Filomat 31:13 (2017), 4297–4304
https://doi.org/10.2298/FIL1713297A

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Graphs with Large Total Geodetic Number

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**Abstract.** For two vertices $u$ and $v$ of a graph $G$, the set $I[u, v]$ consists of all vertices lying on some $u - v$ geodesic in $G$. If $S$ is a set of vertices of $G$, then $I[S]$ is the union of all sets $I[u, v]$ for $u, v \in S$. A set of vertices $S \subseteq V(G)$ is a *total geodetic set* if $I[S] = V(G)$ and the subgraph $G[S]$ induced by $S$ has no isolated vertex. The *total geodetic number*, denoted by $g_t(G)$, is the minimum cardinality among all total geodetic sets of $G$. In this paper, we characterize all connected graphs $G$ of order $n \geq 3$ with $g_t(G) = n - 1$.

## 1. Introduction

In this paper, we continue the study of total geodetic number in graphs. For notation and graph theory terminology, we follow West [20]. Specifically, let $G$ be a simple connected graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the order and the size of the graph $G$, respectively. For two vertices $x$ and $y$ in a (connected) graph $G$, the *distance* $d_G(x, y)$ is the length of a shortest $x - y$ path in $G$. The *girth* of a graph $G$, denoted by *girth*$(G)$, is the length of its shortest cycle. The girth of a graph with no cycle is defined to be 0. The *join* $H \vee K$ of two disjoint graphs $H$ and $K$ is the graph obtained from their union by adding new edges joining each vertex of $V(H)$ to every vertex of $V(K)$. For a vertex $x$ of $G$, the *eccentricity* $e_G(x)$ is the distance between $x$ and a vertex farthest from $x$. The minimum eccentricity among the vertices of $G$ is the *radius*, *rad*$(G)$, and the maximum eccentricity is its *diameter*, *diam*$(G)$. A $x - y$ path of length $d_G(x, y)$ is called a $x - y$ *geodesic*. The *geodetic interval* $I[x, y]$ is the set consisting of $x$, $y$ and all vertices lying in some $x - y$ geodesic of $G$, and for a nonempty subset $S$ of $V(G)$, we define $I[S] = \cup_{x,y \in S} I[x, y]$.

A subset $S$ of vertices of $G$ is a *geodetic set* (or just GS) if $I[S] = V$. The *geodetic number* $g(G)$ is the minimum cardinality of a geodetic set of $G$. A $g(G)$*-set* is a geodetic set of $G$ of size $g(G)$. The geodetic sets of a connected graph were introduced by Harary, Loukakis and Tsouros [14], as a tool for studying metric properties of connected graphs. It was shown in [7] that the determination of $g(G)$ is an NP-hard problem and its decision problem is NP-complete. The geodetic number and its variants have been studied by several authors (see for example [1, 2, 6, 7, 9–13, 15–19, 21]).

A set of vertices $S \subseteq V(G)$ is a *total geodetic set* (or just TGS) if $I[S] = V(G)$ and the subgraph $G[S]$ induced by $S$ has no isolated vertex. The minimum cardinality among all total geodetic sets of $G$ is called the *total geodetic number* and is denoted by $g_t(G)$. The total geodetic number of a connected graph was introduced by Abdollahzadeh Ahangar and Samodivkin in [5]. Very recently, Abdollahzadeh Ahangar and Najimi

2010 *Mathematics Subject Classification*. 05C12.

*Keywords*. Geodetic set, Geodetic number, Total geosetic set, Total geodetic number
Received: 01 January 2016; Accepted: 24 July 2017
Communicated by Francesco Belardo

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introduced the concept of total restrained geodetic number in [3], and continued the study on this new parameter by Abdollahzadeh Ahangar et al. in [4].

A vertex of $G$ is *simplicial* if the subgraph induced by its neighborhood is complete. In particular, every endvertex is simplicial. The set of all simplicial vertices of a graph $G$ is denoted by $Ext(G)$. A vertex of $G$ is a *stem* if it is adjacent to an endvertex. The sets of all endvertices and all stems are denoted by $L(G)$ and $Stem(G)$, respectively. In $K_2$ a vertex is both an endvertex and a stem. A vertex $v$ in $G$ is said to be a *$g_t$-fixed vertex* of $G$ if $v$ belongs to every $g_t(G)$-set. The set of all $g_t$-fixed vertices of a graph $G$ is denoted by $Fi_t(G)$.

The main aim of this paper is to characterize all connected graphs $G$ of order $n \geq 3$ with $g_t(G) = n - 1$.

We make use of the following results in this paper.

**Theorem 1.1.** ([8]) Let $G$ be a connected graph of order $n \geq 3$. Then $g(G) = n - 1$ if and only if $G$ is the join of $K_1$ and pairwise disjoint complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_r}$, that is, $G = (K_{n_1} \cup K_{n_2} \cup \ldots K_{n_r}) \vee K_1$, where $r \geq 2$, $n_1, n_2, \ldots, n_r$ are positive integers with $n_1 + n_2 + \ldots + n_r = n - 1$.

**Proposition 1.2.** ([5]) For $n \geq 3$, $g_t(C_n) = 4$ when $n \geq 5$ and $g_t(C_n) = 3$ when $n \in \{3, 4\}$.

**Observation 1.3.** ([5]) Let $G$ be a connected graph of order $n \geq 2$. Then $Stem(G) \cup Ext(G) \subseteq Fi_t(G)$. In particular, $g_t(G) \geq |Ext(G)| + |Stem(G)|$.

**Proposition 1.4.** Let $G$ be a connected graph of order $n$. Then $g_t(G) = n$ if and only if every vertex of $G$ is simplicial or a stem.

*Proof.* The sufficiency follows from Observation 1.3. To prove the necessity, assume that $g_t(G) = n$. If $G$ has a vertex $v$ that is not simplicial or a stem, then obviously $V(G) - \{v\}$ is a total geodetic set of $G$ which leads to a contradiction. Therefore, every vertex of $G$ is simplicial or a stem. □

## 2. Upper bounds on the total geodetic number

In this section we give several sufficient conditions for a graph of order $n$ to have total geodetic number at most $n - 2$.

A *cut-vertex* is a vertex whose deletion results in a graph with more components than the original graph. The set of all cut-vertices of $G$, is denoted by $Cut(G)$.

**Proposition 2.1.** Let $G$ be a connected graph of order $n$. If there exist two vertices $u_1, u_2 \in Cut(G) - Stem(G)$ with $d(u_1, u_2) \geq 2$, then $g_t(G) \leq n - 2$.

*Proof.* By the assumption, each component of $G - u_1$ and $G - u_2$ have order at least 2. Let $I$ be the set of all isolated vertices of $G - \{u_1, u_2\}$. If $I = \emptyset$, then clearly $V(G) - \{u_1, u_2\}$ is a total geodetic set of $G$ that implies $g_t(G) \leq n - 2$. If $|I| \geq 2$, then clearly $V(G) - I$ is a total geodetic set of $G$ implying that $g_t(G) \leq n - 2$. Assume now that $I = \{w\}$. It follows that $\deg(w) = 2$ and $w$ is adjacent to $u_1$ and $u_2$. Suppose $G_1$ is a component of $G - u_1$ not containing $u_2$ and $G_2$ is a component of $G - u_2$ not containing $u_1$. Let $w_1 \in V(G_1)$ and $w_2 \in V(G_2)$. Then clearly $|V(G_i)| \geq 2$ for $i = 1, 2$ and every $w_1 - w_2$ geodesic contains $u, v$ and $w$. It follows that $V(G) - \{u, v, w\}$ is a total geodetic set of $G$ yielding $g_t(G) \leq n - 3$. This completes the proof. □

Next result is an immediate consequence of Proposition 2.1.

**Corollary 2.2.** If $G$ is a graph of order $n$ with $g_t(G) = n - 1$, then the induced subgraph $G[\mathrm{Cut}(G) - \mathrm{Stem}(G)]$ is complete.

A *vertex cut* of a connected graph $G$ is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. The *connectivity* of $G$, written $\kappa(G)$, is the minimum size of a vertex set $S$ such that $G - S$ is disconnected or has only one vertex.

**Proposition 2.3.** For any connected graph $G$ of order $n$ different from $K_n$,

$$g_t(G) \leq n - \kappa(G) + 1.$$

*Proof.* We may assume that $\kappa(G) \geq 2$, otherwise the result is immediate. Let $S = \{v_1, v_2, \ldots, v_{\kappa(G)}\}$ be a vertex cut of $G$ and let $G_1, G_2, \ldots, G_l$; $(l \geq 2)$ be the components of $G - S$. For each $v_i \in S$, there is a vertex $u_{i_j} \in G_j$ such that $v_i u_{i_j} \in E(G)$, otherwise $S - \{v_i\}$ is a vertex cut of $G$, which is a contradiction.

If $|V(G_i)| \geq 2$ for each $i$, then clearly $V(G) - S$ is a total geodetic set of $G$ and hence $g_t(G) \leq n - \kappa(G)$. Assume that $|V(G_i)| = 1$ for some $i$, say $i = 1$, and let $V(G_1) = \{w\}$. Then $w$ must be adjacent to all vertices of $S$. It is easy to verify that $V(G) - \{v_2, v_3, \ldots, v_{\kappa(G)}\}$ is a total geodetic set of $G$ implying that $g_t(G) \leq n - \kappa(G) + 1$. This completes the proof. □

An immediate consequence of Proposition 2.3 now follows.

**Corollary 2.4.** For any connected graph $G$ of order $n$ with $g_t(G) = n - 1$, $\kappa(G) \leq 2$.

**Proposition 2.5.** Let $k$ be a positive integer and $G$ a connected graph of order $n$ different from $K_n$. If $\delta(G) \geq k + 1$ and $G$ has at least $k$ vertices of degree $n - 1$, then $g_t(G) \leq n - k$.

*Proof.* Let $S$ be a set of vertices of degree $n-1$ and size $k$. Since $G \neq K_n$, $G$ has two non-adjacent vertices $u$ and $v$ and all vertices of $S$ lay on some $u - v$ geodesic. On the other hand, we deduce from $\delta(G) \geq k+1$ that $G - S$ has no isolated vertex. It follows that $V(G) - S$ is a TGS of $G$ which implies that $g_t(G) \leq |V(G) - S| \leq n - k$. □

**Proposition 2.6.** Let $G$ be a graph of order $n$ with $\delta(G) \geq 3$. If $G$ has two non-adjacent vertices $u, v$ such that $\deg(u) + \deg(v) \geq n$, then $g_t(G) \leq n - 2$.

*Proof.* Since $\deg(u) + \deg(v) \geq n$, we have $|N(u) \cap N(v)| \geq 2$. Suppose that $\{w_1, w_2\} \subseteq N(u) \cap N(v)$. It follows from $\delta(G) \geq 3$ that $V(G) - \{w_1, w_2\}$ is a TGS of $G$ which implies that $g_t(G) \leq n - 2$, as desired. □

Next results are immediate consequence of Proposition 2.6.

**Corollary 2.7.** Let $G$ be a graph of order $n \geq 5$ different from $K_n$. If $\delta(G) \geq \lceil \frac{n}{2} \rceil$, then $g_t(G) \leq n - 2$.

**Corollary 2.8.** For any graph $G$ of order $n \geq 5$ with $g_t(G) = n - 1$, $\delta(G) \leq \lfloor \frac{n}{2} \rfloor$.

## 3. Graphs with large geodetic number

In this section, we classify all connected graphs of order $n$ whose total geodetic number is $n - 1$.

**Theorem 3.1.** Let $G$ be a connected graph of order $n$ and let $G' = G - (\text{Ext}(G) \cup \text{Stem}(G))$. Then $g_t(G) = n - 1$ if and only if one of the following statements hold:

(i) $G = C_4$ or $G = C_5$.

(ii) $G'$ is a complete graph of order at least two and every pair of vertices of $G'$ has a common neighbor of degree 2.

(iii) $G'$ is $K_2$ with vertex set $\{x_1, x_2\}$, at least one of induced subgraphs $G[N(x_1) - \{x_2\}]$ and $G[N(x_2) - \{x_1\}]$ is a complete graph and $d(y_1, y_2) \leq 2$ for every vertex $y_1 \in N(x_1) - N(x_2)$ and $y_2 \in N(x_2) - N(x_1)$.

(iv) $G'$ is $K_3$ with vertex set $\{x_1, x_2, x_3\}$, every pair $x_i$ and $x_j$ but one, say $x_2, x_3$, have a common neighbor $w_{i,j}$ of degree two for $1 \leq i, j \leq 3$, $\min\{\deg(x_2), \deg(x_3)\} = 3$, and if $\deg(x_i) \geq 4$ for some $i \in \{2, 3\}$, then $N(x_i) - \{x_1\} \subseteq N(x_1)$.

(v) $G'$ is $P_3 = x_1 x_2 x_3$, $\deg(x_2) = 2$ and there is no vertex $y$ for which $yx_1 \in E(G)$ and $d(y, x_3) = 3$ or $yx_3 \in E(G)$ and $d(y, x_1) = 3$.

(vi) $G'$ is $C_4 = (x_1 x_2 x_3 x_4)$, two consecutive vertices of $C_4$, say $x_3$ and $x_4$ have degree two, $N[x_1] - \{x_4\} = N[x_2] - \{x_3\}$, and $x_1, x_2$ have a common neighbor of degree 2.

**Proof.** If $G = C_4$ or $G = C_5$, then clearly $g_t(G) = n - 1$, by Proposition 1.2. If $G$ satisfies (ii) and $V(G) = \{x_1, x_2, \ldots, x_\ell\}$, then every $g_t(G)$-set must contain all $x_i$ but one. It follows from Observation 1.3 and Proposition 1.4 that $g_t(G) = n - 1$. If $G$ satisfies (iii), then by Proposition 1.4, we have $g_t(G) \leq n - 1$. We may assume that $G[N(x_2) - \{x_1\}]$ is a complete graph. It follows that $x_2$ only lies on $y_1x_2x_1$ geodesic where $y_1 \in N(x_2)$, since $d(y_1, y_2) \leq 2$ for each $y_2 \in N(x_1)$. This implies that every $g_t(G)$-set contains $x_1$ or $x_2$ and so $g_t(G) = n - 1$.

Let $G$ satisfy (iv). By Observation 1.3, $\mathrm{Ext}(G) \cup \mathrm{Stem}(G) \subseteq S$. Since $x_1$ and $x_2$ have a common neighbor of degree two, we have $|S \cap \{x_1, x_2\}| \geq 1$. Similarly $|S \cap \{x_1, x_3\}| \geq 1$. If $x_1 \notin S$, then $x_2, x_3 \in S$ and the result follows as above. Let $x_1 \in S$. By the assumptions, we may assume that $\deg(x_2) = 3$. If $\deg(x_3) = 3$, then clearly $S \cap \{x_2, x_3\} \neq \emptyset$ which implies that $g_t(G) = |S| = n - 1$. Suppose $\deg(x_3) \geq 4$. Then it follows from $N(x_3) - \{x_1\} \subseteq N(x_1)$ that $x_2 \in S$, which implies that $g_t(G) = |S| \geq n - 1$ by Proposition 1.4. Thus $g_t(G) = n - 1$.

Now let $G$ satisfy (v). Assume that $S$ is a $g_t(G)$-set. By Observation 1.3, we have $Ext(G) \cup Stem(G) \subseteq S$. If $x_2 \in S$, then since $G[S]$ has no isolated vertex, we must have $x_1 \in S$ or $x_3 \in S$. So $|S| \geq n - 1$ and it follows from Proposition 1.4 that $g_t(G) = |S| = n - 1$. Hence, we assume $x_2 \notin S$. If $x_1 \notin S$ (the case $x_3 \notin S$ is similar), then we must have a geodetic path $x_3x_2x_1u$, which is a contradiction with the assumptions. Thus $x_1, x_3 \in S$ and so $g_t(G) = n - 1$.

Finally, let $G$ satisfy (vi). If $G = C_4$, then clearly $g_t(G) = n - 1$, by Proposition 1.2. Let $n \geq 5$, two consecutive vertices of $C_4$, say $x_3$ and $x_4$, have degree two, $N[x_1] - \{x_4\} = N[x_2] - \{x_3\}$, and $x_1, x_2$ have a common neighbor $w$ of degree 2. Let $S$ be a $g_t$-set of $G$. Since $w \in S$, we may assume that $x_1 \in S$. If $x_3, x_4 \in S$, then $|S| \geq n - 1$ and we deduce from Proposition 1.4 that $g_t(G) = |S| = n - 1$. Let $|S \cap \{x_3, x_4\}| = 1$. If $x_4 \notin S$, then clearly $x_3 \in S$ and since $\deg(x_3) = 2$ and $S$ is a TGS, we conclude that $x_2 \in S$. Hence $|S| \geq n - 1$ which implies that $g_t(G) = |S| = n - 1$. Let $x_4 \in S$. If $S \cap \{x_2, x_3\} = \emptyset$, then there is a geodetic path $ux_2x_3x_4$ in $G$ which leads to which is a contradiction because $d_G(u, x_4) = 2$. Therefore $x_2 \in S$ and so $g_t(G) = |S| = n - 1$.

Conversely, let $g_t(G) = n - 1$ and let $G - (\mathrm{Ext}(G) \cup \mathrm{Stem}(G)) = G'$. We proceed with some claims:

**Claim 1:** $\delta(G) = 1$ or $\mathrm{diam}(G) \leq 3$.

*Proof.* Suppose, to the contrary, that $\delta(G) \geq 2$ and $\mathrm{diam}(G) \geq 4$. Let $P = v_1, v_2, \ldots, v_k$ be a diametral path in $G$. Clearly $P$ is a $v_1 - v_k$ geodesic path. If $v_i, v_{i+1}$ have no common neighbor of degree 2 for some $2 \leq i \leq k-2$, then clearly $V(G) - \{v_i, v_{i+1}\}$ is a total geodetic set of $G$ implying that $g_t(G) \leq n - 2$ which is a contradiction. Assume that $v_i$ and $v_{i+1}$ have at least a common neighbor of degree 2. If $v_i$ and $v_{i+2}$ have no common neighbor of degree 2 for some $2 \leq i \leq k - 3$, then clearly $V(G) - \{v_i, v_{i+2}\}$ is a total geodetic set of $G$ which implies that $g_t(G) \leq n - 2$, which is a contradiction. Assume $w$ is a common neighbor of $v_2$ and $v_4$ of degree 2. Then it is easy to see that $V(G) - \{w, v_3\}$ is a total geodetic set of $G$ and hence $g_t(G) \leq n - 2$, a contradiction, as well. This proves Claim 1. □

**Claim 2:** If $n \geq 6$ and $\delta(G) \geq 2$, then every induced cycle of $G$ has length at most 4.

*Proof.* Suppose, to the contrary, that $C = (v_1v_2 \ldots v_k)$ is an induced cycle in $G$ with $k \geq 5$. If $n = k$, then $V(G) - \{v_1, v_2\}$ is a TGS of $G$ implying that $g_t(G) \leq n - 2$, which is a contradiction. Suppose that $n \geq k + 1$. Since $G$ is connected, we may assume $uv_1 \in E(G)$ for some vertex $u \notin V(C)$. If $v_2$ and $v_k$ have a common neighbor $w$ of degree 2, then obviously $V(G) - \{v_1, w\}$ is a total geodetic set of $G$ implying that $g_t(G) \leq n - 2$, which is a contradiction again. Otherwise, $V(G) - \{v_2, v_k\}$ is a total geodetic set of $G$ that implies $g_t(G) \leq n - 2$, a contradiction, as well. This proves Claim 2. □

**Claim 3:** If $G \neq C_5$ has an induced $k$-cycle $C = (v_1v_2 \ldots v_k)$ where $k \geq 5$, then $v_i, v_{i+1} \in Stem(G)$ for some $1 \leq i \leq k$.

*Proof.* Suppose, to the contrary, that no two consecutive vertices of $C$ do not belong to $\mathrm{Stem}(G)$. Since $g_t(G) = n-1$, we must have $G \neq C$. By the assumptions, we have $|\mathrm{Stem}(G) \cap V(C)| \leq \lfloor \frac{k}{2} \rfloor$. If $\mathrm{Stem}(G) \cap V(C) = \emptyset$, then an argument similar to that described in the proof of Claim 2 leads to $g_t(G) \leq n - 2$, which is a contradiction. Assume that $\mathrm{Stem}(G) \cap V(C) \neq \emptyset$ and $v_1 \in \mathrm{Stem}(G)$. Let $S$ be the set consisting of all common neighbors of $v_2$ and $v_k$ which has degree 2. If $|S| \geq 2$, then $V(G) - S$ is a total geodetic set of $G$ that leads to a contradiction. If $S = \emptyset$, then obviously $V(G) - \{v_2, v_k\}$ is a total geodetic set of $G$ and so $g_t(G) \leq n-2$, which is a contradiction. Assume that $S = \{w\}$. If $v_3 \notin \mathrm{Stem}(G)$, then obviously $V(G) - \{w, v_3\}$ is a total geodetic set of $G$ and hence $g_t(G) \leq n-2$, a contradiction. Let $v_3 \in \mathrm{Stem}(G)$. Then $v_4 \notin \mathrm{Stem}(G)$. By repeating the above argument, we may assume that $v_2$ and $v_4$ have exactly one common neighbor of degree 2, say $w'$. It is easy to see that $V(G) - \{w, w'\}$ is a TGS of $G$ and hence $g_t(G) \leq n-2$, which is a contradiction. This proves Claim 3. □

**Claim 4:** $G - (\mathrm{Ext}(G) \cup \mathrm{Stem}(G))$ is a connected graph with diameter at most two.

*Proof.* First we show that $G' = G - (\mathrm{Ext}(G) \cup \mathrm{Stem}(G))$ is connected. Suppose, to the contrary, that $G'$ is disconnected and let $G_1$ and $G_2$ be two components of $G'$. Let $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ such that $d_G(u_1, u_2) = d_G(V(G_1), V(G_2))$. Since $v_i$ is not simplicial, $v_i$ has two neighbors $w_{i_1}$ and $w_{i_2}$ such that $w_{i_1}w_{i_2} \notin E(G)$, for $i = 1, 2$. Since $v_i \notin \mathrm{Stem}(G)$, every neighbor of $v_i$ has degree at least two and so $V(G) - \{v_1, v_2\}$ is a total geodetic set of $G$, which is a contradiction.

Now, we show that $\mathrm{diam}(G') \leq 2$. Assume, to the contrary, that $\mathrm{diam}(G') \geq 3$. Let $uw_1w_2 \ldots w_kv$ $(k \geq 2)$ be a diametral path in $G'$. Since $u$ and $v$ are not stems and simplicial, we can see that $V(G) - \{u, v\}$ is a total geodetic set of $G$, which is a contradiction. This proves Claim 4. □

**Claim 5:** $G' = G - (\mathrm{Ext}(G) \cup \mathrm{Stem}(G))$ is $P_3, C_4, C_5$ or a complete graph.

*Proof.* By Claim 5, $G'$ is a connected graph with diameter $\mathrm{diam}(G') \leq 2$. If $\mathrm{diam}(G') = 1$, then $G'$ is a complete graph, and we are done. Let $\mathrm{diam}(G') = 2$. It follows that $girth(G') \leq 5$. First let $girth(G') = 0$. Then $G'$ is a tree. Since $\mathrm{diam}(G') = 2$, $G'$ is a star $K_{1,r}$ $(r \geq 2)$. If $r \geq 3$ and $u, v$ are two leaves of $G'$, then clearly $V(G) - \{u, v\}$ is a total geodetic set of $G$, which is a contradiction. Thus $r = 2$ which implies that $G' = P_3$. Now let $girth(G') > 0$ and $C = (x_1x_2 \ldots x_t)$ be a cycle of $G'$ with $t = girth(G')$. If $t = 5$, then we deduce from Claim 3 that $G = C_5$. Let $t \leq 4$. We consider two cases.

**Case 1.** $t = 4$.
We claim that $|V(G')| = 4$. Suppose, to the contrary, that $|V(G')| \geq 5$. Since $G'$ is connected, there is a vertex $u_1 \in V(G') - V(C)$ such that $u_1x_i \in E(G')$ for some $i$, say $i = 1$. From $G'$ being triangle-free, we have $u_1x_2, u_1x_4 \notin E(G)$. If there is path $x_1u_1 \ldots u_kx_3$ in $G$ and $x_2, x_4$ have no common neighbor of degree two, then $V(G) - \{x_2, x_4\}$ is a TGS of $G$, which is a contradiction. If there is path $x_1u_1 \ldots u_kx_3$ in $G$ and $x_2, x_4$ have a common neighbor $w$ of degree two, then $V(G) - \{w, x_1\}$ is a TGS of $G$, which is a contradiction. Assume that there is no such a path in $G$ and hence $u_1x_3 \notin E(G)$. It follows from $u_1 \notin \mathrm{Ext}(G) \cup \mathrm{Stem}(G)$ that $V(G) - \{u_1, x_3\}$ is a TGS of $G$ which is a contradiction. Therefore $V(G') = 4$ and so $G' = C_4$, as desired.

**Case 2.** $t = 3$.
Let $\{x_1, x_2, \ldots, x_k\}$ be the vertex set of a largest clique in $G'$. Since $G'$ is a connected graph with $\mathrm{diam}(G') = 2$, there is a vertex $u_1 \in V(G') - \{x_1, x_2, \ldots, x_k\}$ such that $u_1x_i \in E(G')$ for some $i$. By the choice of $\{x_1, x_2, \ldots, x_k\}$, $u_1$ is not adjacent to all $x_i$. Assume without loss of generality that $\{x_1, x_2, \ldots, x_k\} \cap N(u_1) = \{x_1, x_2, \ldots, x_r\}$ where $r < k$. Since $u_1$ is not a simplicial vertex, $u_1$ has two non-adjacent neighbors $u'$ and $u''$. We may assume without loss of generality that $u' \notin \{x_1, x_2, \ldots, x_r\}$. Consider two subcases.

**Subcase 2.1.** $r \geq 2$.
If $x_i$ and $x_j$ have no common neighbor of degree two for some $1 \leq i \neq j \leq r$, then $V(G) - \{x_i, x_j\}$ is a TGS of $G$, which is a contradiction. Hence, we assume $x_i$ and $x_j$ have a common neighbor $w_{i,j}$ of degree two for each $1 \leq i \neq j \leq r$. If $u_1$ and $x_i$ have a common neighbor $w$ of degree two for some $r+1 \leq i \leq k$, then $V(G) - \{w, x_1\}$ is a TGS of $G$, which is a contradiction. Let $u_1$ and $x_i$ have no common neighbor of degree two for each $r+1 \leq i \leq k$. Then $V(G) - \{u_1, x_k\}$ is a TGS of $G$ which is a contradiction.

**Subcase 2.2.** $r = 1$.
If $u_1, x_i$ has a common neighbor $w$ of degree 2 and $u_1, x_j$ has a common neighbor $w'$ of degree 2 for some $2 \leq i, j \leq k$ (possibly $i = j$), then $V(G) - \{w_i, w_j\}$ is a TGS of $G$, which is a contradiction. If $u_1$ has exactly one common neighbor $w$ of degree two with some $x_2, x_3, \ldots, x_k$, say $w \in N(u_1) \cap N(x_2)$, then $V(G) - \{w, x_k\}$ is a TGS of $G$, which is a contradiction. Henceforth, we assume that $u_1$ has no common neighbor of degree two with $x_2, x_3, \ldots, x_k$. Then $V(G) - \{u_1, x_3\}$ is a TGS of $G$, which is a contradiction. This completes the proof. □

**Claim 6:** If $G' \cong K_2$ and $V(G) = \{x_1, x_2\}$, then one of the following statements hold:

**(a)** there exists a vertex $w$ with $N(w) = \{x_1, x_2\}$,

**(b)** at least one of induced subgraphs $G[N(x_1) - \{x_2\}]$ and $G[N(x_2) - \{x_1\}]$ is a complete graph and $d(y_1, y_2) \leq 2$ for every vertex $y_1 \in N(x_1) - N(x_2)$ and $y_2 \in N(x_2) - N(x_1)$.

*Proof.* If (a) holds, there is nothing to prove. Assume that (a) does not hold. If both of induced subgraphs $G[N(x_1) - \{x_2\}]$ and $G[N(x_2) - \{x_1\}]$ are not complete, then $V(G) - \{x_1, x_2\}$ is a TGS of $G$, which is a contradiction. Hence, at least one of the induced subgraphs $G[N(x_1) - \{x_2\}]$ and $G[N(x_2) - \{x_1\}]$ is complete. If $d(y_1, y_2) \geq 3$ for some $y_1 \in N(x_1) - N(x_2)$ and $y_2 \in N(x_2) - N(x_1)$, then $V(G) - \{x_1, x_2\}$ is a TGS of $G$, which is a contradiction. Thus (b) holds and the proof is complete. □

**Claim 7:** If $G' \cong K_3$ and $V(K_3) = \{x_1, x_2, x_3\}$, then one of the following holds:

**(a)** $x_i$ and $x_j$ have a common neighbor of degree two for each $1 \leq i, j \leq 3$,

**(b)** every pair $x_i$ and $x_j$ but one, say $x_2$ and $x_3$, have a common neighbor $w_{i,j}$ of degree two for $1 \leq i, j \leq 3$, $\min\{\deg(x_2), \deg(x_3)\} = 3$, and if $\deg(x_i) \geq 4$ for some $i \in \{2, 3\}$, then $N(x_i) - \{x_1\} \subseteq N(x_1)$.

*Proof.* Let (a) does not hold. The proof is achieved by means of contradiction. Assume first that $x_i$ and $x_j$ have no common neighbor of degree 2 for each $1 \leq i, j \leq 3$. If $N(x_1) - \{x_2\} \subseteq N(x_2)$, then since $x_1$ is not a simplicial vertex, $V(G) - \{x_1, x_2\}$ is a TGS of $G$, which is a contradiction. Assume that $x_1^1$ is a neighbor of $x_1$ such that $x_1^1 \notin N(x_2)$. If $x_1^1 x_3 \in E(G)$, then clearly $V(G) - \{x_1, x_3\}$ is a TGS of $G$, which is a contradiction. Suppose that $x_1^1 x_3 \notin E(G)$. If $N(x_2) - \{x_3\} \subseteq N(x_3)$, then $V(G) - \{x_2, x_3\}$ is a TGS of $G$, which is a contradiction. Let $x_2^1 \in N(x_2) \setminus N(x_3)$. Then $V(G) - \{x_1, x_2\}$ is a TGS of $G$ which is a contradiction.

Now let $x_1, x_3$ and $x_2, x_3$ have no common neighbor of degree two and $x_1, x_2$ have a common neighbor $w_{1,2}$ of degree two. If $N(x_3) - \{x_2\} \subseteq N(x_2)$, since $x_3$ is not a simplicial vertex, then $V(G) - \{x_2, x_3\}$ is a TGS of $G$, which is a contradiction. Assume that $x_3^1$ is a neighbor of $x_3$ such that $x_3^1 \notin N(x_2)$. If $x_3^1 x_1 \in E(G)$, then obviously $V(G) - \{x_1, x_3\}$ is a TGS of $G$, which is a contradiction. If $x_3^1 x_1 \notin E(G)$, then $d_G(x_1, x_3^1) = 3$ which implies that $V(G) - \{x_1, x_3\}$ is a TGS of $G$, which is a contradiction. Thus every pair $x_i$ and $x_j$ but one, say $x_2, x_3$, have a common neighbor $w_{i,j}$ of degree two for $1 \leq i, j \leq 3$.

If $\min\{\deg(x_2), \deg(x_3)\} \geq 4$, then clearly $V(G) - \{x_2, x_3\}$ is a TGS of $G$ which is a contradiction. Hence, $\min\{\deg(x_2), \deg(x_3)\} = 3$. Assume that $\deg(x_2) = 3$. Finally, if $N(x_3) \nsubseteq N(x_1)$ and $x_3^1 \in N(x_3) \setminus N(x_1)$, then $d_G(x_3, w_{1,2}) = 2$ and $V(G) - \{x_2, x_3\}$ is a TGS of $G$, which is a contradiction. So $G$ satisfies (b). □

**Claim 8:** If $G' \cong K_l$ ($l \geq 4$) and $V(K_l) = \{x_1, x_2, \ldots, x_l\}$, then $x_i$ and $x_j$ have a common neighbor of degree two for each $1 \leq i, j \leq l$.

*Proof.* Assume, to the contrary, that $x_i$ and $x_j$ have no common neighbor of degree two for some $1 \leq i, j \leq l$, say $i = 1, j = 2$. Let $w_1$ be a common neighbor of $x_1, x_3$ and $w_2$ be a common neighbor of $x_2, x_4$. Then $d_G(w_1, w_2) = 3$ which implies that $V(G) - \{x_1, x_2\}$ is a TGS of $G$, which is a contradiction. This completes the proof. □

**Claim 9:** If $G' \cong x_1x_2x_3$, then $\deg(x_2) = 2$ and there is no vertex $y$ for which $yx_1 \in E(G)$ and $d(y, x_3) = 3$ or $yx_3 \in E(G)$ and $d(y, x_1) = 3$.

*Proof.* Let $G - (\text{Ext}(G) \cup \text{Stem}(G)) = x_1x_2x_3$. If $\deg(x_2) \geq 3$, then $V(G) - \{x_1, x_3\}$ is a TGS of $G$, which is a contradiction. Thus $\deg(x_2) = 2$. If there is a vertex $y \in N(x_1)$ such that $d(y, x_3) = 3$ (the case $y \in N(x_3)$ such that $d(y, x_1) = 3$ is similar), then $V(G) - \{x_1, x_2\}$ is a TGS of $G$ which is which is a contradiction. This completes the proof. □

**Claim 10:** If $G' \cong (x_1x_2x_3x_4)$, then either $G = C_4$ or $n \geq 5$, two consecutive vertices of $C_4$, say $x_3$ and $x_4$, have degree two, $N[x_1] - \{x_4\} = N[x_2] - \{x_3\}$, and $x_1, x_2$ have a common neighbor of degree 2.

*Proof.* Let $G - (Ext(G) \cup Stem(G)) = (x_1x_2x_3x_4)$. If $n = 4$, then clearly $G = C_4$. Suppose that $n \geq 5$. We first show that two consecutive adjacent vertices of cycle $(x_1x_2x_3x_4)$ have degree 2. If $\deg(x_1) \geq 3$ and $\deg(x_3) \geq 3$, then $V(G) - \{x_2, x_4\}$ is a TGS of $G$, and if $\deg(x_2) \geq 3$ and $\deg(x_4) \geq 3$, then $V(G) - \{x_1, x_2\}$ is a TGS of $G$, which is a contradiction. Assume without loss of generality that $\deg(x_3) = \deg(x_4) = 2$. If $x_1$ and $x_2$ have no common vertex of degree two, then $V(G) - \{x_1, x_2\}$ is a TGS of $G$, which is a contradiction. □

In view of the above Claims, the proof of Theorem 3.1 is complete. □

An immediate consequence of Theorems 1.1, 3.1 and Observation 1.3 now follows:

**Corollary 3.2.** Let $G$ be a connected graph of order $n$. Then $g(G) = g_t(G) = n - 1$ if and only if $G$ is the join of $K_1$ and pairwise disjoint complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_r}$, that is, $G = (K_{n_1} \cup K_{n_2} \cup \ldots K_{n_r}) \vee K_1$, where $r \geq 2$, $n_i \geq 2$, for $1 \leq i \leq r$, with $n_1 + n_2 + \ldots + n_r = n - 1$.

**Corollary 3.3.** If $G$ is a graph of order $n$ with $g_t(G) = n - 1$, then the difference $g_t(G) - g(G)$ is at most $|V(G')| - 1$, where $G' = G - (Ext(G) \cup Stem(G))$.

*Proof.* Assume that $G' = G - (Ext(G) \cup Stem(G))$. Since $g_t(G) = n - 1$, by Theorem 3.1 one of the conditions (i)-(vi) hold. If $G$ satisfies (i), then clearly $g(C_4) = 2$ and $g(C_5) = 3$. Thus $g_t(G) - g(G) = 2$, and so the result follows. If $G$ satisfies (ii), then $Ext(G) \cup Stem(G)$ is the unique $g$-set of $G$, and so $g(G) = |Ext(G) \cup Stem(G)| = |V(G)| - |V(G')|$. Thus $g_t(G) - g(G) = n - 1 - n + |V(G')| = |V(G')| - 1$, and so the result follows.

If $G$ satisfies (iii), then by Theorem 1.1, we have $g(G) \leq n - 2$. On the other hand, $V(G) - V(K_2) = Ext(G) \cup Stem(G)$ is a $g$-set of $G$. Hence $g_t(G) - g(G) = 1$.

Let $G$ satisfy (iv). If $\deg(x_2) \geq 4$ and $\deg(x_3) \geq 4$, then $g(G) = |Ext(G)| + |Stem(G)|$. Thus $g_t(G) - g(G) = n - 1 - n + |V(G')| = |V(G')| - 1$. Otherwise, at least one of $x_2$ or $x_3$ must be included in any $g$-set of $G$ and so $g(G) \geq n - 2$. On the other hand, Theorem 1.1 implies that $g(G) = n - 2$. Hence $g_t(G) - g(G) = 1$.

Let $G$ satisfy (v). Clearly, $x_2$ belongs to any $g$-set of $G$ yielding $g(G) = n - 2$. Thus $g_t(G) - g(G) = 1$.

Assume $G$ satisfies (vi). Since $\deg(x_3) = \deg(x_4) = 2$, we conclude from the structure of $G$ that any $g$-set of $G$ contains $x_3$ or $x_4$. This implies that $g(G) \geq |Ext(G) \cup Stem(G)| + 1 = n - 3$. Thus $g_t(G) - g(G) \leq 2$ and the proof is complete. □

We conclude the paper by giving the following result:

**Proposition 3.4.** The difference $g_t(G) - g(G)$ can be arbitrarily large.

*Proof.* For each integer $n \geq 3$, let $G$ be a graph obtained from $K_{1,r}$ by subdivision all leaves once. Clearly, $g(G) = r$, and $g_t(G) = 2r$. Thus, $g_t(G) - g(G) = r$. □

**Acknowledgements**

The author is grateful to anonymous referees for their remarks and suggestions that helped improve the manuscript.

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