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Fuzzy Semiprime and Fuzzy Prime Subsets of Ordered Groupoids

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Abstract. A fuzzy subset f of an ordered groupoid (or groupoid) S is called fuzzy semiprime if $f(x) \geq f(x^2)$ for every $x \in S$; it is called fuzzy prime if $f(xy) \leq \min\{f(x), f(y)\}$ for every $x, y \in S$ (Definition 1). Following the terminology of semiprime subsets of ordered groupoids (or groupoids) and the terminology of ideal elements of *poe*-groupoids (: ordered groupoids possessing a greatest element), a fuzzy subset f of an ordered groupoid (or groupoid) should be called fuzzy semiprime if for every fuzzy subset g of S such that $g^2 := g \circ g \preceq f$, we have $g \preceq f$; it should be called prime if for any fuzzy subsets h, g of S such that $h \circ g \preceq f$ we have $h \preceq f$ of $g \preceq f$ (Definition 2). And this is because if S is a groupoid or an ordered groupoid, then the set of all fuzzy subsets of S is a *poe*-groupoid. What is the relation between these two definitions? that is between the Definition 1 (the usual definition we always use) and the Definition 2 given in this paper? The present paper gives the related answer.

1. Introduction

An ordered groupoid (: *po*-groupoid), denoted by $(S, \cdot, \leq)$, is an ordered set $(S, \leq)$ endowed with a multiplication "$\cdot$" which is compatible with the ordering (that is, $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for every $c \in S$). If the multiplication is associative, then S is called an ordered semigroup (: *po*-semigroup). A *poe*-groupoid is an ordered groupoid having a greatest element usually denoted by "e" ($e \geq a$ for all $a \in S$) (cf. for example [2]). Following L. Zadeh [12], the founder of fuzzy sets, if S is an ordered groupoid (or groupoid), a fuzzy subset of S (or a fuzzy set in S) is a mapping f of S into the closed interval $[0, 1]$ of real numbers. For a nonempty subset A of an ordered groupoid (or groupoid) S, the characteristic function f_A is the fuzzy subset on S defined by

$$f_A : S \to \{0,1\} \mid x \to \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \notin A. \end{cases}$$

If $(S, .)$ is a groupoid, f, g fuzzy subsets of S and $a \in S$, the multiplication $f \circ g$ on S is the fuzzy subset of S defined as follows:

$$(f \circ g)(a) := \begin{cases} \bigvee\limits_{x,y \in S,\ xy = a} \min\{f(x), g(y)\} & \text{if there exist } x, y \in S \text{ such that } a = xy \\ 0 \quad \text{if there are no } x, y \in S \text{ such that } a = xy. \end{cases}$$

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A groupoid S with the multiplication defined above and the order relation "$\preceq$" defined by $f \preceq g$ if and only if $f(x) \leq g(x)$ for all $x \in S$, is a *poe*-groupoid [4]. If S is an ordered groupoid (or groupoid), $x \in S$ and $\lambda \in [0,1]$, the mapping

$$x_\lambda : S \to [0,1] \mid y \to \begin{cases} \lambda & \text{if} \quad y = x \\ 0 & \text{if} \quad y \neq x \end{cases}$$

is called a *fuzzy point* of S. If S is an ordered groupoid, then for an element a of S, we define

$$A_a = \{(x,y) \in S \times S \mid a \leq xy\}.$$

For two fuzzy subsets f, g of the ordered groupoid S, the multiplication $f \circ g$ is the fuzzy subset of S defined by:

$$(f \circ g)(a) := \begin{cases} \bigvee\limits_{(x,y)\in A_a} \min\{f(x), g(y)\} & \text{if} \quad A_a \neq \emptyset \\ 0 & \text{if} \quad A_a = \emptyset, \end{cases}$$

and the order relation is defined as follows:

$$f \preceq g \text{ if and only if } f(x) \leq g(x) \text{ for all } x \in S.$$

If f, g are fuzzy subsets of S such that $f \preceq g$ then, for every fuzzy subset h of S, we have $f \circ h \preceq g \circ h$ and $h \circ f \preceq h \circ g$, so the set of all fuzzy subsets of S form a *po*-groupoid, in particular this is a *poe*-groupoid [6]. If S is an ordered semigroup, then the multiplication of fuzzy subsets of S is associative, so the set of all fuzzy subsets of S with the multiplication and the order above is an ordered semigroup, in particular, a *poe*-semigroup [6, 7]. It might be noted that if S is a groupoid or an ordered groupoid, then the *poe*-groupoid of all fuzzy subsets of S has a zero element and S is embedded in the set of all fuzzy subsets of S [7].

According to Clifford and Preston [1; p. 121], a subset T of a semigroup S is called semiprime if for every $a \in S$ such that $a^2 \in T$, we have $a \in T$. Semiprime ideals play an important role in studying the structure of semigroups. As an example, a semigroup S is left regular (resp. right regular) if and only if every left (resp. right) ideal of S is semiprime. A semigroup S is intra-regular if and only if every ideal of S is semiprime. These, in turn, are equivalent to saying that a semigroup S is left (resp. right) regular if and only if it is a union (or disjoint union) of left (resp. right) simple subsemigroups of S. Every left and every right ideal of S is semiprime if and only if S is a union of groups (or disjoint groups). A semigroup S is intra-regular (equivalently, the principal ideals of S constitute a semilattice Y under intersection) if and only if it is a union of simple semigroups. The semiprime subsets of ordered semigroups (groupoids) have been defined in [3] in the same way: A subset T of an ordered groupoid S is called semiprime if for every $A \subseteq S$ such that $A^2 \subseteq T$, we have $A \subseteq T$. Exactly as in semigroups, semiprime subsets of ordered semigroups play an important role in studying the structure of ordered semigroups.

A fuzzy subset f of a semigroup S is called *semiprime* if $f(x) \geq f(x^2)$ for every $x \in S$. This concept has been introduced by N. Kuroki in [9], as he was the first who showed that a nonempty subset A of a semigroup S is semiprime if and only if its characteristic function f_A is fuzzy semiprime [9]. Kehayopulu and Tsingelis were the first who introduced and investigated the fuzzy ordered groupoids in [5]. Following Kuroki, they kept the same definition of semiprime subsets of an ordered groupoid as a fuzzy subset f of S satisfying $f(x) \geq f(x^2)$ for every $x \in S$ [5]. Since then, many papers on semigroups and ordered semigroups appeared adapting this definition as the definition of semiprime fuzzy subsets both for semigroups and ordered semigroups. It might be also noted that a fuzzy subset f of a groupoid S is semiprime if and only if for every $x \in S$ and every $\lambda \in [0,1]$ such that $x_\lambda \circ x_\lambda \preceq f$ implies $x_\lambda \preceq f$ [8].

On the other hand, an element t of a *poe*-groupoid (or *poe*-semigroup) S is called semiprime if for every $a \in S$ such that $a^2 \leq t$, we have $a \leq t$ [2]. And the same definition of semiprime elements is the usual definition for ordered groupoids in general. As this is the case for ordered semigroups, in addition, since the fuzzy subsets of groupoids (resp. ordered semigroups) form an ordered groupoid [4] (resp. ordered semigroup [6, 7]), one should expect that in the theory of fuzzy ordered semigroups (semigroups without order or groupoids) the fuzzy semiprime subset should be defined in a similar way. That is, if S is a groupoid (semigroup or ordered semigroup), then a fuzzy subset f of S should be called fuzzy semiprime if for any fuzzy subset g of S such that $g \circ g \preceq f$, we have $g \preceq f$. A fuzzy subset f of S should be called fuzzy prime if for any two fuzzy subsets g and h of S such that $g \circ h \preceq f$, we have $g \preceq f$ or $h \preceq f$. In spite of that, in the existing bibliography, for an ordered semigroup S, a fuzzy subset f of S is called fuzzy semiprime if $f(x) \geq f(x^2)$ for every $x \in S$; it is called prime if $f(xy) \leq \max\{f(x), f(y)\}$ for all $x, y \in S$, and this is the usual definitions the authors always use. However, Xie and Tang, in Definition 4.1 in [11] call a fuzzy ideal of an ordered semigroup S prime if for any two fuzzy ideals g and h of S, $g \circ h \preceq f$ implies $g \preceq f$ or $h \preceq f$. And Tang in Definition 4.1 in [10] calls a fuzzy subset of an ordered semigroup semiprime if for any fuzzy ideal g of S, $g^2 \preceq f$ implies $g \preceq f$. It is natural to ask what is the relation between these two definitions. The present paper gives the related answer. We prove that if a fuzzy subset f of an ordered groupoid S is semiprime (in the usual sense), then for any fuzzy subset g of S such that $g \circ g \preceq f$, we have $g \preceq f$, and that the converse statement does not hold in general. Moreover, for the ordered groupoids satisfying the condition

$$x \leq yz \Longrightarrow \min\{f(y^2), f(z^2)\} \leq f(x),$$

the two definitions are equivalent. We prove that if a fuzzy subset f of an ordered groupoid S is prime (in the usual sense) and h, g fuzzy subsets of S such that $h \circ g \preceq f$, then this, in general, does not mean that $h \preceq f$ of $g \preceq f$. Conversely, if f is a fuzzy subset of an ordered groupoid and g, h fuzzy subsets of S such that $g \circ h \preceq f$ implies $h \preceq f$ of $g \preceq f$, this does not mean that the fuzzy subset f is fuzzy prime in general.

2. About Semiprime Fuzzy Subsets

Proposition 1. *Let $(S, \cdot, \leq)$ be an ordered groupoid and f a fuzzy subset of S. Then*

$$f(x) \leq (f \circ f)(x^2) \text{ for every } x \in S.$$

Proof. Let $x \in S$. Since $(x, x) \in A_{x^2}$, we have $A_{x^2} \neq \emptyset$ and

$$\begin{aligned}(f \circ f)(x^2) &= \bigvee_{(u,v)\in A_{x^2}} \min\{f(u), f(v)\}\\ &\geqslant \min\{f(x), f(x)\}\\ &= f(x),\end{aligned}$$

so $f(x) \leq (f \circ f)(x^2)$. □

Proposition 2. *Let $(S, \cdot, \leq)$ be an ordered groupoid and f, g fuzzy subsets of S such that $g \circ g \preceq f$. Then*

$$g(x) \leq f(x^2) \text{ for every } x \in S.$$

Proof. Let $x \in S$. Since g is a fuzzy subset of S, by Proposition 1, we have $g(x) \leq (g \circ g)(x^2)$. Since $g \circ g \preceq f$, we have $(g \circ g)(x^2) \leq f(x^2)$. Thus we have $g(x) \leq f(x^2)$. □

Definition 3. [5, 8] If S is an ordered groupoid, a fuzzy subset f of S is called *fuzzy semiprime* if $f(x) \geq f(x^2)$ for every $x \in S$.

Theorem 4. *Let S be an ordered groupoid and f a fuzzy subset of S. We consider the following statements:*
(1) *f is fuzzy semiprime.*
(2) *If g is a fuzzy subset of S such that $g \circ g \preceq f$, then $g \preceq f$.*
Then (1) ⇒ (2). *The implication* (2) ⇒ (1) *does not hold in general.*

Proof. (1) ⟹ (2). Let g be a fuzzy subset of S such that $g \circ g \preceq f$ and $x \in S$. By Proposition 2, we have $g(x) \leq f(x^2)$. Since f is fuzzy semiprime, we have $f(x) \geq f(x^2)$, Then we have $g(x) \leq f(x)$ and (2) holds.

Condition (2) does not always imply (1). In fact:

The set $S = \{n \in N \mid n \geq 2\} = \{2, 3, 4, ...\}$ of natural numbers with the usual multiplication and the usual order is an ordered groupoid (in particular, an ordered semigroup). Let f be the fuzzy subset of S defined by:

$$f \ : \ (S, ., \leq) \ \to \ [0,1] \mid x \to \begin{cases} 0 & \text{if} \quad x = 2 \\ 1 & \text{if} \quad x > 2. \end{cases}$$

Condition (2) *is satisfied.* Indeed: Let g be a fuzzy subset of S such that $g \circ g \preceq f$ and let $x \in S$.

(I) Let $x = 2$. We consider the set $A_2 = \{(m, n) \in S \times S \mid 2 \leq mn\}$. Since $(2, 2) \in A_2$, we have $A_2 \neq \emptyset$ and

$$(g \circ g)(2) = \bigvee_{(m,n) \in A_2} \min\{g(m), g(n)\} \geqslant \min\{g(2), g(2)\} = g(2).$$

Since $g \circ g \preceq f$, we have $(g \circ g)(2) \leq f(2)$, so we have $g(2) \leq f(2)$.

(II) Let $x > 2$. Then $f(x) = 1$. On the other hand, since g is a fuzzy subset of S, we have $g(x) \leq 1$. Thus we have $g(x) \leq f(x)$.
By (I) and (II), we have $g(x) \leq f(x)$ for every $x \in S$, so we have $g \preceq f$.

Condition (1) *does not hold.* In fact, we have $f(2) = 0$ and $f(2.2) = f(4) = 1$, so $f(2) \ngeq f(2^2)$. □

It is natural to ask under what conditions the implication (2) ⇒ (1) is satisfied. The next theorem gives a related answer.

Theorem 5. *Let S be an ordered groupoid and f a fuzzy subset of S such that*
(a) $a \leq xy \Longrightarrow \min\{f(x^2), f(y^2)\} \leq f(a)$ $(a, x, y \in S)$ *and*
(b) *if g is a fuzzy subset of S such that* $g \circ g \preceq f$, *then* $g \preceq f$.
Then f is fuzzy semiprime.

Proof. Let $x \in S$. We consider the fuzzy subset g of S defined by:

$$g : (S, \cdot, \leq) \to [0, 1] \mid x \to f(x^2).$$

We have $g \circ g \preceq f$. In fact: Let $a \in S$. If $A_a = \emptyset$, then $(g \circ g)(a) = 0 \leq f(a)$. If $A_a \neq \emptyset$, then $(g \circ g)(a) = \bigvee_{(x,y) \in A_a} \min\{g(x), g(y)\}$. On the other hand,

$$\min\{g(x), g(y)\} \leq f(a) \text{ for every } (x, y) \in A_a.$$

Indeed, if $(x, y) \in A_a$, then $a \leq xy$ and, by (a), $\min\{f(x^2), f(y^2)\} \leq f(a)$, that is $\min\{g(x), g(y)\} \leq f(a)$. Therefore we have $(g \circ g)(a) \leq f(a)$. This is for every $a \in S$, thus we obtain $g \circ g \preceq f$. By condition (b), we get $g \preceq f$, then $f(x) \geq g(x) = f(x^2)$ for every $x \in S$, so S is fuzzy semiprime. □

Remark 6. Fuzzy semiprime subsets of ordered semigroups do not satisfy the condition (a) of Theorem 5 in general. In fact: Let $S = [0, 1]$ be the ordered semigroup of real numbers with the usual multiplication, order of the real numbers and f the identity mapping on S. Then f is a fuzzy semiprime fuzzy subset of S. Indeed: If $x \in S$ then, since $0 \leq x \leq 1$, we have $x^2 \leq x$, so $f(x) \geq f(x^2)$.
S does not satisfy condition (a). In fact: $\frac{1}{10} \leq \frac{1}{6} = \frac{1}{2} . \frac{1}{3}$, but

$$\begin{aligned} \min\{f((\frac{1}{2})^2), f((\frac{1}{3})^2)\} &= \min\{f(\frac{1}{4}), f(\frac{1}{9})\} = \min\{\frac{1}{4}, \frac{1}{9}\} \\ &= \frac{1}{9} \nleq \frac{1}{10} = f(\frac{1}{10}). \end{aligned}$$

3. About Prime Fuzzy Subsets

Definition 7. [5, 8] If S is an ordered groupoid, a fuzzy subset f of S is called *prime* if

$$f(xy) \leq \max\{f(x), f(y)\} \text{ for every } x, y \in S.$$

Theorem 8. *Let S be an ordered semigroup and f a fuzzy subset of S. We consider the statements:*

(1) *f is fuzzy prime.*
(2) *If h and g are fuzzy subsets of S such that $h \circ g \preceq f$, then $h \preceq f$ or $g \preceq f$.*

Then (1) *does not imply* (2) *in general; and* (2) *does not imply* (1) *in general.*

Proof. Let $S = \{a, b, c\}$ be the ordered semigroup defined by the multiplication " $\cdot$ " and the order " $\leq$ " below:

$\cdot$	a	b	c
a	a	a	a
b	a	a	a
c	a	b	c

$$\leq = \{(a,a), (b,b), (c,c), (a,b)\}$$

Let f, g and h be the fuzzy subsets of S defined by:

$$f : S \to [0,1] \mid x \to f(x) = \begin{cases} \frac{1}{2} & if \;\; x = a \;\; or \;\; x = b \\ 0 & if \;\; x = c \end{cases}$$

$$g : S \to [0,1] \mid x \to g(x) = \begin{cases} \frac{1}{2} & if \;\; x = a \\ 1 & if \;\; x = b \\ 0 & if \;\; x = c \end{cases}$$

$$h : S \to [0,1] \mid x \to h(x) = \tfrac{1}{2}$$

Then f is a fuzzy prime subset of S since $f(xy) \leq \max\{f(x), f(y)\}$ for all $x, y \in S$. Moreover, $(g \circ h)(x) \leq f(x)$ for all $x \in S$. Indeed: If $x = a$, then

$$\begin{aligned} A_a : &= \{(x,y) \in S \times S \mid a \leq xy\} \\ &= \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b)\}, \end{aligned}$$

and

$$(g \circ h)(a) = \bigvee_{(y,z) \in A_a} \min\{g(y), h(z)\} = \sup\{0, \frac{1}{2}\} = \frac{1}{2} \leq f(a).$$

If $x = b$, then
$A_b := \{(x,y) \in S \times S \mid a \leq xy\} = \{(c,b)\}$, and

$$(g \circ h)(b) = \bigvee_{(y,z) \in A_b} \min\{g(y), h(z)\} = \sup\{0\} = 0 \leq \frac{1}{2} = f(b).$$

If $x = c$, then $A_c := \{(c,c)\}$, and

$$(g \circ h)(b) = \bigvee_{(y,z) \in A_c} \min\{g(y), h(z)\} = \sup\{0\} = 0 \leq \frac{1}{2} = f(c).$$

Thus $f \circ g \preceq f$, but $g \npreceq f$ since $g(b) = 1 \nleq \frac{1}{2} = f(b)$ and $h \npreceq f$ since $h(c) = \frac{1}{2} \nleq 0 = f(c)$.

Condition (2) does not imply condition (1) in general. In fact:
We consider the set $S = \{n \in N \mid n \geq 2\}$ and the fuzzy subset f of S defined by $f(x) = 0$ if $x = 2$ and $f(x) = 1$

if $x > 2$ (considered in the proof of Theorem 4). Let g and h be fuzzy subsets of S such that $g \circ h \preceq f$. Then $g \preceq f$ or $h \preceq f$. Indeed: If $g(2) \le h(2)$, then

$$(g \circ h)(2) = \bigvee_{(m,n)\in A_2} \min\{g(m), h(n)\} \ge \min\{g(2), h(2)\} = g(2),$$

so $(g \circ h)(2) \ge g(2)$. Since $g \circ h \preceq f$, we have $(g \circ h)(2) \le f(2)$, thus we have $g(2) \le f(2)$. If $x > 2$, then $f(x) = 1$. Since g is a fuzzy subset of S, we have $g(x) \le 1$, so we have $g(x) \le f(x)$. Since $g(x) \le f(x)$ for every $x \in S$, we have $g \preceq f$. If $h(2) \le g(2)$, in a similar way, we get $h \preceq f$. So condition (2) of the theorem is satisfied. On the other hand, $f(2.2) = f(4) = 1$ and $\min\{f(2), f(2)\} = 0$, so $f(2.2) \nleq \min\{f(2), f(2)\}$, and condition (2) does not hold. □

4. Conclusion

As a conclusion, let us give the two definitions below: Definitions 1 and 3 are the definitions in the existing bibliography we always use. The Definitions 2 and 4 are based on the definition of semiprime subsets (or semiprime ideal elements) of ordered groupoids. The definitions which actually should be.
In the following, S denotes an ordered groupoid and $g^2 := g \circ g$.

Definition 1. A fuzzy subset f of S is called *fuzzy semiprime* if

$$f(x) \ge f(x^2)$$

for every $x \in S$.

Definition 2. A fuzzy subset f of S is called *fuzzy semiprime* if

For any fuzzy subset g of S such that $g^2 \preceq f$, we have $g \preceq f$.

Then Definition 1 implies Definition 2, but Definition 2 does not imply Definition 1 in general. In particular, if the fuzzy subset f of S has the property

$$a \le xy \implies \min\{f(x^2), f(y^2)\} \le f(a),$$

then the two definitions are equivalent.

Definition 3. A fuzzy subset f of S is called *fuzzy prime* if

$$f(xy) \le \max\{f(x), f(y)\}$$

for every $x, y \in S$.

Definition 4. A fuzzy subset f of S is called *fuzzy prime* if

For any fuzzy subsets g, h of S such that $g \circ h \preceq f$, we have $g \preceq f$ or $h \preceq f$.

The Definition 3 does not imply Definition 4 and the Definition 4 does not imply the Definition 3 in general.

If we consider a groupoid instead of an ordered groupoid, then again the Definitions 1 and 3 and the Definitions 2 and 4 are not equivalent.

Problem. Find conditions under which the definitions 3 and 4 are equivalent.

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