0)})\otimes\alpha_M^{-1}([\alpha_H^{-2}(h_{22})\alpha_H^{-2}(n_{(1)})][S^{-1}(\alpha_H^{-2}(h_{12}))\alpha_H^{-2}(h_{11})]\cdot\alpha_M(m))\\
= & \alpha_N^{-1}(h_{21}\cdot n_{(0)})\otimes\alpha_M^{-1}([\alpha_H^{-2}(h_{22})\alpha_H^{-2}(n_{(1)})][S^{-1}(1_1)\varepsilon(\alpha_H^{-2}(h_1 1_2))]\cdot\alpha_M(m))\\
= & \alpha_N^{-1}(\alpha_H^{-1}(h_{12})1_3\cdot n_{(0)})\otimes\alpha_M^{-1}([\alpha_H^{-2}(h_2)1_4\alpha_H^{-2}(n_{(1)})][S^{-1}(1_1)\varepsilon(\alpha_H^{-2}(h_{11}1_2))]\cdot\alpha_M(m))\\
= & \alpha_N^{-1}(h_1 1_2\cdot n_{(0)})\otimes\alpha_M^{-1}([\alpha_H^{-2}(h_2)1_3\alpha_H^{-2}(n_{(1)})]S^{-1}(1_1)\cdot\alpha_M(m))\\
= & h_1\cdot(1_2\cdot\alpha_N^{-2}(n_{(0)}))\otimes\alpha_H^{-1}(h_2)[1_3\alpha_H^{-4}(n_{(1)})S^{-1}(1_1)]\cdot m\\
= & h_1\cdot\alpha_N^{-1}(n_{(0)})\otimes\alpha_M^{-1}(h_2)\alpha_H^{-2}(n_{(1)})\cdot m\\
= & h\cdot c_{M,N}(m\otimes n).
\end{aligned}$$

Next we prove that $c_{M,N}$ is H-colinear.

$$\begin{aligned}
& \rho_{N\otimes M}c_{M,N}(m\otimes n)\\
= & \rho_{N\otimes M}(\alpha_N^{-1}(n_{(0)})\otimes\alpha_M^{-1}(\alpha_H^{-1}(n_{(1)})\cdot m))\\
= & \alpha_N^{-1}(n_{(0)(0)})\otimes(\alpha_H^{-2}(n_{(1)})\cdot\alpha_M^{-1}(m))_{(0)}\otimes\alpha_H^{-2}((\alpha_H^{-2}(n_{(1)})\cdot\alpha_M^{-1}(m))_{(1)}\alpha_N^{-1}(n_{(0)(1)}))\\
= & \alpha_N^{-1}(n_{(0)(0)})\otimes\alpha_H^{-3}(n_{(1)21})\cdot\alpha_M^{-1}(m_{(0)})\otimes\alpha_H^{-2}((\alpha_H^{-4}(n_{(1)22})\alpha_H^{-2}(m_{(1)})\\
& S^{-1}(n_{(1)1}))\alpha_N^{-1}(n_{(0)(1)}))\\
= & \alpha_N^{-1}(n_{(0)(0)})\otimes\alpha_H^{-2}(n_{(1)1})\cdot\alpha_M^{-1}(m_{(0)})\otimes\alpha_H^{-2}((\alpha_H^{-2}(n_{(1)2})\alpha_H^{-1}(m_{(1)}))\\
& [S^{-1}(\alpha_H^{-3}(n_{(0)(1)2}))\alpha_N^{-3}(n_{(0)(1)(1)})])\\
= & n_{(0)}\otimes\alpha_H^{-3}(n_{(1)1})1_2\cdot\alpha_M^{-1}(m_{(0)})\otimes\alpha_H^{-2}((\alpha_H^{-3}(n_{(1)2})1_3\alpha_H^{-1}(m_{(1)}))S^{-1}(1_1))\\
= & \alpha_N^{-1}(n_{(0)(0)})\otimes\alpha_H^{-3}(n_{(0)(1)})1_2\cdot\alpha_M^{-1}(m_{(0)})\otimes\alpha_H^{-2}((\alpha_H^{-2}(n_{(1)})1_3\alpha_H^{-1}(m_{(1)}))S^{-1}(1_1))\\
= & \alpha_N^{-1}(n_{(0)(0)})\otimes\alpha_H^{-2}(n_{(0)(1)})(1_2\cdot\alpha_M^{-2}(m_{(0)}))\otimes\alpha_H^{-2}(n_{(1)}[1_3\alpha_H^{-2}(m_{(1)})]S^{-1}(1_1))\\
= & \alpha_N^{-1}(n_{(0)(0)})\otimes\alpha_H^{-2}(n_{(0)(1)})\cdot\alpha_M^{-1}(m_{(0)})\otimes\alpha_H^{-2}(n_{(1)}m_{(1)})\\
= & (c_{M,N}\otimes id_H)\rho_{M\otimes N}(m\otimes n).
\end{aligned}$$

As for (4.2), for any $m\in M, n\in N$ and $p\in P$, we have

$$\begin{aligned}
& (a_{N,P,M}\circ c_{M,N\otimes P}\circ a_{M,N,P})((m\otimes_t n)\otimes_t p)\\
= & a_{N,P,M}(\alpha_N^{-1}(n_{(0)})\otimes p_{(0)}\otimes\alpha_H^{-4}(\alpha_H(p_{(1)})n_{(1)})\cdot\alpha_M^{-2}(m))\\
= & \alpha_N^{-2}(n_{(0)})\otimes(p_{(0)}\otimes\alpha_H^{-3}(\alpha_H(p_{(1)})n_{(1)})\cdot\alpha_M^{-1}(m))\\
= & (id_N\otimes_t(c_{M,P}))(\alpha_N^{-2}(n_{(0)})\otimes(\alpha_H^{-2}(n_{(1)})\cdot\alpha_M^{-1}(m)\otimes\alpha_P(p))\\
= & ((id_N\otimes_t c_{M,P})\circ a_{N,M,P}\circ(c_{M,N}\otimes_t id_P))((m\otimes_t n)\otimes_t p),
\end{aligned}$$

we can check that (4.1) in the similar way.

Lemma 4.3. $c_{M,N}$ is bijective with inverse

$$c_{M,N}^{-1}(n\otimes m)=\alpha_M^{-1}(\alpha_H^{-1}(S(n_{(1)}))\cdot m)\otimes\alpha_N^{-1}(n_{(0)}).$$

Proof. First, we prove that $c_{M,N}\circ c_{M,N}^{-1}=id$. For any $m\in M$ and $n\in N$, we have

$$
\begin{aligned}
& c_{M,N}c_{M,N}^{-1}(n\otimes m)\\
= \ & c_{M,N}(\alpha_M^{-1}(\alpha_H^{-1}(S(n_{(1)}))\cdot m)\otimes\alpha_N^{-1}(n_{(0)}))\\
= \ & c_{M,N}(\alpha_H^{-2}(S(n_{(1)}))\cdot\alpha_M^{-1}(m)\otimes\alpha_N^{-1}(n_{(0)}))\\
= \ & \alpha_N^{-2}(n_{(0)(0)})\otimes\alpha_N^{-3}(n_{(0)(1)})\cdot[\alpha_H^{-3}(S(n_{(1)}))\cdot\alpha_M^{-2}(m)]\\
= \ & \alpha_N^{-2}(n_{(0)(0)})\otimes[\alpha_N^{-4}(n_{(0)(1)})\alpha_H^{-3}(S(n_{(1)}))]\cdot\alpha_M^{-1}(m)\\
= \ & \alpha_N^{-1}(n_{(0)})\otimes[\alpha_N^{-4}(n_{(1)1})\alpha_H^{-4}(S(n_{(1)2}))]\cdot\alpha_M^{-1}(m)\\
= \ & \alpha_N^{-1}(n_{(0)})\otimes 1_2\varepsilon(1_1n_{(1)})\cdot\alpha_M^{-1}(m)\\
= \ & 1'_1\alpha_N^{-2}(n_{(0)})\otimes 1_2\varepsilon(1_11'_2n_{(1)})\cdot\alpha_M^{-1}(m)\\
= \ & 1_1\alpha_N^{-1}(n)\otimes 1_2\cdot\alpha_M^{-1}(m)\\
= \ & n\otimes m.
\end{aligned}
$$

Then, we note that the following relation holds, for all $m\in M$,

$$
\begin{aligned}
& m_{(0)}\otimes m_{(1)}\\
= \ & \alpha_H^{-1}(1_{21})\cdot\alpha_M^{-1}(m_{(0)})\otimes[\alpha_H^{-2}(1_{22})\alpha_H^{-2}(m_{(1)})]S^{-1}(1_1)\\
= \ & 1'_1\cdot(1_2\cdot\alpha_M^{-1}(m_{(0)}))\otimes[\alpha_H^{-2}(1'_2)\alpha_H^{-2}(m_{(1)})]S^{-1}(1_1)\\
= \ & 1'_1\cdot(1_2\cdot\alpha_M^{-1}(m_{(0)}))\otimes\alpha_H^{-1}(1'_2)[\alpha_H^{-2}(m_{(1)})S^{-1}(\alpha_H^{-1}(1_1))]\\
= \ & 1'_1\cdot(1_2\cdot\alpha_M^{-1}(m_{(0)}))\otimes 1'_2[\alpha_H^{-2}(m_{(1)})S^{-1}(1_1)]\\
= \ & 1_2\cdot\alpha_M^{-1}(m_{(0)})\otimes\alpha_H^{-2}(m_{(1)})S^{-1}(1_1).
\end{aligned}
$$

Finally, we check that $c_{M,N}^{-1}\circ c_{M,N}=id$. For any $m\in M$ and $n\in N$, we have

$$
\begin{aligned}
& c_{M,N}^{-1}c_{M,N}(m\otimes n)\\
= \ & c_{M,N}^{-1}(\alpha_N^{-1}(n_{(0)})\otimes\alpha_M^{-1}(\alpha_H^{-1}(n_{(1)})\cdot m))\\
= \ & \alpha_H^{-3}(S(n_{(0)(1)}))\cdot[\alpha_H^{-3}(n_{(1)})\cdot\alpha_M^{-2}(m)]\otimes\alpha_N^{-2}(n_{(0)(0)})\\
= \ & [\alpha_H^{-4}(S(n_{(0)(1)}))\alpha_H^{-3}(n_{(1)})]\cdot\alpha_M^{-1}(m)\otimes\alpha_N^{-2}(n_{(0)(0)})\\
= \ & [\alpha_H^{-4}(S(n_{(1)1}))\alpha_H^{-4}(n_{(1)2})]\cdot\alpha_M^{-1}(m)\otimes\alpha_N^{-1}(n_{(0)})\\
= \ & \varepsilon_s(n_{(1)})\cdot\alpha_M^{-1}(m)\otimes\alpha_N^{-1}(n_{(0)})\\
= \ & 1_1S((S^{-1}\varepsilon_s(n_{(1)})))\cdot\alpha_M^{-1}(m)\otimes 1_2\cdot\alpha_N^{-1}(n_{(0)})\\
= \ & S(1_2)S((S^{-1}\varepsilon_s(n_{(1)})))\cdot\alpha_M^{-1}(m)\otimes S(1_1)\cdot\alpha_N^{-1}(n_{(0)})\\
= \ & S((S^{-1}\varepsilon_s(n_{(1)}))1_2)\cdot\alpha_M^{-1}(m)\otimes S(1_1)\cdot\alpha_N^{-1}(n_{(0)})\\
= \ & S((S^{-1}\varepsilon_s(n_{(1)}))S(1_1))\cdot\alpha_M^{-1}(m)\otimes S^2(1_2)\cdot\alpha_N^{-1}(n_{(0)})\\
= \ & S^2(1_1)\cdot\alpha_M^{-1}(m)\otimes S^2(1_2S^{-1}\varepsilon_s(n_{(1)}))\cdot\alpha_N^{-1}(n_{(0)})\\
= \ & S^2(1_1)\cdot\alpha_M^{-1}(m)\otimes S^2(1_2)\cdot\alpha_N^{-1}(n)\\
= \ & m\otimes n.
\end{aligned}
$$

Theorem 4.4. ${}_H\mathcal{WYD}^H$ is a braided monoidal category.

We can make now the connection between Yetter-Drinfeld modules over weak Hom-Hopf algebras and modules over quasitriangular weak Hom-Hopf algebras.

Definition 4.5.[16] Let (H,α) be a weak Hom-bialgebra. If there exists $R=R^{(1)}\otimes R^{(2)}\in\Delta^{op}(1)(H\otimes_{\Bbbk}H)\Delta(1)$, such that the following conditions hold:

$$\begin{cases} (1)\ (\alpha\otimes\alpha)R = R; \\ (2)\ R\Delta(h) = \Delta^{op}(h)R; \\ (3)\ \text{there exists } \overline{R}\in \Delta(1)(H\otimes_{\Bbbk} H)\Delta^{op}(1),\ \text{such that } R\overline{R} = \Delta^{op}(1),\ \overline{R}R = \Delta(1); \\ (4)\ \alpha(R^{(1)})\otimes R^{(2)}_1\otimes R^{(2)}_2 = \alpha^{-1}(r^{(1)}R^{(1)})\otimes R^{(2)}\otimes r^{(2)}; \\ (5)\ R^{(1)}_1\otimes R^{(1)}_2\otimes \alpha(R^{(2)}) = r^{(1)}\otimes R^{(1)}\otimes \alpha^{-1}(r^{(2)}R^{(2)}), \end{cases}$$

where $h\in H$, $r = R = R^{(1)}\otimes R^{(2)} = r^{(1)}\otimes r^{(2)}$, then R is called an *R-matrix* of H, $\overline{R}$ is called the *weak inverse of* R. (H,R) is called a *quasitriangular weak Hom-bialgebra*.

Proposition 4.6. Let (H,α_H,R) be a quasitriangular weak Hom-Hopf algebra, then we have

(i) Let (M,α_M) be a left H-module with action $H\otimes M\to M, h\otimes_t m\mapsto h\cdot m$. Define the linear map $\rho_M : M\to M\otimes_t H, \rho_M(m) = m_{(0)}\otimes_t m_{(1)} := R^{(2)}\cdot m\otimes_t \alpha_H(R^{(1)})$. Then (M,α_M) with these structures is a Yetter-Drinfeld module over H.

(ii) Let (N,α_N) be another left H-module with action $H\otimes_t N\to N, h\otimes_t n\mapsto h\cdot n$, regarded as a Yetter-Drinfeld module as in (i), via the map $\rho_N : N\to N\otimes H, \rho_N(n) = n_{(0)}\otimes n_{(1)} := r^{(2)}\cdot m\otimes_t \alpha_H(r^{(1)})$. We regard $(M\otimes_t N,\alpha_M\otimes\alpha_N)$ as a left H-module via the standard action $h\cdot(m\otimes n) = h_1\cdot m\otimes_t h_2\cdot n$ and then we regard $(M\otimes_t N,\alpha_M\otimes\alpha_N)$ as a Yetter-Drinfeld module as in (i). Then this Yetter-Drinfeld module $(M\otimes_t N,\alpha_M\otimes\alpha_N)$ coincides with the Yetter-Drinfeld module $M\otimes_t N$ defined as in Proposition 4.1.

Proof. First we have to prove that (M,α_M) is a right H-comodule; $\rho(\alpha(m)) = \alpha_M(m_{(0)})\otimes\alpha(m_{(1)})$ is easy and left to the reader, we check

$$\begin{aligned} (\alpha_M\otimes\Delta)\rho_M(m) &= \alpha_M(R^{(2)}\cdot m)\otimes\Delta(\alpha_H(R^{(1)})) \\ &= \alpha_H(R^{(2)})\cdot\alpha_M(m)\otimes\alpha_H(R^{(1)}_1)\otimes\alpha_H(R^{(1)}_2) \\ &= r^{(2)}R^{(2)}\cdot\alpha_M(m)\otimes\alpha^2_H(r^{(1)})\otimes\alpha^2_H(R^{(1)}) \\ &= \alpha_H(r^{(2)})\cdot(R^{(2)}\cdot m)\otimes\alpha^2_H(r^{(1)})\otimes\alpha^2_H(R^{(1)}) \\ &= r^{(2)}\cdot(R^{(2)}\cdot m)\otimes\alpha_H(r^{(1)})\otimes\alpha^2_H(R^{(1)}) \\ &= \rho_M(R^{(2)}\cdot m)\otimes\alpha^2_H(R^{(1)}) \\ &= (\rho_M\otimes\alpha_M)\rho_M(m). \end{aligned}$$

Now we check the Yetter-Drinfeld condition (3.3):

$$\begin{aligned} (h_2\cdot m)_{(0)}\otimes(h_2\cdot m)_{(1)}\alpha^2_H(h_1) &= R^{(2)}\cdot(h_2\cdot m)\otimes\alpha_H(R^{(1)})\alpha^2_H(h_1) \\ &= \alpha_H(R^{(2)})\cdot(h_2\cdot m)\otimes\alpha^2_H(R^{(1)})\alpha^2_H(h_1) \\ &= (R^{(2)}h_2)\cdot\alpha_M(m)\otimes\alpha^2_H(R^{(1)}h_1) \\ &= (h_1R^{(2)})\cdot\alpha_M(m)\otimes\alpha^2_H(h_2R^{(1)}) \\ &= \alpha_H(h_1)\cdot(R^{(2)}\cdot m)\otimes\alpha^2_H(h_2)\alpha^2_H(R^{(1)}) \\ &= \alpha_H(h_1)\cdot m_{(0)}\otimes\alpha^2_H(h_2)\alpha_H(m_{(1)}). \end{aligned}$$

(ii) We only need to prove that the two comodule structures on $M\otimes_t N$ coincide, that is, for all $m\in M$ and $n\in N$,

$$m_{(0)}\otimes n_{(0)}\otimes\alpha^{-2}_H(n_{(1)}m_{(1)}) = R^{(2)}\cdot(m\otimes n)\otimes\alpha_H(R^{(1)}),$$

that is

$$R^{(2)}\cdot m\otimes r^{(2)}\cdot n\otimes\alpha^{-2}_H(\alpha_H(r^{(2)})\alpha_H(R^{(2)})) = R^{(2)}_1\cdot m\otimes R^{(2)}_2\cdot n\otimes\alpha_H(R^{(1)}),$$

which is equivalent to

$$\alpha_H(R^{(2)})\cdot m\otimes\alpha_H(r^{(2)})\cdot n\otimes r^{(2)}R^{(2)} = R^{(2)}_1\cdot m\otimes R^{(2)}_2\cdot n\otimes\alpha_H(R^{(1)}).$$

Proposition 4.7. Let (H,α_H,R) be a quasitriangular weak Hom-Hopf algebra. Denote by $Rep(H)$ the category whose objects are left H-modules and whose morphisms are H-linear maps. Then $Rep(H)$ is a braided monoidal subcategory of ${}_H\mathcal{WYD}^H$, with tensor product defined as in Proposition 4.1, associativity constraints defined by the formula $a_{M,N,P}((m\tilde{\otimes}n)\tilde{\otimes}p) = \alpha_M^{-1}(m)\tilde{\otimes}(n\tilde{\otimes}\alpha_P(p))$ for any $M,N,P \in Rep(H)$, and braiding $c_{M,N} : M\otimes_t M \to N\otimes_t M,\ \ m\otimes_t n \mapsto R^{(2)}\cdot\alpha_N^{-1}(n)\otimes_t R^{(1)}\cdot\alpha_M^{-1}(m)$, with inverse $c_{M,N}^{-1} : N\otimes_t M \to M\otimes_t M,\ \ n\otimes_t m \mapsto \overline{R}^{(1)}\cdot\alpha_M^{-1}(m)\otimes_t \overline{R}^{(2)}\cdot\alpha_N^{-1}(n)$, for any $(M,\alpha_M),(N,\alpha_N)\in Rep(H)$.

5. A Braided Monoidal Category ${}_H\mathcal{WYD}^H$ II

Modules over quasitriangular weak Hom-Hopf algebras become Yetter-Drinfeld modules over weak Hom-Hopf algebras are proved in Section 4. Similarly, comodules over coquasitriangular weak Hom-Hopf algebras become Yetter-Drinfeld modules over weak Hom-Hopf algebras; inspired by this, we can introduce a second braided monoidal category structure on ${}_H\mathcal{WYD}^H$. We include these facts here for completeness. Each of the next results is the analogue of a result in Section 4; their proofs are similar to those of their analogues and are left to the reader.

Proposition 5.1. Let (H,α_H) be a weak Hom-Hopf algebra.

(i) Let $(M,\alpha_M),(N,\alpha_N)\in {}_H\mathcal{WYD}^H$, with notation as above, and the tensor product $M\tilde{\otimes}N$ is obtained by

$$M\tilde{\otimes}N = \{m\tilde{\otimes}n = m_0\otimes_{\Bbbk} n_0\varepsilon(m_1n_1) \mid m\in M, n\in N\},$$

with structures:

$$h\cdot(m\tilde{\otimes}n) = \alpha_H^{-2}(h_1)\cdot m\tilde{\otimes}\alpha_H^{-2}(h_2)\cdot n,$$
$$m\tilde{\otimes}n \mapsto (m\tilde{\otimes}n)_{(0)}\otimes(m\tilde{\otimes}n)_{(1)} = (m_{(0)}\tilde{\otimes}n_{(0)})\tilde{\otimes}n_{(1)}m_{(1)}.$$

(ii) ${}_H\mathcal{WYD}^H$ is a braided monoidal category, with tensor product $\tilde{\otimes}$ as in (i) and associativity constraints $a_{M,N,P}$ and quasi-braiding $c_{M,N}$ defined as follows: for any (M,α_M), $(N,\alpha_N),(P,\alpha_P)\in {}_H\mathcal{WYD}^H$, define an associativity constraint by

$$a_{M,N,P}((m\tilde{\otimes}n)\tilde{\otimes}p) = \alpha_M^{-1}(m)\tilde{\otimes}(n\tilde{\otimes}\alpha_P(p)),\ m\in M,\ n\in N,\ p\in P,$$

$$c_{M,N} : M\tilde{\otimes}N \to M\tilde{\otimes}M,\ \ c_{M,N}(m\tilde{\otimes}n) = \alpha_N^{-1}(n_{(0)})\tilde{\otimes}\alpha_M^{-1}(\alpha_H^{-1}(n_{(1)})\cdot m).$$

with inverse

$$c_{M,N}^{-1}(n\tilde{\otimes}m) = \alpha_M^{-1}(\alpha_H^{-1}(S(n_{(1)}))\cdot m)\tilde{\otimes}\alpha_N^{-1}(n_{(0)}).$$

Definition 5.2.[16] Let (H,α) be a weak Hom-bialgebra, if there is a linear map $\sigma : H\otimes_{\Bbbk} H\to\Bbbk$, such that the following conditions hold:

$$\begin{cases}
(1)\ \ \sigma(a,b) = \varepsilon(b_1a_1)\sigma(a_2,b_2)\varepsilon(a_3b_3);\\
(2)\ \ \sigma(a_1,b_1)a_2b_2 = b_1a_1\sigma(a_2,b_2);\\
(3)\ \ \text{there exists } \sigma'\in hom_{\Bbbk}(H\otimes H,\Bbbk),\ \text{such that } \sigma(a_1,b_1)\sigma'(a_2,b_2) = \varepsilon(ab),\\
\quad\ \ \text{and } \sigma'(a_1,b_1)\sigma(a_2,b_2) = \varepsilon(ba);\\
(4)\ \ \sigma(\alpha(a),\alpha(b)) = \sigma(a,b);\\
(5)\ \ \sigma(\alpha(a),bc) = \sigma(a_1,\alpha(c))\sigma(a_2,\alpha(b));\\
(6)\ \ \sigma(ab,\alpha(c)) = \sigma(\alpha(a),c_1)\sigma(\alpha(b),c_2),
\end{cases}$$

where $a,b,c\in H$, then σ is called an *coquasitriangular form* of H, σ' is called the *weak convolution inverse of* σ. (H,σ) is called a coquasitriangular weak Hom-bialgebra.

Proposition 5.3. Let (H,α_H,σ) be a coquasitriangular weak Hom-Hopf algebra, then we have

(i) Let (M,α_M) be a left H-comodule with coaction $M\to M\tilde{\otimes}H, \rho_M(m) = m_{(0)}\tilde{\otimes}m_{(1)}$. Define the linear map $H\otimes M\to M, h\tilde{\otimes}m\mapsto h\cdot m := \sigma(\alpha_H(h),m_{(1)})m_{(0)}$. Then (M,α_M) with these structures is a Yetter-Drinfeld module over H.

(ii) Let (N,α_N) be another left H-comodule with coaction $N\to N\tilde{\otimes}H, \rho_N(n) = n_{(0)}\tilde{\otimes}n_{(1)}$. Define the linear map $H\otimes N\to N, h\tilde{\otimes}n\mapsto h\cdot n := \sigma(\alpha_H(h),n_{(1)})n_{(0)}$. We regard $(M\tilde{\otimes}N,\alpha_M\otimes\alpha_N)$ as a left H-module via the standard action $h\cdot(m\tilde{\otimes}n)=\alpha_H^{-2}(h_1)\cdot m\tilde{\otimes}\alpha_H^{-2}(h_2)\cdot n$ and then we regard $(M\tilde{\otimes}N,\alpha_M\otimes\alpha_N)$ as a Yetter-Drinfeld module as in (i). Then this Yetter-Drinfeld module $(M\tilde{\otimes}N,\alpha_M\otimes\alpha_N)$ coincides with the Yetter-Drinfeld module $M\tilde{\otimes}N$ defined as in Proposition 5.1.

Theorem 5.4. Let (H,α_H,σ) be a coquasitriangular weak Hom-Hopf algebra. Denote by $Corep(H)$ the category whose objects are right H-comodules (M,α_M) and morphisms are morphisms of right H-comodules. Then $Corep(H)$ is a braided monoidal subcategory of ${}_H\mathcal{WYD}^H$, with tensor product defined as in Proposition 5.1, associativity constraints defined by the formula $a_{M,N,P}((m\tilde{\otimes}n)\tilde{\otimes}p)=\alpha_M^{-1}(m)\tilde{\otimes}(n\tilde{\otimes}\alpha_P(p))$ for any $M,N,P\in Corep(H)$, and braiding $c_{M,N}:M\tilde{\otimes}N\to N\tilde{\otimes}M,\ \ m\tilde{\otimes}n\mapsto\alpha_M^{-1}(n_0)\tilde{\otimes}\alpha_M^{-1}(m_0)\sigma(m_1,n_1)$, with inverse $c^{-1}1_{M,N}:N\tilde{\otimes}M\to M\tilde{\otimes}N,\ \ n\tilde{\otimes}m\mapsto\alpha_M^{-1}(m_0)\tilde{\otimes}\alpha_N^{-1}(n_0)\sigma'(m_1,n_1)$, for any $(M,\alpha_M),(N,\alpha_N)\in Corep(H)$.

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