Filomat 31:13 (2017), 3975–3997
https://doi.org/10.2298/FIL1713975A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Continued Fraction of Ramanujan and Some Ramanujan-Weber Class Invariants

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Abstract. On Page 36 of his "lost" notebook, Ramanujan recorded four q-series representations of the famous Rogers-Ramanujan continued fraction. In this paper, we establish two q-series representations of Ramanujan's continued fraction found in his "lost" notebook. We also establish three equivalent integral representations and modular equations for a special case of this continued fraction. Furthermore, we derive continued-fraction representations for the Ramanujan-Weber class invariants g_n and G_n and establish formulas connecting g_n and G_n. We obtain relations between our continued fraction with the Ramanujan-Göllnitz-Gordon and Ramanujan's cubic continued fractions. Finally, we find some algebraic numbers and transcendental numbers associated with a certain continued fraction $A(q)$ which is related to Ramanujan's continued fraction $F(a, b, \lambda; q)$, the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$ and the Dedekind eta function $\eta(s)$.

1. Introduction, Definitions and Preliminary Results

Throughout this paper, we use the following notation:

$$(\lambda; q)_\mu := \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right)$$

for arbitrary (real or complex) numbers q, λ and μ (*with* $|q| < 1$), so that

$$(\lambda; q)_n = \begin{cases} 1 & (n = 0) \\ (1-\lambda)(1-\lambda q)\cdots(1-\lambda q^{n-1}) & (n \in \mathbb{N}) \end{cases}$$

2010 *Mathematics Subject Classification*. Primary 11A55, 11F20; Secondary 11R04.
Keywords. q-Continued fractions; q-Series; Modular equations; Ramanujan-Weber class invariants; Algebraic numbers; Transcendental numbers; Dedekind eta function.
Received: 27 February 2016; Accepted: 09 July 2016
Communicated by Dragan S. Djordjević
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and

$$(\lambda;q)_\infty = \lim_{n\to\infty}(\lambda;q)_n = \prod_{j=0}^{\infty}(1-\lambda q^j).$$

As usual, here and in what follows, we have

$$\mathbb{N} = \{1,2,3,\cdots\} \quad \text{and} \quad \mathbb{N}_0 := \mathbb{N}\cup\{0\} = \{0,1,2,\cdots\}.$$

We also denote by $\mathfrak{h}$ the complex upper-half plane:

$$\mathfrak{h} := \{\tau : \tau \in \mathbb{C} \qquad \text{and} \qquad \Im(\tau) > 0\},$$

where $\mathbb{C}$ denotes the set of complex numbers.

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \qquad (|ab|<1). \tag{1}$$

The well-known Jacobi triple-product identity [1, Entry 19] in Ramanujan's notation is given by

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2}$$

We recall here the following three most interesting special cases of (1) [1, Entry 22]:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{4}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{5}$$

Also, following Ramanujan's work, we define

$$\chi(q) := (-q;q^2)_\infty. \tag{6}$$

The Rogers-Ramanujan continued fraction $R(q)$ is defined by

$$R(q) := \frac{q^{1/5}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\cdots \qquad (|q|<1), \tag{7}$$

which first appeared in a paper by Rogers [30] in 1894. This continued fraction has many representations. For example, it can be expressed in terms of infinite products as follows:

$$R(q) = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{8}$$

The identity (8) was proved by Rogers [30] and was also discovered by Ramanujan [28, Vol. II, Chapter 16, Section 15] (see also [1]). Ramanujan [29, p. 50] also gave 2- and 5-dissections of this continued fraction and its reciprocal, which were first proved by Andrews [7] and Hirschhorn [21], respectively. Furthermore, in

his "lost" notebook [29, p. 36], Ramanujan stated four q-series representations for $R(q)$ (see [8, p. 121, Entry 4.5.1]).
In particular, on Page 46 in his "lost" notebook [29], Ramanujan claimed that

$$R(q) = \frac{\sqrt{5}-1}{2} \exp\left(-\frac{1}{5}\int_q^1 \frac{(1-t)^5(1-t^2)^5\cdots}{(1-t^5)(1-t^{10})\cdots}\,\frac{dt}{t}\right) \qquad (0 < q < 1). \tag{9}$$

The identity (9) was proved by Andrews [7]. More recently, Adiga, Kim, Naika and Madhusudhan [5] established two integral representations for Ramanujan's cubic continued fraction.

The following q-series identity is Heine's q-analogue of the Gauss ${}_2F_1$ summation formula [20] (see also [35, p. 348, Eq. 9.4(277)]):

$$\sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}\left(\frac{c}{ab}\right)^n = \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/(ab);q)_\infty} \qquad \left(\left|\frac{c}{ab}\right| < 1\right). \tag{10}$$

For some recent usages of various q-results (especially q-hypergeometric summation theorems), one may be referred (for example) to [33] and [34].

In his "lost" notebook [29], Ramanujan recorded many interesting continued fraction identities including (for example)

$$\frac{G(aq,\lambda q,b;q)}{G(a,\lambda,b;q)} = \frac{1}{1+}\,\frac{aq+\lambda q}{1+}\,\frac{bq+\lambda q^2}{1+}\,\frac{aq^2+\lambda q^3}{1+}\,\frac{bq^2+\lambda q^4}{1+}\,\cdots \tag{11}$$

$$= \frac{1}{1+aq+}\,\frac{\lambda q - abq^2}{1+aq^2+bq+}\,\frac{\lambda q^2 - abq^4}{1+aq^3+bq^2+}\,\cdots, \tag{12}$$

where

$$G(a,\lambda,b;q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-\lambda/a;q)_n\, a^n}{(q;q)_n(-bq;q)_n}. \tag{13}$$

We find it to be convenient here to use the following notations:

$$F_1(a,b,\lambda;q) := \frac{G(aq,\lambda q,b;q)}{G(a,\lambda,b;q)} \tag{14}$$

and

$$F_1(a,b,-b;q) =: F(a,b;q) := \frac{G(aq,-bq,b;q)}{G(a,-b,b;q)}. \tag{15}$$

For proofs of (11) and (12), see the work of Bhargava and Adiga [13].

The identities (11) and (12) have been generalized and proved by many mathematicians including (among others) Bhargava *et al.* [15] and Andrews [6] (see also the related recent works by Adiga *et al.* [3], [12] and [19]).

Here, in our present investigation, we need each of the following lemmas in order to prove our main theorems. The proof of Lemma 1 can be found in [9, p. 13, Entry 1.3.2] *with* b replaced by $-b$. A slightly different proof of Lemma 1 is given below.

Lemma 1. *For any complex numbers a and b,*

$$\sum_{n=0}^{\infty} \frac{(b/a;q)_n\, a^n\, q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n} = \frac{(-aq;q)_\infty}{(-bq;q)_\infty} \qquad (|q| < 1). \tag{16}$$

Proof. Letting $a \to \infty$ in (10) and then replacing c by $-cq$, we find that

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(b;q)_n}{(-cq;q)_n(q;q)_n}\left(\frac{c}{b}\right)^n = \frac{(-cq/b;q)_\infty}{(-cq;q)_\infty}. \tag{17}$$

Putting $b = c/a$ in (17) and then setting $c \mapsto b$ in the resulting identity, we obtain (16). □

Lemma 2. *Each of the following identities holds true*:

$$(aq^{-m};q)_n = \frac{(a;q)_n(qa^{-1};q)_m}{(a^{-1}q^{1-n};q)_m} q^{-mn} \tag{18}$$

and

$$(aq^{1-n};q)_\infty = (-a)^n q^{-n(n-1)/2}(a^{-1};q)_n(aq;q)_\infty. \tag{19}$$

The first assertion (18) of Lemma 2 can be found in [18, p. 6, Eq. (1.2.38)] and the second one (19) can be verified fairly easily.

The purpose of the present paper is to derive some q-series representations of Ramanujan's continued fraction $F(a,b;q)$. In Section 2, we establish two q-series representations of Ramanujan's continued fraction $F(a,b;q)$. We also establish some relations between special cases of $F(a,b;q)$. In Section 3, we study the special case $N(q) := F(-1,1;q)$ of Ramanujan's continued fraction given by (11) and establish three equivalent integral representations of $N(q)$ and some modular equations for the continued fraction $N(q)$. In Section 4, we find continued fraction representations for the Ramanujan-Weber class invariants g_n and G_n and obtain several relations between them. We also derive relations of the continued fraction $N(q)$ with the Ramanujan-Göllnitz-Gordon and Ramanujan's cubic continued fractions. In Section 5, we obtain some algebraic numbers and transcendental numbers involving $A(q) := N(q)+1$, the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$ and the Dedekind eta function $\eta(s)$. The last section of this paper (Section 6) is devoted to several concluding remarks and observations.

2. A Set of Main Results

In this section, we establish two new q-series representations of the Ramanujan's continued fraction $F(a,b;q)$ and we use these representations to establish many relations between some particular cases of $F(a,b;q)$.

Theorem 1. *For any complex numbers a and $b \neq 0$, and for $|q| < 1$,*

$$\begin{aligned} F(a,b;q) = &-\frac{a}{b}\sum_{n=1}^{\infty} \frac{q^n}{\left(\frac{aq}{b};q\right)_n(-aq;q)_n} \\ &+ \frac{(-bq;q)_\infty}{(-aq;q)_\infty\left(\frac{aq}{b};q\right)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}}{(-bq;q)_n(q;q)_n}b^n. \end{aligned} \tag{20}$$

Proof. Making use of (13), we may write

$$\begin{aligned} G(aq,-bq,b;q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(b/a;q)_n\,(aq)^n}{(q;q)_n(-bq;q)_n} \\ &= \frac{1}{\left(1-\frac{b}{aq}\right)}\sum_{n=0}^{\infty}\frac{\left(\frac{b}{aq};q\right)_n\left(1-\frac{b}{a}q^{n-1}\right)q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}(aq)^n \\ &= -\frac{aq}{b\left(1-\frac{aq}{b}\right)}\cdot\frac{(-aq^2;q)_\infty}{(-bq;q)_\infty} + \frac{1}{\left(1-\frac{aq}{b}\right)}\sum_{n=0}^{\infty}\frac{\left(\frac{b}{aq};q\right)_n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}\left(aq^2\right)^n, \end{aligned}$$

where we have employed Lemma 1 in the last step with a replaced by aq. Repeating the above manipulations m times, we find that

$$G(aq,-bq,b;q) = -\frac{a}{b(-bq;q)_\infty}\sum_{k=1}^{m}\frac{\left(-aq^{k+1};q\right)_\infty}{\left(\frac{aq}{b};q\right)_k}q^k + \frac{1}{\left(\frac{aq}{b};q\right)_m}\sum_{n=0}^{\infty}\frac{\left(\frac{b}{aq^m};q\right)_n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}\left(aq^{m+1}\right)^n.$$

Using (18) with a replaced by $\frac{b}{a}$ and after some simplifications, we obtain

$$G(aq,-bq,b;q) = -\frac{a(-aq;q)_\infty}{b(-bq;q)_\infty}\sum_{k=1}^{m}\frac{q^k}{\left(\frac{aq}{b};q\right)_k(-aq;q)_k} + \sum_{n=0}^{\infty}\frac{\left(\frac{b}{a};q\right)_n q^{n(n+1)/2}(aq)^n}{\left(\frac{a}{b}q^{1-n};q\right)_m(q;q)_n(-bq;q)_n}.$$

Letting $m\to\infty$ in the above identity, and then using (19), we deduce

$$G(aq,-bq,b;q) = \frac{(-aq;q)_\infty}{(-bq;q)_\infty}\left[-\frac{a}{b}\sum_{n=1}^{\infty}\frac{q^n}{\left(\frac{aq}{b};q\right)_n(-aq;q)_n} + \frac{(-bq;q)_\infty}{(-aq;q)_\infty\left(\frac{aq}{b};q\right)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}}{(-bq;q)_n(q;q)_n}b^n\right]. \tag{21}$$

Finally, by employing Lemma 1 and (14) in (21), we evidently complete the proof of Theorem 1. □

To prove our next theorem, we need the following lemma, which can be found in Chapter 16 [1, p. 8, Eq. (8.1)] and is called Sears' transformation [31, p. 174, Eq. (10.1)].

Lemma 3. *Let*

$$\left|\frac{de}{abc}\right|,\quad \left|\frac{e}{a}\right| \qquad and \qquad |q|<1.$$

Then

$$\sum_{k=0}^{\infty}\frac{(a;q)_k(b;q)_k(c;q)_k}{(d;q)_k(e;q)_k(q;q)_k}\left(\frac{de}{abc}\right)^k = \frac{(e/a;q)_\infty\,(de/bc;q)_\infty}{(e;q)_\infty\,(de/abc;q)_\infty}\cdot\sum_{k=0}^{\infty}\frac{(a;q)_k(d/b;q)_k(d/c;q)_k}{(d;q)_k(de/bc;q)_k(q;q)_k}\left(\frac{e}{a}\right)^k. \tag{22}$$

Theorem 2. *For any nonzero complex numbers a and b, and for $|q|<1$,*

$$F(a,b;q) = \frac{(-aq;q)_\infty(aq/b;q)_\infty}{(-aq;q)_\infty(aq/b;q)_\infty-1}\left[-\frac{a}{b}\sum_{n=1}^{\infty}\frac{q^n}{\left(\frac{aq}{b};q\right)_n(-aq;q)_n} + \frac{aq(-bq;q)_\infty}{(-aq;q)_\infty\left(\frac{aq}{b};q\right)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}}{(-bq;q)_n(q;q)_n(1+aq^{n+1})}(bq)^n\right]. \tag{23}$$

Proof. Letting a and b tend to ∞ on both sides of (22), and then replacing c, d and e by $-aq$, $-bq$ and $-aq^2$, respectively, we find after some simplifications that

$$G(aq,-bq,b;q)=(-aq;q)_\infty\sum_{k=0}^{\infty}\frac{(-b)^k q^{k(k+1)}}{(-bq;q)_k(q;q)_k(1+aq^{k+1})}. \tag{24}$$

Now, employing (24) and (14) in (21), and after some simplifications, we complete the proof of Theorem 2. □

We need the following interesting result asserted by Lemma 4 below.

Lemma 4. *Each of the following q-identities holds true*:

$$\sum_{n=0}^{\infty}\frac{q^n}{(-aq;q)_n(-bq;q)_n}=\frac{1}{(-aq;q)_\infty(-bq;q)_\infty}\sum_{m=0}^{\infty}(-1)^{m+1}a^{-m-1}b^m q^{m(m+1)/2} + \left(1+\frac{1}{a}\right)\sum_{m=0}^{\infty}\frac{(-1)^m a^{-m}b^m q^{m(m+1)/2}}{(-bq;q)_m} \qquad (a\neq 0) \tag{25}$$

and

$$(-bq;q)_\infty\sum_{n=0}^{\infty}\frac{(-\lambda/a;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}=(-aq;q)_\infty\sum_{n=0}^{\infty}\frac{(-\lambda/b;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(-aq;q)_n}. \tag{26}$$

The q-identities (25) and (26) are due to Ramanujan [29]; their proofs can be found in [9, Entry 6.3.1, p. 115] and [8, Entry 6.2.2, p. 146], respectively. Recently, the identity (25) was proved by Somashekara and Mamta [32] as a special case of a more general formula. A proof of (26) can also be found in [14].

In Theorem 3 below, we establish some functional equations involving the continued fraction $F(a,b;q)$.

Theorem 3. *Each of the following assertions holds true*:

(i) *If* $a\neq -1$ *and* $b\neq 0$, *then*

$$F(a,b;q)-\left(1-\frac{a}{b}\right)(1+a)F\left(aq^{-1},b;q\right)=\frac{a}{b}. \tag{27}$$

(ii) *If* $a\neq -1$, $a\neq b$ *and* $b\neq 0$, *then*

$$F\left(-\frac{a}{bq},b^{-1};q\right)+bF\left(aq^{-1},b;q\right) = \frac{1}{(\frac{a}{b};q)_\infty(-a;q)_\infty}\left\{F\left(0,b^{-1};q\right)+bF(0,b;q)\right\}. \tag{28}$$

(iii) *If* $a\neq b$ *and* $b\neq 0$, *then*

$$F(a,b;q)+b^{-1}(1+a)\left(1-\frac{a}{b}\right)F(-\frac{a}{bq},b^{-1};q) = \frac{a}{b}+\frac{1}{(\frac{a}{b}q;q)_\infty(-aq;q)_\infty}\left\{F(0,b;q)+b^{-1}F(0,b^{-1};q)\right\}. \tag{29}$$

Proof. First of all, one can easily verify that

$$G(aq,-bq,b;q)-\left(1-\frac{a}{b}\right)G(a,-bq,b;q)=\frac{a}{b}\,G(a,-b,b;q).$$

Dividing both sides of the above identity by $G(a,-b,b;q)$ and then using the following identity:

$$\frac{G(a,b,-b;q)}{G(aq^{-1},b,-b;q)}=\frac{1}{1+a}\qquad (a\neq -1),$$

which follows from Lemma 1, we obtain (27).

To prove (28), we observe that Theorem 1 can be written in the form:

$$F(a,b;q)=-\frac{a}{b}\sum_{n=1}^{\infty}\frac{q^n}{(\frac{a}{b}q;q)_n(-aq;q)_n}+\frac{F(0,b;q)}{(\frac{a}{b}q;q)_\infty(-aq;q)_\infty}. \tag{30}$$

We may also rewrite Theorem 1 in the following form:

$$F(-ab^{-1},b^{-1};q)=a\sum_{n=1}^{\infty}\frac{q^n}{(\frac{a}{b}q;q)_n(-aq;q)_n}+\frac{F(0,b^{-1};q)}{(\frac{a}{b}q;q)_\infty(-aq;q)_\infty}. \tag{31}$$

Upon adding (30) and (31), if we simplify and replace a by aq^{-1}, we deduce (28).

The identity (29) follows easily from (27) and (28) by eliminating $F\left(aq^{-1},b;q\right)$. We have thus completed the *first* proof of Theorem 3.

Alternatively, upon replacing a and b in (25) by $-\frac{a}{\beta}$ and a, respectively, in (25), if we replace β by b and use a special case of (26), we find that

$$\sum_{n=0}^{\infty}\frac{q^n}{(\frac{a}{b}q;q)_n(-aq;q)_n}=\frac{b\,F(0,b;q)}{a(\frac{a}{b}q;q)_\infty(-aq;q)_\infty}+\left(1-\frac{b}{a}\right)\sum_{m=0}^{\infty}\frac{b^m q^{m(m+1)/2}}{(-aq;q)_m}. \tag{32}$$

Using (26), (14) and Lemma 1, we get

$$\sum_{m=0}^{\infty}\frac{b^m q^{m(m+1)/2}}{(-aq;q)_m}=(1+a)F(aq^{-1},b;q)\qquad (a\neq -1). \tag{33}$$

By means of (30), (32) and (33), we deduce the assertion (27) of Theorem 3.

Replacing b by $-\frac{a}{b}$ in (25) and using a special case of (26), we obtain

$$\sum_{n=0}^{\infty}\frac{q^n}{(\frac{a}{b}q;q)_n(-aq;q)_n}=-\frac{F(0,b^{-1};q)}{a(\frac{a}{b}q;q)_\infty(-aq;q)_\infty}+\left(1+\frac{1}{a}\right)\sum_{m=0}^{\infty}\frac{(b^{-1})^m q^{m(m+1)/2}}{(\frac{a}{b}q;q)_m}. \tag{34}$$

Using (26), (14) and Lemma 1, we find that

$$\sum_{m=0}^{\infty}\frac{(b^{-1})^m q^{m(m+1)/2}}{(\frac{a}{b}q;q)_m}=\left(1-\frac{a}{b}\right)F(-\frac{a}{bq},b^{-1};q)\qquad (a\neq b). \tag{35}$$

By means of (32) to (35), we complete the proof of the assertion (28) of Theorem 3.

The proof of the assertion (29) of Theorem 3 follows easily from (30), (34) and (35). This evidently completes the *second* proof of Theorem 3. □

Upon setting $a = 0$ and $b = 1$ in Theorem 1, if we employ the q-binomial theorem in the resulting identity, we find that

$$F(0,1;q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{36}$$

By applying Theorem 3 and (36), it is easy to deduce the following corollary.

Corollary. *Each of the follwing q-identities holds true*:

$$F(a,1;q) - (1-a^2)F(aq^{-1},1;q) = a \qquad (a \neq -1), \tag{37}$$

$$F(a,-1;q) - (1+a)^2 F(aq^{-1},-1;q) = -a \qquad (a \neq -1), \tag{38}$$

$$F(-aq^{-1},1;q) + F(aq^{-1},1;q) = \frac{2(q^2;q^2)_\infty}{(a^2;q^2)_\infty (q;q^2)_\infty} \qquad (a \neq \pm 1) \tag{39}$$

and

$$F(a,1;q) + (1-a^2)F(-aq^{-1},1;q) = a + \frac{2(q^2;q^2)_\infty}{(a^2q^2;q^2)_\infty (q;q^2)_\infty} \qquad (a \neq 1). \tag{40}$$

The continued fraction $M(q)$ given by

$$M(q) := \frac{q^{1/8}}{1+}\,\frac{-q}{1+}\,\frac{q-q^2}{1+}\,\frac{-q^3}{1+}\,\frac{q^2-q^4}{1+}\,\cdots \qquad (|q|<1) \tag{41}$$

was studied by Adiga and Kim [4]. They established an integral representation of $M(q)$ and obtained its explicit evaluations. In addition, they derived its relationship with the Ramanujan-Göllnitz-Gordon continued fraction. Motivated by these developments, in the next sections, we study the continued fraction $N(q)$ given by

$$N(q) := \frac{1}{1-}\,\frac{2q}{1-}\,\frac{q^2-q}{1-}\,\frac{q^3+q^2}{1-}\,\frac{q^4-q^2}{1-}\,\cdots \qquad (|q|<1). \tag{42}$$

3. Integral Representations and Modular Equations for the Continued Fraction $N(q)$

In this section, we initiate our study of the continued fraction $N(q)$ defined by (42) by establishing three equivalent integral representations and some modular equations for $N(q)$.

Our continued fraction $N(q)$ can be expressed in terms of infinite product as follows:

$$N(q) = F(-1,1;q) = \frac{2}{(q;q^2)_\infty} - 1. \tag{43}$$

The expression for $N(q)$ given by (43) can be verified in two different ways. Firstly, by using (39) with $a = q$. Secondly, by using (40) with $a = -1$.

The following theorem follows easily by applying the principle of mathematical induction on k with the help of (37).

Theorem 4. *For any positive integer k,*

$$(q^2;q^2)_k \sum_{n=1}^{k} \frac{q^n}{(q^2;q^2)_n} = (q^2;q^2)_k F(-1,1;q) - F(-q^k,1;q). \tag{44}$$

In Lemma 5 below, we establish relations involving $N(q)$ and some theta functions.

Lemma 5. *For the continued fraction $N(q)$ defined by (42), it is asserted that*

$$2N(q^2) = N(q)N(-q) + N(q) + N(-q) - 1, \tag{45}$$

$$2\,\frac{f(-q^2)}{f(-q)} = N(q) + 1, \tag{46}$$

$$8\,\frac{\psi(q)}{\varphi(-q)} = \Big(N(q) + 1\Big)^3, \tag{47}$$

$$4\,\frac{f(-q^2)}{\varphi(-q)} = \Big(N(q) + 1\Big)^2, \tag{48}$$

$$\frac{\psi(q)}{\psi(-q)} = \frac{N(q) + 1}{N(-q) + 1}, \tag{49}$$

$$\frac{\varphi(q)}{\varphi(-q)} = \left(\frac{N(q) + 1}{N(-q) + 1}\right)^2, \tag{50}$$

$$2\,\frac{f(-q)}{\varphi(-q)} = N(q) + 1 \tag{51}$$

and

$$4\,\frac{\psi(q)}{\varphi(-q^2)} = \Big(N(q^2) + 1\Big)\Big(N(q) + 1\Big). \tag{52}$$

Proof. Identity (43) can be written in the following form:

$$(q;q^2)_\infty = \frac{2}{N(q) + 1}. \tag{53}$$

Upon replacing q by $-q$, if we multiply the resulting identity by (53), we find that

$$2\Big(N(q^2) + 1\Big) = \Big(N(q) + 1\Big)\Big(N(-q) + 1\Big), \tag{54}$$

which is the same as (45).

In a similar way, the identities (46) to (52) follow immediately from (2) to (5) and (53). □

Next, by using (46), one can easily verify the following identity by applying the principle of mathematical induction on n:

$$\frac{f\left(-q^{2^n}\right)}{f(-q)} = 2^{-n}A\left(q^{2^{n-1}}\right)A\left(q^{2^{n-2}}\right)\cdots A(q^4)A(q^2)A(q), \tag{55}$$

where

$$A(q) := N(q) + 1.$$

In the following theorem, we establish three equivalent integral representations for the continued fraction $N(q)$.

Theorem 5. *For* $0 < q < 1$,

$$N(q) + 1 = C_1 \exp\left(\int \frac{1}{8q}\left\{\varphi^4(q) - 16q^2\frac{\chi'(-q^2)}{\chi(-q^2)} - 1\right\}dq\right), \tag{56}$$

$$N(q) + 1 = C_2 \exp\left(\int \frac{1}{16q}\left\{\varphi^4(q) + 8q\frac{\chi'(-q)}{\chi(-q)} - 16q^2\frac{\chi'(-q^2)}{\chi(-q^2)} - 1\right\}dq\right) \tag{57}$$

and

$$N(q) + 1 = C_3 \exp\left(\int \frac{1}{12q}\left\{\varphi^4(q) + 4q\frac{\chi'(-q)}{\chi(-q)} - 16q^2\frac{\chi'(-q^2)}{\chi(-q^2)} - 1\right\}dq\right), \tag{58}$$

where $\varphi(q)$ *and* $\chi(q)$ *are defined as in* (3) *and* (6), *respectively, and* C_1, C_2 *and* C_3 *are some constants.*

Proof. Using (43), we obtain

$$\log\left(N(q) + 1\right) = \log 2 - \sum_{n=1}^{\infty}\log(1 - q^{2n-1}), \tag{59}$$

which, upon differentiating both sides with respect to q, yields

$$\frac{d}{dq}\left\{\log\left(N(q) + 1\right)\right\} = \sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-2}}{1 - q^{2n-1}}. \tag{60}$$

Furthermore, the logarithmic derivative of

$$\chi(-q) = \frac{1}{(-q;q)_\infty}$$

is given by

$$\frac{\chi'(-q)}{\chi(-q)} = \sum_{n=1}^{\infty}\frac{nq^{n-1}}{1 + q^n}. \tag{61}$$

Using (60) and (61) in the following known identity [11, p. 61]:

$$\varphi^4(q) = 1 + 8\sum_{n=1}^{\infty}\left(\frac{(2n-1)q^{2n-1}}{1 - q^{2n-1}} + \frac{2nq^{2n}}{1 + q^{2n}}\right), \tag{62}$$

we obtain

$$\frac{d}{dq}\left\{\log\left(N(q) + 1\right)\right\} = \frac{1}{8q}\left(\varphi^4(q) - 16q^2\frac{\chi'(-q^2)}{\chi(-q^2)} - 1\right). \tag{63}$$

Integrating both sides of (63) and then exponentiating, we get the assertion (56). The identities (57) and (58) follow in a similar way on using (48) and (47), respectively. This completes the proof of Theorem 5. □

Theorem 6. *Let*

$$A(q) := N(q) + 1.$$

Then

$$A^2(q) + A^2(-q) = \frac{2A(q^4)A^3(q^2)}{A^2(-q^4)}, \tag{64}$$

$$A^2(q) - A^2(-q) = 4^{-1}qA^2(q^8)A(q^4)A^3(q^2), \tag{65}$$

$$A^4(q) + A^4(-q) = 8\,\frac{A^6(q^2)}{A^4(-q^2)}, \tag{66}$$

$$A^4(q) - A^4(-q) = 8^{-1}qA^4(q^4)A^6(q^2) \tag{67}$$

and

$$A^8(q) - A^8(-q) = 2^{-4}\,qA^{16}(q^2). \tag{68}$$

Proof. Replacing q by $-q$ in the reciprocal of (48) and then adding the resulting identity to the reciprocal of (48), we obtain

$$\frac{1}{4f(-q^2)}\{\varphi(q) + \varphi(-q)\} = \frac{1}{A^2(-q)} + \frac{1}{A^2(q)}. \tag{69}$$

Using the following identity, which is due to Ramanujan [28] (see, for example, [1, Entry 25(i)]):

$$\varphi(q) + \varphi(-q) = 2\,\varphi(q^4), \tag{70}$$

if we employ (48) (with q replaced by $-q^4$) and (54) in (70), we find that

$$8\,\frac{f(-q^8)}{f(-q^2)} = \left(\frac{A\left(-q^4\right)}{A\left(q^2\right)}\right)^2 \left\{A^2(q) + A^2(-q)\right\}. \tag{71}$$

Employing (46) in the left hand-side of the above identity and after some simplifications, we arrive at (64).

The proof of (65) follows in a similar way by using the following known result [1, Entry 25(ii)]:

$$\varphi(q) - \varphi(-q) = 4q\,\psi(q^8), \tag{72}$$

together with the identities (51) and (52).

The proof of (66) follows also in a similar way by means of the following result [1, Entry 25(vi)]:

$$\varphi^2(q) + \varphi^2(-q) = 2\,\varphi^2(q^2) \tag{73}$$

in conjunction with (48) and (54).

The q-identity (67) follows immediately from (64) and (65) with the help of (54).

The proof of (68) follows easily from (66) and (67). This obviously completes the proof of Theorem 6. □

Theorem 7. *Suppose that*

$$A(q) := N(q) + 1.$$

Then

$$A^2(q)A^2(-q^3) + A^2(-q)A^2(q^3) = 8\,\frac{A^2(q^2)A^2(q^6)}{A(-q^2)A(-q^6)}, \tag{74}$$

$$A^2(q)A^2(-q^3) - A^2(-q)A^2(q^3) = qA^2(q^2)A^2(q^6)A(q^4)A(q^{12}), \tag{75}$$

$$A(q)A(-q^5) + A(-q)A(q^5) = \frac{A(q^2)A(q^4)A^2(q^{10})}{A(q^{20})}, \tag{76}$$

$$A(q)A(-q^5) - A(-q)A(q^5) = \frac{qA^2(q^2)A(q^{10})A(q^{20})}{A(q^4)}, \tag{77}$$

$$A(q)A(-q^9) + A(-q)A(q^9) = 4\,\frac{A(q^6)f^2(-q^{12})}{f(-q^2)f(-q^{18})}, \tag{78}$$

$$A(q^8)A(-q^4) + qA(q^{24})A(-q^{12}) = 4\,\frac{A(-q^4)A(-q^{12})}{A(-q)A(q^3)}, \tag{79}$$

$$4\,A^2(q^4) + qA^4(q^8) = 16\,\frac{A^2(q)A^2(q^8)}{A(q^4)A^3(q^2)}, \tag{80}$$

$$\left(\frac{A(q^2)}{A(-q^5)}\right)^2 - q\left(\frac{A(q^{10})}{A(-q)}\right)^2 = \frac{4}{A\,(q^5)\,A\,(q)}, \tag{81}$$

$$\left(\frac{A(q^4)}{A(-q^6)}\right)^2 + q\left(\frac{A(q^{12})}{A(-q^2)}\right)^2 = \frac{4}{A\,(-q^3)\,A\,(-q)}, \tag{82}$$

$$\left(\frac{A(q^2)}{A(-q^9)}\right)^2 - q^2\left(\frac{A(q^{18})}{A(-q)}\right)^2 = \frac{f^2(-q^3)}{f(-q^2)f(-q^{18})}, \tag{83}$$

$$8 - qA^2(q^8)A^2(-q^4) = 32\,\frac{A^2(-q^4)}{A^2(q)A(q^2)A(q^4)} \tag{84}$$

and

$$8 + qA(-q^2)A(q^4)A(-q^6)A(q^{12}) = 32\,\frac{A(-q^2)A(-q^6)}{A^2(-q)A^2(q^3)}. \tag{85}$$

Proof. The following theta function identity can be found in [17]:

$$2\varphi\left(-q^4\right)\varphi\left(-q^{12}\right)=\varphi\left(-q^3\right)\varphi(q)+\varphi\left(q^3\right)\varphi(-q). \tag{86}$$

Dividing both sides of (86) by $f(-q^2)f(-q^6)$ and then using (48), we obtain

$$2\,\frac{f(-q^8)f(-q^{24})}{A^2(q^4)A^2(q^{12})f(-q^2)f(-q^6)}=\frac{1}{A^2(q^3)A^2(-q)}+\frac{1}{A^2(-q^3)A^2(q)}. \tag{87}$$

Using (46) and (54) in the above identity, after some simplifications, we deduce (74).

The proofs of the identities (75) to (85) follow in a similar way on using Lemma 5 and the following theta function identities:

$$4q\psi\left(-q^2\right)\psi\left(-q^6\right)=\varphi\left(-q^3\right)\varphi(q)-\varphi\left(q^3\right)\varphi(-q), \tag{88}$$

$$2f\left(-q^8\right)f\left(q^{10}\right)=\psi\left(-q^5\right)\psi(q)+\psi\left(q^5\right)\psi(-q), \tag{89}$$

$$2qf\left(-q^{40}\right)f\left(q^2\right)=\psi\left(-q^5\right)\psi(q)-\psi\left(q^5\right)\psi(-q), \tag{90}$$

$$2f\left(-q^{24}\right)f\left(q^6\right)=\psi\left(-q^9\right)\psi(q)+\psi\left(q^9\right)\psi(-q), \tag{91}$$

$$\psi\left(-q^3\right)\psi(q)=\varphi\left(-q^{24}\right)\psi\left(-q^4\right)+q\varphi\left(-q^8\right)\psi\left(-q^{12}\right), \tag{92}$$

$$\psi\left(q^8\right)\varphi(q)=\psi^2(q^4)+q\varphi\left(q^8\right)\psi\left(q^{16}\right), \tag{93}$$

$$\varphi\left(q^5\right)\psi\left(q^2\right)=f\left(-q^5\right)f(-q)+q\varphi(q)\psi\left(q^{10}\right), \tag{94}$$

$$\psi\left(q^3\right)\psi(q)=\varphi\left(q^6\right)\psi\left(q^4\right)+q\varphi\left(q^2\right)\psi\left(q^{12}\right), \tag{95}$$

$$\varphi\left(q^9\right)\psi\left(q^2\right)=f^2\left(-q^3\right)+q^2\varphi(q)\psi\left(q^{18}\right), \tag{96}$$

$$\varphi\left(-q^4\right)\varphi(-q)=\varphi^2\left(-q^8\right)-2q\psi^2\left(-q^4\right) \tag{97}$$

and

$$\varphi\left(-q^3\right)\varphi(q)=2q\psi\left(-q^6\right)\psi\left(-q^2\right)+\varphi\left(-q^{12}\right)\varphi\left(-q^4\right), \tag{98}$$

respectively, where

$$f(q)=\frac{f^3(-q^2)}{f(-q)f(-q^4)}.$$

The q-identities (88) to (91), (94) and (96) are due to Bulkhali [17] and the other q-identities are due to Adiga and Bulkhali [2]. This completes the proof of Theorem 7. □

Theorem 8. *For any positive integers α and β,*

$$A^2(q^\alpha)A^2\left(q^\beta\right) - A^2(-q^\alpha)A^2\left(-q^\beta\right) = \frac{q^\beta}{4}B(q)\left\{\left(\frac{A\left(q^{8\beta}\right)}{A(-q^{4\alpha})}\right)^2 + q^{\alpha-\beta}\left(\frac{A\left(q^{8\alpha}\right)}{A(-q^{4\beta})}\right)^2\right\} \tag{99}$$

and

$$A^2(q^\alpha)A^2\left(q^\beta\right) + A^2(-q^\alpha)A^2\left(-q^\beta\right) = \frac{1}{32}B(q)\left\{\left(\frac{8}{A(-q^{4\alpha})A(-q^{4\beta})}\right)^2 + q^{\alpha+\beta}\left[A\left(q^{8\alpha}\right)A\left(q^{8\beta}\right)\right]^2\right\}, \tag{100}$$

where

$$B(q) := A\left(q^{4\alpha}\right)A\left(q^{4\beta}\right)A^3\left(q^{2\alpha}\right)A^3\left(q^{2\beta}\right).$$

Proof. Adding (70) and (72), we obtain

$$\varphi(q^4) + 2q\psi(q^8) = \varphi(q). \tag{101}$$

Also, on replacing q by q^α and q^β in (101), and then multiplying the resulting identities, we find that

$$\varphi(q^\alpha)\varphi(q^\beta) = \varphi(q^{4\alpha})\varphi(q^{4\beta}) + 2q^\alpha\psi(q^{8\alpha})\varphi(q^{4\beta}) + 2q^\beta\varphi(q^{4\alpha})\psi(q^{8\beta}) + 4q^{\alpha+\beta}\psi(q^{8\alpha})\psi(q^{8\beta}). \tag{102}$$

Now, if we replace q^α and q^β in (102) by $-q^\alpha$ and $-q^\beta$, respectively, and then subtract the resulting identity from (102), we get

$$\varphi(q^\alpha)\varphi(q^\beta) - \varphi(-q^\alpha)\varphi(-q^\beta) = 4q^\beta\left\{\varphi(q^{4\alpha})\psi(q^{8\beta}) + q^{\alpha-\beta}\varphi(q^{4\beta})\psi(q^{8\alpha})\right\}. \tag{103}$$

Dividing both sides of this last equation (103) by $f\left(-q^{2\alpha}\right)f\left(-q^{2\beta}\right)$ and using (46), (48), (54) and the following simply-provable identity:

$$2\psi(q) = A(q)f(-q^2),$$

we obtain (99).

Our demonstration of the q-identity (100) follows in a similar way, so we omit the details. □

4. Invariants of the Ramanujan-Weber Class

In this section, we find continued fraction representations for the Ramanujan-Weber class invariants g_n and G_n and obtain several relations between them. Furthermore, we derive relations between our continued fraction $A(q)$ with the Ramanujan-Göllnitz-Gordon and Ramanujan's cubic continued fractions.

If $q = e^{-\pi\sqrt{n}}$, where n is a positive integer, the two class invariants G_n and g_n are defined by (see, for details, [10])

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \qquad \text{and} \qquad g_n := 2^{-1/4}q^{-1/24}\chi(-q). \tag{104}$$

In his *first* notebook [28], Ramanujan recorded the values for 107 class invariants. Moreover, on Pages 294–299 in his *second* notebook [28], Ramanujan gave a table of values for 77 class invariants, three of which are not found in the first notebook. By the time when Ramanujan wrote his paper [27], he was aware of Weber's work [36], and so his table of 46 class invariants in [27] does not contain any that are found in

Weber's book [36]. Hence Ramanujan calculated a total of 116 class invariants. Berndt [10, pp. 189-204, Chapter 34] gave a table of all values of each of the two class invariants G_n and g_n, which were found by Ramanujan. Many other authors have found several new values of G_n and g_n.

Using (43) and (104), one can easily find that

$$A(q) = 2^{3/4} q^{-1/24} g_n^{-1} \qquad \text{and} \qquad A(-q) = 2^{3/4} q^{-1/24} G_n^{-1}. \tag{105}$$

By applying the above two identities, it is clear that we can find values of $A(q)$ and $A(-q)$ using known values for g_n and G_n.

We can obtain the continued fraction representation of the Ramanujan-Weber class invariants g_n and G_n using (42) and (105) as follows:

$$g_n = 2^{3/4} \frac{e^{\pi\sqrt{n}/24}}{1+} \frac{1}{1} - \frac{2e^{-\pi\sqrt{n}}}{1-} \frac{e^{-2\pi\sqrt{n}} - e^{-\pi\sqrt{n}}}{1-} \frac{e^{-3\pi\sqrt{n}} + e^{-2\pi\sqrt{n}}}{1-} \cdots \tag{106}$$

and

$$G_n = 2^{3/4} \frac{e^{\pi\sqrt{n}/24}}{1+} \frac{1}{1-} \frac{-2e^{-\pi\sqrt{n}}}{1-} \frac{e^{-2\pi\sqrt{n}} + e^{-\pi\sqrt{n}}}{1-} \frac{-e^{-3\pi\sqrt{n}} + e^{-2\pi\sqrt{n}}}{1-} \cdots . \tag{107}$$

Ramanujan [27] stated the following two identities:

$$g_{4n} = 2^{1/4} g_n G_n \tag{108}$$

and

$$(g_n G_n)^8 \left(G_n^8 - g_n^8\right) = \frac{1}{4}. \tag{109}$$

We can easily verify that (108) and (109) are equivalent to (54) and (68), respectively.

Many mathematicians have established several relations between g_n and G_n (see, for example, the works by Borwein and Borwien [16], Berndt [10] and Naika [26]). In Theorem 9 below, we establish some new relations between G_n and g_n.

Theorem 9. *Each of the following relations hold true between G_n and g_n:*

$$g_{9n}^2 G_n^2 + g_n^2 G_{9n}^2 = 2^{1/2} G_{4n} G_{36n}, \tag{110}$$

$$g_{9n}^2 G_n^2 - g_n^2 G_{9n}^2 = \frac{2^{1/2}}{g_{16n} g_{144n}}, \tag{111}$$

$$g_{25n} G_n + g_n G_{25n} = 2^{1/2} \frac{G_{100n}}{g_{16n}}, \tag{112}$$

$$g_{25n} G_n - g_n G_{25n} = 2^{1/4} \frac{g_{16n}}{g_{4n} g_{400n}}, \tag{113}$$

$$\frac{1}{g_n G_{81n}} + \frac{1}{g_{81n} G_n} = 2^{5/4} q^{1/6} \frac{f^2(-q^{12})}{g_{36n} f(-q^2) f(-q^{18})}, \tag{114}$$

$$\frac{G_{144n}}{g_{64n}}+\frac{G_{16n}}{g_{576n}}=2^{1/2}g_{9n}G_n, \tag{115}$$

$$2^{1/2}g_{64n}^4+g_{16n}^2=2\left(\frac{g_{64n}}{g_n}\right)^2\left(g_{4n}g_{16n}\right)^3, \tag{116}$$

$$\frac{G_{25n}^2}{g_{4n}^2}-\frac{G_n^2}{g_{100n}^2}=2^{1/2}g_ng_{25n}, \tag{117}$$

$$\frac{G_{36n}^2}{g_{16n}^2}+\frac{G_{4n}^2}{g_{144n}^2}=2^{1/2}G_nG_{9n}, \tag{118}$$

$$\frac{G_{81n}^2}{g_{4n}^2}-\frac{G_n^2}{g_{324n}^2}=q^{-7/12}\frac{f^2(-q^3)}{f(-q^2)f(-q^{18})}, \tag{119}$$

$$G_{16n}^2-\frac{1}{g_{64n}^2}=2^{1/2}g_n^2g_{4n}g_{16n}, \tag{120}$$

$$G_{4n}G_{36n}+\frac{1}{g_{16n}g_{144n}}=2^{1/2}g_{9n}^2G_n^2, \tag{121}$$

$$\left(G_ng_{9n}\right)^5\left(G_{9n}g_n\right)-\left(G_ng_{9n}\right)\left(G_{9n}g_n\right)^5=1 \tag{122}$$

and

$$\left(G_ng_{25n}\right)^3\left(G_{25n}g_n\right)-\left(G_ng_{25n}\right)\left(G_{25n}g_n\right)^3=1. \tag{123}$$

Proof. The relations (110) to (121) follow from the q-identities (74) to (85), respectively, by using (105) and (108). The relation (122) follows upon multiplying (110) by (111) and using (108). Similarly, the relation (123) follows from (112) and (113). □

Theorem 10 below is a consequence of Theorem 8 with help of (105) and (108).

Theorem 10. *It is asserted that*

$$G_{\alpha^2n}^2G_{\beta^2n}^2-g_{\alpha^2n}^2g_{\beta^2n}^2=\left(\frac{G_{16\alpha^2n}^2}{g_{64\beta^2n}^2}+\frac{G_{16\beta^2n}^2}{g_{64\alpha^2n}^2}\right)\mathcal{B} \tag{124}$$

and

$$G_{\alpha^2n}^2G_{\beta^2n}^2+g_{\alpha^2n}^2g_{\beta^2n}^2=\left(G_{16\alpha^2n}^2G_{16\beta^2n}^2+\frac{1}{g_{64\alpha^2n}^2g_{64\beta^2n}^2}\right)\mathcal{B}, \tag{125}$$

where

$$\mathcal{B}:=\frac{2^{-1/2}}{G_{4\alpha^2n}G_{4\beta^2n}g_{4\alpha^2n}^2g_{4\beta^2n}^2}.$$

We now turn to the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$, which is defined by

$$H(q) := \frac{q^{1/2}}{1+q+} \, \frac{q^2}{1+q^3+} \, \frac{q^4}{1+q^5+} \, \frac{q^6}{1+q^7+} \cdots \qquad (|q| < 1). \tag{126}$$

In his notebook [28, p. 229], Ramanujan presented the following two identities for $H(q)$:

$$\frac{1}{H(q)} - H(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)} \tag{127}$$

and

$$\frac{1}{H(q)} + H(q) = \frac{\varphi(q)}{q^{1/2}\psi(q^4)}. \tag{128}$$

In light of the relations (127) and (128), we remark that $A(q)$ and $H(q)$ are related by the following equations:

$$\frac{1}{H(q)} - H(q) = \frac{64}{q^{1/2}A^4(-q^2)A^2(q^2)} \tag{129}$$

and

$$\frac{1}{H(q)} + H(q) = \frac{128}{q^{1/2}A^2(-q)A^3(q^2)A^2(-q^2)}. \tag{130}$$

From the equation (129), we can compute $H(q)$ by using the known values of $A(q^2)$ and $A(-q^2)$. On the other hand, we can also compute $A(q^2)$ by using the known values of $H(q)$ and $A(-q^2)$.

On Page 366 of his "lost" notebook [29], Ramanujan recorded the following continued fraction:

$$G(q) := \frac{q^{1/3}}{1+} \, \frac{q+q^2}{1+} \, \frac{q^2+q^4}{1+} \, \frac{q^3+q^6}{1+} \cdots \qquad (|q| < 1), \tag{131}$$

which is known as Ramanujan's cubic continued fraction. Ramanujan [29, p. 366] gave some identities for $G(q)$ including (for example) the following identity:

$$G(q) = q^{1/3} \, \frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3}.$$

The above identity can be written in terms of our continued fraction as follows:

$$4G(q)A(q) - q^{1/3}A^3(q^3) = 0.$$

Recently, Adiga and Kim [4] showed that

$$M(q) = q^{1/8} \, \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

where $M(q)$ is given by (41). In view of this last identity, we remark that $M(q)$ and $A(q)$ are related by the following equation:

$$f(-q^2) = 2q^{-1/8} \, \frac{M(q)}{A(q)}, \tag{132}$$

where $f(q)$ is defined as in (5). From the equation (132), we can compute the theta function $f(-q^2)$ by using the known values of $A(q)$ and $M(q)$. On the other hand, we can also compute $A(q)$ by using the known values of $f(-q^2)$ and $M(q)$. For example, we have the following result.

Theorem 11. (see [10, Entry 2, p. 326]) *It is asserted that*

$$f(-e^{-4\pi}) = 2^{-7/8} e^{\pi/6} \frac{\pi^{1/4}}{\Gamma(3/4)} \tag{133}$$

and

$$f(-e^{-8\pi}) = 2^{-11/8} e^{\pi/3} (\sqrt{2}-1)^{1/2} (4+3\sqrt{2})^{1/8} \frac{\pi^{1/4}}{\Gamma(3/4)}. \tag{134}$$

Proof. Making use of the value $G_1 = 1$ [10, p. 189] in (108) and (109) and then using (105), we find that

$$A(e^{-2\pi}) = 2^{5/8} e^{\pi/12} \tag{135}$$

and

$$A(e^{-4\pi}) = \frac{2^{3/8} e^{\pi/6}}{\left(1+\frac{3}{4}\sqrt{2}\right)^{1/8}}. \tag{136}$$

We now recall that Adiga and Kim [4] found that

$$M(e^{-2\pi}) = 2^{-5/4} \frac{\pi^{1/4}}{\Gamma(3/4)} \tag{137}$$

and

$$M(e^{-4\pi}) = 2^{-7/4} (\sqrt{2}-1)^{1/2} \frac{\pi^{1/4}}{\Gamma(3/4)}. \tag{138}$$

Using the above four equalities in (132), we obtain the results asserted by Theorem 11. □

5. Associated Algebraic and Transcendental Numbers

We begin this section by recalling that Berndt *et al.* [12] proved that the singular values $q^{-\frac{1}{60}}\, G(q)$ and $q^{\frac{11}{60}}\, H(q)$ are algebraic numbers in Abelian extensions of $Q(\tau)$. More recently, Griffin *et al.* [19] showed that the Rogers-Ramanujan continued fraction $R(q)$ has the special property that its singular values are algebraic integer units.

The Rogers-Ramanujan continued fraction $R(q)$ can be expressed by the following quotient of infinite products:

$$R(q) = \frac{q^{\frac{1}{60}} \prod_{m=1}^{\infty} \left(1-q^{5m-1}\right)\left(1-q^{5m-4}\right)}{q^{-\frac{11}{60}} \prod_{m=1}^{\infty} (1-q^{5m-2})(1-q^{5m-3})}.$$

It is well known that the Rogers-Ramanujan continued fraction $R(q)$ is an *algebraic* number (*cf.* [24, Eq. (3.1)]).

By using (127) and (128), we derive the following identities which are related to the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$:

$$\frac{1}{[H(q)]^2} - [H(q)]^2 = \frac{\varphi(q^2)\varphi(q)}{q[\psi(q^4)]^2} \tag{139}$$

and

$$\frac{\frac{1}{H(q)} - H(q)}{\frac{1}{H(q)} + H(q)} = \frac{\varphi(q^2)}{\varphi(q)}. \tag{140}$$

It would be of interest to investigate whether the left-hand sides of (139) and (140) are algebraic or transcendental numbers.

Let k be an imaginary quadratic field and let $\mathfrak{h}$ be the upper-half complex plane. Also let

$$\tau \in k \cap \mathfrak{h} \qquad \text{and} \qquad A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \quad \big(b\,(\text{mod}\, d)\big).$$

Denoting by $|A|$ the determinant of the matrix A, we set

$$\Phi_A(\tau) = \left(\frac{|A|}{d}\right)^{12} \frac{\Delta(A\tau)}{\Delta(\tau)}, \tag{141}$$

where

$$A\tau = \frac{a\tau + b}{d}$$

and

$$\Delta(\tau) = (2\pi)^{12}\, q^2 \prod_{m=1}^{\infty} (1 - q^{2m})^{24}.$$

We recall from [22], [23], [24] and [25] that, for any

$$\tau \in k \cap \mathfrak{h},$$

the value of $\Phi_A(\tau)$ is an algebraic integer which divides $|A|^{12}$.

In this section, we use the notations employed in the work of Kim and Koo [23]. We first recall that the Dedekind eta function $\eta(\tau)$ is defined as follows:

$$\eta(\tau) := e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n \tau}\right),$$

where $\tau \in \mathfrak{h}$. However, following the work of Kim and Koo [23], here we define $\eta(\tau)$ by

$$\eta(\tau) := q^{\frac{1}{12}} f(-q^2), \tag{142}$$

where

$$q = e^{\pi i \tau} \quad \text{and} \quad \tau \in \mathfrak{h}.$$

In fact, Kim and Koo [23, Eqs. (3.6) and (3.10)] showed that

$$\frac{q^{\frac{1}{8}} \psi(q)}{\eta(\tau)} \tag{143}$$

and

$$\frac{\varphi(q)}{\eta(\tau)} \tag{144}$$

are algebraic numbers. But, on the other hand, $\eta(\tau)$ and $q = e^{\pi i \tau}$ are transcendental numbers (*cf.* [22], [24], [23] and [25]).

Theorem 12. *The following number*:

$$\frac{\varphi(q^2)}{\varphi(q)}$$

is algebraic.

Proof. From the work of Kim and Koo [23] and also Lang [25], we can easily see that

$$\frac{\eta(2\tau)}{\eta(\tau)} = \frac{\eta\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\tau\right)}{\eta(\tau)} \tag{145}$$

is an algebraic number. We now set

$$\frac{\varphi(q^2)}{\varphi(q)} = \left(\frac{\varphi(q^2)}{\eta(2\tau)}\right)\left(\frac{\eta(\tau)}{\varphi(q)}\right)\left(\frac{\eta(2\tau)}{\eta(\tau)}\right). \tag{146}$$

In view of the algebraic nature of the numbers in (144) and (145), all factors on the right-hand side of (146) are algebraic numbers. Thus, if we let B be the set of algebraic numbers in $\mathbb{C}$, then it is well-known that B is a field. Therefore, the left-hand side of (146) is an algebraic number. This evidently leads us to the desired result asserted by Theorem 12. □

We next observe that, by using a known result [23, Eq. (3.6)], we can deduce that each of the following numbers:

$$\frac{\eta(4\tau)}{q^{\frac{1}{2}}\psi(q^4)} \tag{147}$$

and

$$\frac{\eta(\tau)}{q^{\frac{1}{8}}\psi(q)}$$

is an algebraic number. Here

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} 4\tau = \tau \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} 4\tau = 2\tau.$$

Theorem 13. *The number*

$$\frac{\varphi(q^2)\varphi(q)}{q[\psi(q^4)]^2}$$

is algebraic.

Proof. Our proof of Theorem 13 is similar to that of Theorem 12. Indeed, if we set

$$\frac{\varphi(q^2)\varphi(q)}{q[\psi(q^4)]^2} = \left(\frac{\varphi(q^2)}{\eta(2\tau)}\right)\left(\frac{\varphi(q)}{\eta(\tau)}\right)\left(\frac{\eta(4\tau)}{q^{\frac{1}{2}}\psi(q^4)}\right)^2\left(\frac{\eta(\tau)}{\eta(4\tau)}\right)\left(\frac{\eta(2\tau)}{\eta(4\tau)}\right), \tag{148}$$

then, by virtue of (143), (144), (145) and (147), all factors on the right-hand side of (148) are algebraic numbers. Therefore, the number on the left-hand side of (148) is also an algebraic number, just as asserted by Theorem 13. □

By combining Theorem 12 and Theorem 13 with (139) and (140), we arrive at Theorem 14 below.

Theorem 14. *Each of the following numbers:*

$$\frac{\frac{1}{H(q)} - H(q)}{\frac{1}{H(q)} + H(q)}$$

and

$$\frac{1}{[H(q)]^2} - [H(q)]^2$$

is algebraic.

In their aforecited work, Kim and Koo [23, p. 62, Eq. (3.5)] showed that

$$\frac{\sqrt{2}q^{1/24}}{\prod_{m=1}^{\infty}(1-q^{2m-1})} \quad \text{and} \quad \frac{\sqrt{2}q^{1/24}}{\prod_{m=1}^{\infty}(1+q^{2m-1})} \tag{149}$$

are algebraic integers. Since

$$A(q) = \frac{2}{(q;q^2)_\infty} = \frac{2}{\prod_{m=1}^{\infty}(1-q^{2m-1})} \quad \text{and} \quad A(-q) = \frac{2}{\prod_{m=1}^{\infty}(1+q^{2m-1})}, \tag{150}$$

we are led to the following theorem.

Theorem 15. *The following numbers:*

$$q^{1/24}A(q) \quad \textit{and} \quad q^{1/24}A(-q), \tag{151}$$

are algebraic integers.

We next recall from [23, p. 66] that the following numbers:

$$\sqrt{d}\,\frac{\eta(d\tau)}{\eta(\tau)} \quad \text{and} \quad \frac{\eta(\tau)}{\eta(d\tau)} \tag{152}$$

are algebraic integers for $d \in \mathbb{Z}^+$. Thus, if we replace q by q^2 in (46), it is easy to verify that

$$A(q^2) = 2q^{-1/12}\,\frac{\eta(2\tau)}{\eta(\tau)} \quad \text{and} \quad A(-q^2) = 2q^{-1/12}\,\frac{\eta(\tau)\eta(4\tau)}{\eta^2(2\tau)}. \tag{153}$$

Using (152) in (153), we deduce that the following numbers:

$$q^{1/12}A(q^2) \quad \text{and} \quad q^{1/12}A(-q^2)$$

are algebraic integers. Furthermore, by using (152), we can easily prove Theorem 16 below.

Theorem 16. *The following numbers:*

$$q^{n/12}\,A(q^{2n}) \quad \textit{and} \quad q^{n/12}\,A(-q^{2n})$$

are algebraic integers.

By applying Theorem 16, we see that the following numbers:

$$q^{1/3}\ \frac{A^2(q^2)A^2(q^6)}{A(-q^2)A(-q^6)},$$

$$q^{4/3}\ [A(q^2)]^2[A(q^6)]^2A(q^4)A(q^{12}),$$

$$q^{1/4}\ \frac{A(q^2)A(q^4)[A(q^{10})]^2}{A(q^{20})},$$

$$q^{5/4}\ \frac{[A(q^2)]^2A(q^{10})A(q^{20})}{A(q^4)},$$

$$q^{5/12}\ \frac{A(q^6)[f(-q^{12})]^2}{f(-q^2)f(-q^{18})},$$

$$q^{1/2}\ \left\{A(q^8)A(-q^4)+qA(q^{24})A(-q^{12})\right\},$$

$$q^{1/3}\ \frac{[A(q)]^2[A(q^8)]^2}{A(q^4)[A(q^2)]^3},$$

$$\frac{[A(-q^4)]^2}{[A(q)]^2A(q^2)A(q^4)}$$

and

$$\frac{A(-q^2)A(-q^6)}{[A(-q)]^2[A(q^3)]^2}$$

are algebraic integers.

It was shown in [23, p. 59] that the number $\eta(\tau)$ is transcendental. Thus, by (132) and (142), the number

$$q^{-1/24}\frac{M(q)}{A(q)}$$

is transcendental. Also, by applying Theorem 15, we conclude that the following number:

$$\frac{M(q)}{\eta(\tau)}$$

is an algebraic integer.

6. Concluding Remarks and Observations

This article is motivated essentially by the fact that, on Page 36 of his "lost" notebook, Ramanujan recorded four q-series representations of the famous Rogers-Ramanujan continued fraction. In our present investigation, we have established two q-series representations of Ramanujan's continued fraction found in his "lost" notebook. We also have obtained three equivalent integral representations and modular equations for a special case of this continued fraction. Furthermore, we have derived continued fraction representations for the Ramanujan-Weber class invariants g_n and G_n and established formulas connecting g_n and G_n. We have deduced several relations between our continued fraction with the Ramanujan-Göllnitz-Gordon and Ramanujan's cubic continued fractions. Finally, we have found some algebraic numbers and transcendental numbers associated with a certain continued fraction $A(q)$ which is related to Ramanujan's continued fraction $F(a, b, \lambda; q)$, the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$ and the Dedekind eta function $\eta(s)$. Several interesting (known or new corollaries and consequences of our main results (Theorems 1 to 16) have also been indicated.

Acknowledgements

The first-named and the second-named authors are thankful to the University Grants Commission of the Government of India for financial support under Grant F.510/2/SAP-DRS/2011. The third-named author was supported by the University Scientific Research Projects Unit of the University of Akdeniz.

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$$1 + \frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)}x + \frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})}x^2 + \cdots,$$

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