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Some New Star-Selection Principles in Topology

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Abstract. Motivated by the recent works of Kočinac who initiated investigation of star selection principles, we introduce and study some new types of star-selection principles. Also some open problems are posed.

1. Introduction and Preliminaries

Classical selection principles, based on the diagonalization arguments, have a long history going back to the works by Borel [2], Menger [12], Hurewicz [5], Rothberger [13], and others. Scheepers [16] began a systematic investigation of selection principles, which motivated a large number of researchers for investigation on selection principles and their applications.

Throughout the paper, $[X]^{<\omega}$ (respectively $[X]^{\leq\omega}$) will denote the collection of all finite (respectively countable) subsets of a set X.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of an infinite set X.

In 1925, Hurewicz [5] introduced two selection principles (in notation from [16]) $\mathcal{S}_{fin}(\mathcal{A},\mathcal{B})$ (derived from a property introduced by Menger [12]) and $\mathcal{U}_{fin}(\mathcal{A},\mathcal{B})$.

$\mathcal{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the following selection hypothesis :

> For each sequence $\{A_n : n \in \omega\}$ of elements of $\mathcal{A}$, there is a sequence $\{B_n : n \in \omega\}$ of finite sets such that for each $n \in \omega$, $B_n \subset A_n$ and $\bigcup\{B_n : n \in \omega\} \in \mathcal{B}$.

$\mathcal{U}_{fin}(\mathcal{A},\mathcal{B})$ denotes the following selection hypothesis :

> For each sequence $\{A_n : n \in \omega\}$ of elements of $\mathcal{A}$, there is a sequence $\{B_n : n \in \omega\}$ of finite sets such that for each $n \in \omega$, we have $B_n \subset A_n$ and $\{\bigcup B_n : n \in \omega\} \in \mathcal{B}$.

In 1938, Rothberger [13] introduced the selection principle $\mathcal{S}_1(\mathcal{A},\mathcal{B})$.

$\mathcal{S}_1(\mathcal{A},\mathcal{B})$ denotes the following selection hypothesis :

> For each sequence $\{A_n : n \in \omega\}$ of elements of $\mathcal{A}$, there is a sequence $\{B_n : n \in \omega\}$ such that for each $n \in \omega$, we have $B_n \in A_n$ and $\{B_n : n \in \omega\} \in \mathcal{B}$.

Scheepers [15] mentioned the selection principle $\mathcal{S}_{ctbl}(\mathcal{A},\mathcal{B})$ as a natural companion of the above selection principles, where $\mathcal{S}_{ctbl}(\mathcal{A},\mathcal{B})$ denotes the following selection hypothesis:

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For each sequence $\{A_n : n \in \omega\}$ of elements of $\mathcal{A}$, there is a sequence $\{B_n : n \in \omega\}$ such that for each $n \in \omega$, B_n is a countable subset of A_n and $\bigcup_{n\in\omega} B_n \in \mathcal{B}$.

By a space, we mean a topological space and for different notions in topology we follow [4].

For a set X, let $\mathcal{U}$ be a collection of subsets of X and $A \subset X$; then star of A with respect to $\mathcal{U}$ is denoted and defined by $St(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. For $x \in X$, we write $St(x, \mathcal{U})$ instead of $St(\{x\}, \mathcal{U})$. A space X is said to be *star-compact* if for every open cover $\mathcal{U}$ of X there exists a finite set $A \subset X$ such that $St(A, \mathcal{U}) = X$ [3, 11]. A space X is said to be *star-Lindelöf* if for every open cover $\mathcal{U}$ of X there exists a countable set $A \subset X$ such that $St(A, \mathcal{U}) = X$ [3, 11]. From the above definitions, it is clear that every star-compact space is star-Lindelöf, but the converse is not necessarily true [1].

In 1999, Kočinac [6, 7] introduced the following selection principles in connection with the star operator.
$\mathcal{S}_1^*(\mathcal{A}, \mathcal{B})$ denotes the following selection hypothesis:

For each sequence $\{\mathcal{U}_n : n \in \omega\}$ of elements of $\mathcal{A}$, there exists a sequence $\{U_n : n \in \omega\}$ such that for each $n \in \omega$, $U_n \in \mathcal{U}_n$ and $\{St(U_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

$\mathcal{S}_{fin}^*(\mathcal{A}, \mathcal{B})$ denotes the following selection hypothesis:

For each sequence $\{\mathcal{U}_n : n \in \omega\}$ of elements of $\mathcal{A}$, there exists a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\omega}\{St(V, \mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$.

$\mathcal{U}_{fin}^*(\mathcal{A}, \mathcal{B})$ denotes the following selection hypothesis:

For every sequence $\{\mathcal{U}_n : n \in \omega\}$ of members of $\mathcal{A}$, there exists a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{St(\cup\mathcal{V}_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

Song [18–22], Kočinac [6–10], Sakai [14], Tsaban [23] and many others have made investigations on these selection principles and interesting results have been obtained.

Let $\mathcal{K}$ be a family of subsets of a space X. Then:
$\mathcal{SS}_{\mathcal{K}}^*(\mathcal{A}, \mathcal{B})$ represents the following selection hypothesis:

For every sequence $\{\mathcal{U}_n : n \in \omega\}$ of elements of $\mathcal{A}$, there exists a sequence $\{K_n : n \in \omega\}$ of elements of $\mathcal{K}$ such that $\{St(K_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$ (see [6]).

When $\mathcal{K}$ is the collection of all one-point [resp., finite, compact] subsets of X, we write $\mathcal{SS}_1^*(\mathcal{A}, \mathcal{B})$ [resp., $\mathcal{SS}_{fin}^*(\mathcal{A}, \mathcal{B})$, $\mathcal{SS}_{comp}^*(\mathcal{A}, \mathcal{B})$] instead of $\mathcal{SS}_{\mathcal{K}}^*(\mathcal{A}, \mathcal{B})$ (see [6]).

Now let us mention the definitions of the games which are naturally associated to the selection principles mentioned above.

$G_{fin}(\mathcal{A}, \mathcal{B})$ denotes an infinitely long game for two players, ONE and TWO, who play a round for each non-negative integer. In the n-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing a finite set $B_n \subset A_n$. The play $\{A_0, B_0, A_1, B_1, ..., A_n, B_n, ...\}$ is won by TWO if $\bigcup_{n\in\omega} B_n \in \mathcal{B}$; otherwise, ONE wins (see [15]).

$G_1(\mathcal{A}, \mathcal{B})$ denotes an infinitely long game for two players, ONE and TWO, who play a round for each non-negative integer. In the n-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing an element $b_n \in A_n$. The play $\{A_0, b_0, A_1, b_1, ..., A_n, b_n, ...\}$ is won by TWO if $\{b_n : n \in \omega\} \in \mathcal{B}$; otherwise, ONE wins (see [15]).

$G_1^*(\mathcal{A}, \mathcal{B})$ denotes an infinitely long game for two players, ONE and TWO, who play a round for each non-negative integer. In the n-th round ONE chooses a set $\mathcal{U}_n \in \mathcal{A}$, TWO responds by choosing an element $U_n \in \mathcal{U}_n$. The play $\{\mathcal{U}_0, U_0, \mathcal{U}_1, U_1, ..., \mathcal{U}_n, U_n, ...\}$ is won by TWO if $\{St(U_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$; otherwise, ONE wins (see [6]).

$G_{fin}^*(\mathcal{A}, \mathcal{B})$ denotes an infinitely long game for two players, ONE and TWO, who play a round for each non-negative integer. In the n-th round ONE chooses a set $\mathcal{U}_n \in \mathcal{A}$, and then TWO responds by choosing a finite set $\mathcal{V}_n \subset \mathcal{U}_n$. The play $\{\mathcal{U}_0, \mathcal{V}_0, \mathcal{U}_1, \mathcal{V}_1, ..., \mathcal{U}_n, \mathcal{V}_n, ...\}$ is won by TWO if $\bigcup_{n\in\omega}\{St(V, \mathcal{V}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$; otherwise, ONE wins (see [6]).

If X is a space, then $\mathcal{S}G_1^*(\mathcal{A},\mathcal{B})$ denotes an infinitely long game for two players, ONE and TWO, who play a round for each non-negative integer. In the n-th round ONE chooses a set $\mathcal{U}_n \in \mathcal{A}$, TWO responds by choosing an element $x_n \in X$. The play $\{\mathcal{U}_0, x_0, \mathcal{U}_1, x_1, ..., \mathcal{U}_n, x_n, ...\}$ is won by TWO if $\{St(x_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$; otherwise, ONE wins see (see [6]).

If X is a space, then $\mathcal{S}G_{fin}^*(\mathcal{A},\mathcal{B})$ denotes an infinitely long game for two players, ONE and TWO, who play a round for each non-negative integer. In the n-th round ONE chooses a set $\mathcal{U}_n \in \mathcal{A}$, TWO responds by choosing an finite subset $F_n \subset X$. The play $\{\mathcal{U}_0, F_0, \mathcal{U}_1, F_1, ..., \mathcal{U}_n, F_n, ...\}$ is won by TWO if $\{St(F_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$; otherwise, ONE wins (see [6]).

If X is a space, then $\mathcal{S}G_{comp}^*(\mathcal{A},\mathcal{B})$ denotes an infinitely long game for two players, ONE and TWO, who play a round for each non-negative integer. In the n-th round ONE chooses a set $\mathcal{U}_n \in \mathcal{A}$, TWO responds by choosing an compact subset $K_n \subset X$. The play $\{\mathcal{U}_0, K_0, \mathcal{U}_1, K_1, ..., \mathcal{U}_n, K_n, ...\}$ is won by TWO if $\{St(K_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$; otherwise, ONE wins (see [6]).

2. New Selection Principles

In this section we introduce two selection principles in connection with the star operator : ${}^*\mathcal{U}_1(\mathcal{A},\mathcal{B})$ and ${}^*\mathcal{U}_{fin}(\mathcal{A},\mathcal{B})$.

Definition 2.1. ${}^*\mathcal{U}_1(\mathcal{A},\mathcal{B})$ denotes the following selection principle:

> For each sequence $\{\mathcal{U}_n : n \in \omega\}$ of elements of $\mathcal{A}$, there exists a sequence $\{U_n : n \in \omega\}$ such that for each $n \in \omega$, $U_n \in \mathcal{U}_n$ and $\{St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

Definition 2.2. ${}^*\mathcal{U}_{fin}(\mathcal{A},\mathcal{B})$ denotes the following selection principle:

> For each sequence $\{\mathcal{U}_n : n \in \omega\}$ of elements of $\mathcal{A}$, there exists a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{St(\bigcup_{i\in\omega}(\bigcup \mathcal{V}_i), \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

Proposition 2.3. ${}^*\mathcal{U}_1(\mathcal{A},\mathcal{B}) \Rightarrow^* \mathcal{U}_{fin}(\mathcal{A},\mathcal{B})$.

Proposition 2.4. *If $\mathcal{A}$ and $\mathcal{B}$ are two collections of families of subsets of an infinite set X such that $\mathcal{A} \subset \mathcal{B}$, then*

$$ {}^*\mathcal{U}_1(\mathcal{B},\mathcal{B}) \Rightarrow^* \mathcal{U}_1(\mathcal{A},\mathcal{B}) $$

$$ {}^*\mathcal{U}_1(\mathcal{A},\mathcal{A}) \Rightarrow^* \mathcal{U}_1(\mathcal{A},\mathcal{B}) $$

$$ {}^*\mathcal{U}_1(\mathcal{B},\mathcal{A}) \Rightarrow^* \mathcal{U}_1(\mathcal{A},\mathcal{A}) $$

$$ {}^*\mathcal{U}_1(\mathcal{B},\mathcal{A}) \Rightarrow^* \mathcal{U}_1(\mathcal{B},\mathcal{B}) $$

Proof. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of elements of $\mathcal{A}$. But $\mathcal{A} \subset \mathcal{B}$. Therefore $\{\mathcal{U}_n : n \in \omega\}$ is a sequence of elements of $\mathcal{B}$. Since ${}^*\mathcal{U}_1(\mathcal{B},\mathcal{B})$ holds, there exists a sequence $\{U_n : n \in \omega\}$ such that $U_n \in \mathcal{U}_n$ for each $n \in \omega$ and $\{St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$. Therefore, ${}^*\mathcal{U}_1(\mathcal{A},\mathcal{B})$ holds.

Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of elements of $\mathcal{A}$. Since ${}^*\mathcal{U}_1(\mathcal{A},\mathcal{A})$ holds, there exists a sequence $\{U_n : n \in \omega\}$ such that $U_n \in \mathcal{U}_n$ for each $n \in \omega$ and $\{St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n) : n \in \omega\} \in \mathcal{A}$. But $\mathcal{A} \subset \mathcal{B}$. Thus, $\{St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$. Therefore, ${}^*\mathcal{U}_1(\mathcal{A},\mathcal{B})$ holds.

Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of elements of $\mathcal{A}$. But $\mathcal{A} \subset \mathcal{B}$. Therefore, $\{\mathcal{U}_n : n \in \omega\}$ is a sequence of elements of $\mathcal{B}$. Since ${}^*\mathcal{U}_1(\mathcal{B},\mathcal{A})$ holds, there exists a sequence $\{U_n : n \in \omega\}$ such that $U_n \in \mathcal{U}_n$ for each $n \in \omega$ and $\{St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n) : n \in \omega\} \in \mathcal{A}$. Therefore, ${}^*\mathcal{U}_1(\mathcal{A},\mathcal{A})$ holds.

Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of elements of $\mathcal{B}$. Since ${}^*\mathcal{U}_1(\mathcal{B},\mathcal{A})$ holds, there exists a sequence $\{U_n : n \in \omega\}$ such that $U_n \in \mathcal{U}_n$ for each $n \in \omega$ and $\{St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n) : n \in \omega\} \in \mathcal{A}$. But $\mathcal{A} \subset \mathcal{B}$. Thus, $\{St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$. Therefore, ${}^*\mathcal{U}_1(\mathcal{B},\mathcal{B})$ holds. □

So, we conclude that the selection principle ${}^*\mathcal{U}_1(\mathcal{A},\mathcal{B})$ is monotonic in the second collection and is anti-monotonic in the first collection.

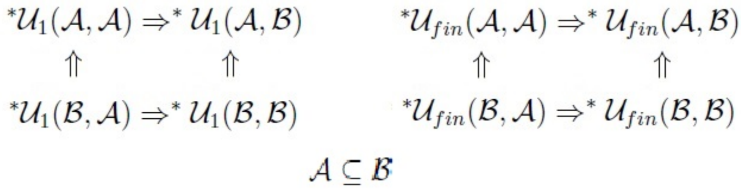

Figure 1: Monotonicity of ${}^*\mathcal{U}_1(\mathcal{A},\mathcal{B})$ and ${}^*\mathcal{U}_{fin}(\mathcal{A},\mathcal{B})$.

Proposition 2.5. *${}^*\mathcal{U}_{fin}(\mathcal{A},\mathcal{B})$ is monotonic in the second collection and anti-monotonic in the first collection.*

Proof. The proof of this proposition is similar to the proof of Proposition 2.4, so omitted. □

In this paper, we emphasize on the cases where $\mathcal{A}$ and $\mathcal{B}$ are the classes of topologically significant open covers of a space X:

$\mathcal{O}$ - the collection of all open covers of X.

Λ - the collection of all large covers of X. An open cover $\mathcal{U}$ of X is a *large cover* if each $x \in X$ belongs to infinitely many members of $\mathcal{U}$.

Ω - the collection of all ω-covers of X. An open cover $\mathcal{U}$ of X is an *ω-cover* if every finite subset of X is contained in a member of $\mathcal{U}$.

Γ - the collection of all γ-covers of X. An open cover $\mathcal{U}$ of X is a *γ-cover* if it is infinite, and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.

$\mathcal{O}^{gp}$ - the collection of all groupable open-covers of X. An open cover $\mathcal{U}$ of X is *groupable* if it can be expressed as a countable union of finite, pairwise disjoint subfamilies of $\mathcal{U}_n$, $n \in \omega$, such that each $x \in X$ belongs to $\bigcup \mathcal{U}_n$ for all but finitely many n.

If the covers are considered to be non-trivial then we have, $\Gamma \subset \Omega \subset \Lambda \subset \mathcal{O}$. Under such condition, we have the following relation diagram (Figure 2):

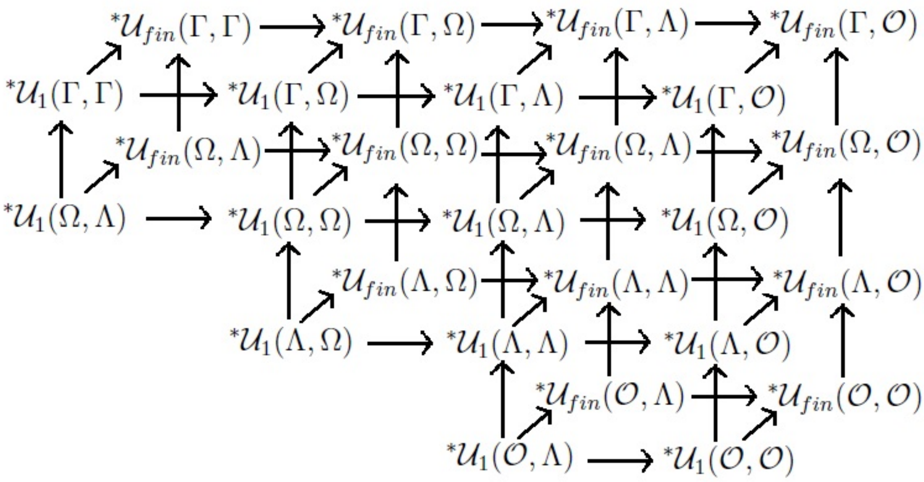

Figure 2: Relation Chart 1

Proposition 2.6. *Every star-Lindelöf space has the property ${}^*\mathcal{U}_1(\mathcal{O},\mathcal{O})$.*

Proof. Let X be a star-Lindelöf space and $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since $\mathcal{U}_0$ is an open cover of X and X is star-Lindelöf, there exists a countable set, $\{x_0, x_1, x_2, ..., x_n, ...\} \subset X$ such that $St(\{x_0, x_1, x_2, ..., x_n, ...\}, \mathcal{U}_0) = X$.

For each $n \in \omega$, we select $U_n \in \mathcal{U}_n$ such that $x_n \in U_n$. Clearly $\{x_0, x_1, x_2, ..., x_n, ...\} \subset \bigcup_{n\in\omega} U_n$. Therefore $St(\bigcup_{n\in\omega} U_n, \mathcal{U}_0) = X$. Thus $\{St(\bigcup_{n\in\omega} U_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X, i.e. ${}^*\mathcal{U}_1(\mathcal{O}, \mathcal{O})$ holds for X. Hence the theorem. □

Corollary 2.7. *Compact spaces, star-compact spaces and Lindelöf spaces have the property* ${}^*\mathcal{U}_1(\mathcal{O}, \mathcal{O})$.

Example 2.8. The converse of Proposition 2.6 is not necessarily true, i.e. there exists a space which has the property ${}^*\mathcal{U}_1(\mathcal{O}, \mathcal{O})$ but is not star-Lindelöf.

Consider the space $X = \mathbb{R}^+ \setminus \{\mathbb{R}^+ \bigcap \mathbb{Q}\ \}$. Let $A = [0, 1] \setminus ([0, 1] \bigcap \mathbb{Q})$. For each $x \in A$, $A_x = \{(n + x) : n \in \omega\} \bigcup A$. Set $Y = \{A_x : x \in A\} \bigcup \{A\}$, and define $\tau(X) = \{\bigcup B : B \in P(Y)\}$. $\tau(X)$ is a topology on X.

Now, let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Therefore, $\mathcal{U}_0$ is an open cover of X. By the construction of the space, every open set other than $\emptyset$ contains A. We select $U_0 \in \mathcal{U}_0$ such that $A \subset U_0$ and for $i \in \omega \setminus \{0\}$, select $U_i \in \mathcal{U}_i$.

Since $St(U_0, \mathcal{U}_0) = X$, $St(\bigcup_{i\in\omega} U_i, \mathcal{U}_0) = X$, so that the set $\{St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Hence, X has the property ${}^*\mathcal{U}_1(\mathcal{O}, \mathcal{O})$.

On the other hand, $\mathcal{U} = \{A_x : x \in A\}$ is an uncountable open cover of X. For each $x, y \in A$, $x \neq y$, $A_x \bigcap A_y = A$ and $\bigcup_{x\in A} A_x = X$, but $\bigcup_{i\in\omega} A_{x_i} \subsetneq X$ for any countable set $\{x_i\}_{i\in\omega} \subset A$.

Let $F = \{y_i\}_{i\in\omega} \subset X$. For each $y_i \in X$, there exists a $x_i \in A$ such that $y_i \in A_{x_i}$. Therefore, $St(F, \mathcal{U}) = \bigcup_{i\in\omega}(A_{x_i}) \neq X$. We find X, not star-Lindelöf eventhough it has the property ${}^*\mathcal{U}_1(\mathcal{O}, \mathcal{O})$.

We obtain the following diagram of implication and non-implication from the above results:

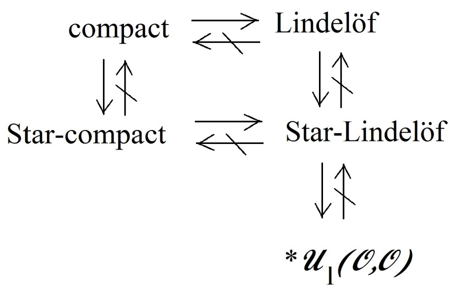

Figure 3: Relation Chart 2

Proposition 2.9. $\mathcal{S}_1^*(\mathcal{A}, \mathcal{O}) \Rightarrow^* \mathcal{U}_1(\mathcal{A}, \mathcal{O})$.

Proof. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of elements of $\mathcal{A}$. Since $\mathcal{S}_1^*(\mathcal{A}, \mathcal{B})$ holds, there exists a sequence $\{U_n : n \in \omega\}$ such that for each $n \in \omega$, $U_n \in \mathcal{U}_n$ and $\{St(U_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{O}$. Hence, $\{St(U_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover for X.

We have, for each $n \in \omega$, $St(U_n, \mathcal{U}_n) \subset St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n)$. Therefore, $\{St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n) : n \in \omega\}$ is also an open cover for X. Thus ${}^*\mathcal{U}_1(\mathcal{A}, \mathcal{O})$ holds. □

But ${}^*\mathcal{U}_1(\mathcal{O}, \mathcal{O}) \nRightarrow \mathcal{S}_1^*(\mathcal{O}, \mathcal{O})$ in general. This follows from the example given below.

Example 2.10. Let $X = (0, 3] \subset \mathbb{R}$. We consider the topology $\tau(X) = \{(x, y] : x, y \in [0, 3) \text{ and } x < y\} \cup \{\emptyset, X\}$, the upper limit topology on X induced from the upper limit topology of $\mathbb{R}$.

We construct a sequence of open covers of X as follows:

$$\mathcal{U}_0 = \{(0, 1], (1, 2], (2, 3]\},$$

$$\mathcal{U}_1 = \left\{\left(0, \frac{1}{2}\right], \left(\frac{1}{2}, \frac{2}{2}\right], \left(\frac{2}{2}, \frac{3}{2}\right], \left(\frac{3}{2}, \frac{4}{2}\right], \left(\frac{4}{2}, \frac{5}{2}\right], \left(\frac{5}{2}, \frac{6}{2}\right]\right\},$$

$$\mathcal{U}_2 = \left\{\left(0, \frac{1}{2^2}\right], \left(\frac{1}{2^2}, \frac{2}{2^2}\right], \left(\frac{2}{2^2}, \frac{3}{2^2}\right], ..., \left(\frac{11}{2^2}, \frac{12}{2^2}\right]\right\},$$

....

$$\mathcal{U}_n = \left\{\left(0, \frac{1}{2^n}\right], \left(\frac{1}{2^n}, \frac{2}{2^n}\right], \left(\frac{2}{2^n}, \frac{3}{2^n}\right], ..., \left(\frac{3.2^n - 1}{2^n}, \frac{3.2^n}{2^n}\right]\right\},$$

....

For each $n \in \omega$, length of each interval contained in $\mathcal{U}_n$ is $\frac{1}{2^n}$. Also, for each $n \in \omega$, $\mathcal{U}_n$ is a pairwise disjoint collection of open sets. Choose $U_n \in \mathcal{U}_n$, then we have $St(U_n, \mathcal{U}_n) = U_n$.

So, length of $St(U_n, \mathcal{U}_n) = \frac{1}{2^n}$, for each $n \in \omega$. If $St(U_n, \mathcal{U}_n)$ covers different portions of X for each $n \in \omega$, it will cover a length of X. The maximum length of the subset of X covered by $\{St(U_n, \mathcal{U}_n) : n \in \omega\}$ is

$$\sum_{n\in\omega} \frac{1}{2^n} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + ... = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + ... = \left(1 - \frac{1}{2}\right)^{-1} = 2.$$

But length of X is 3. Hence, $\{St(u_n, \mathcal{U}_n) : n \in \omega\}$ can not be a cover of X. So, it is not possible to find a sequence $\{U_n : n \in \omega\}$ such that for each $n \in \omega$, $U_n \in \mathcal{U}_n$ and $\{St(U_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover for X. This implies that X does not have the property $\mathcal{S}_1^*(\mathcal{O}, \mathcal{O})$.

We have $\mathbb{R}$ with the upper limit topology is hereditarily Lindelöf, hence X is a Lindelöf space. Thus, by Corollary 2.7, X has the property ${}^*\mathcal{U}_1(\mathcal{O}, \mathcal{O})$.

Proposition 2.11. $\mathcal{U}^*_{fin}(\mathcal{A}, \mathcal{O}) \Rightarrow^* \mathcal{U}_{fin}(\mathcal{A}, \mathcal{O})$.

Proof. The proof is similar to that of Proposition 2.9, so omitted □

In view of the above results, we have the following relation diagram (Figure 4):

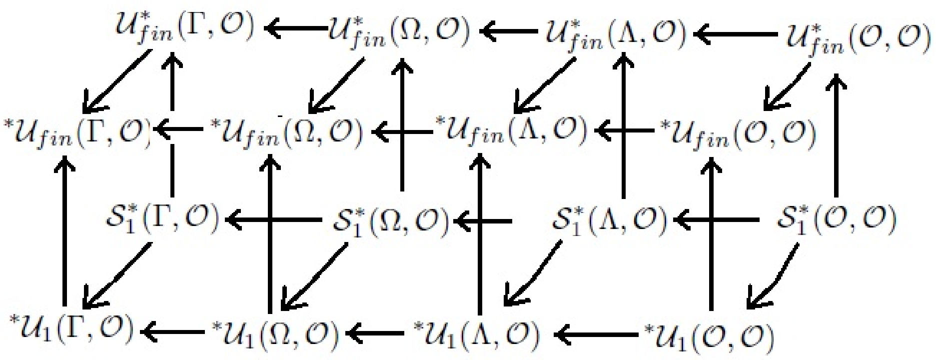

Figure 4: Relation Chart 3

Problem 2.12. *Does there exists a space which has the property ${}^*\mathcal{U}_{fin}(\mathcal{O}, \mathcal{O})$ but does not have the property $\mathcal{U}^*_{fin}(\mathcal{O}, \mathcal{O})$.*

Proposition 2.13. *If a space X is compact, then it has the property ${}^*\mathcal{U}_{fin}(\mathcal{O}, \mathcal{O})$.*

Proof. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers for X. Since X is compact, there exists $\mathcal{A}_n \in [\mathcal{U}_n]^{<\omega}$ for each $n \in \omega$, such that $\mathcal{A}_n$ is a cover for X. Let $x \in X$ be an arbitrary point. For each $n \in \omega$, there exists $A_{n_x} \in \mathcal{A}_n \subset \mathcal{U}_n$ such that $x \in A_{n_x} \in \mathcal{U}_n$. So, $x \in A_{n_x} \subset \bigcup \mathcal{A}_n \Rightarrow A_{n_x} \cap (\bigcup \mathcal{A}_n) \neq \emptyset$, for each $n \in \omega$.

So, $x \in A_{n_x} \subset St(\bigcup \mathcal{A}_n, \mathcal{U}_n)$, for each $n \in \omega$, i.e. $x \in St(\bigcup_{i\in\omega}(\bigcup \mathcal{A}_i), \mathcal{U}_n)$, for each $n \in \omega$. Therefore, $\{St(\bigcup_{i\in\omega}(\bigcup \mathcal{A}_i), \mathcal{U}_n) : n \in \omega\}$ is an open cover for X. Hence X has the property ${}^*\mathcal{U}_{fin}(\mathcal{O}, \mathcal{O})$. □

Proposition 2.14. *If $f : X \to Y$ is a continuous surjection and X has the property $^*\mathcal{U}_1(\mathcal{O},\mathcal{O})$, then Y also has the property $^*\mathcal{U}_1(\mathcal{O},\mathcal{O})$.*

Proof. Let $\{\mathcal{V}_n : n \in \omega\}$ be a sequence of open covers for Y. For each $n \in \omega$, let $\mathcal{U}_n = \{f^{-1}(V) : V \in \mathcal{V}_n\}$ is a sequence of open covers of X. Since, X has the property $^*\mathcal{U}_1(\mathcal{O},\mathcal{O})$, there exists a sequence $\{f^{-1}(V_n) : n \in \omega\}$ where $V_n \in \mathcal{V}_n$ for all $n \in \omega$ such that $f^{-1}(V_n) \in \mathcal{U}_n$, for each $n \in \omega$ and $\{St(\bigcup_{i\in\omega} f^{-1}(V_i), \mathcal{U}_n) : n \in \omega\}$ is an open cover for X.

Let $y \in Y$ be an arbitrary point. Then, there exists $x \in X$ such that $f(x) = y$. Thus, $x \in St(\bigcup_{i\in\omega} f^{-1}(V_i), \mathcal{U}_m)$ for some $m \in \omega$. Therefore, there exists a $f^{-1}(V_y) \in \mathcal{U}_m$ such that $x \in f^{-1}(V_y)$ and $f^{-1}(V_y) \cap (\bigcup_{i\in\omega} f^{-1}(V_i)) \neq \emptyset$. So, $y \in V_y \in \mathcal{V}_m$ and $f^{-1}(V_y) \bigcap f^{-1}(V_i) \neq \emptyset$ for some $i \in \omega$. i.e. $V_y \bigcap V_i \neq \emptyset$.

Thus, $V_y \cap (\bigcup_{i\in\omega} V_i) \neq \emptyset$. $\therefore$ $y \in St(\bigcup_{i\in\omega} V_i, \mathcal{V}_m)$. Hence $\{St(\bigcup_{i\in\omega} V_i, \mathcal{V}_n) : n \in \omega\}$ is an open cover for Y. This completes the proof of the theorem. □

In a similar way we prove the following result.

Proposition 2.15. *If $f : X \to Y$ is a continuous surjection and X has the property $^*\mathcal{U}_{fin}(\mathcal{O},\mathcal{O})$, then Y also has the property $^*\mathcal{U}_{fin}(\mathcal{O},\mathcal{O})$.*

Corollary 2.16. *If the product of two spaces belongs to the class $^*\mathcal{U}_1(\mathcal{O},\mathcal{O})$, then each of them belongs to the class $^*\mathcal{U}_1(\mathcal{O},\mathcal{O})$. Similarly, if the product of two spaces belong to the class $^*\mathcal{U}_{fin}(\mathcal{O},\mathcal{O})$, then each of them belongs to the class $^*\mathcal{U}_{fin}(\mathcal{O},\mathcal{O})$.*

Problem 2.17. *Does there exist spaces which have the property $^*\mathcal{U}_1(\mathcal{O},\mathcal{O})$ but their product do not have the property.*

Proposition 2.18. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are any collection of subsets of X and if $\mathcal{C}$ is a cover for X. If $^*\mathcal{U}_1(\mathcal{A},\mathcal{B})$ and $^*\mathcal{U}_1(\mathcal{B},\mathcal{C})$ holds, then $\{X\} \in \mathcal{C}$.*

Proof. $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of elements of $\mathcal{A}$. Since $^*\mathcal{U}_1(\mathcal{A},\mathcal{B})$ holds, there exists a sequence $\{U_n : n \in \omega\}$ such that for each $n \in \omega$, $U_n \in \mathcal{U}_n$ and $\{St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

Suppose $V_n = St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n)$ for each $n \in \omega$ and $\mathcal{V} = \{V_n : n \in \omega\}$. Now, choose a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that $\mathcal{V}_n = \mathcal{V}$, for each $n \in \omega$. Then $\{\mathcal{V}_n : n \in \omega\}$ is sequence of elements of $\mathcal{B}$. Since $^*\mathcal{U}_1(\mathcal{B},\mathcal{C})$ holds, there exists a sequence $\{V'_n : n \in \omega\}$ such that for each n, $V'_n \in \mathcal{V}_n = \mathcal{V}$ and $\{St(\bigcup_{i\in\omega} V'_i, \mathcal{V}_n) : n \in \omega\} \in \mathcal{C}$. We have

$$\left\{St\left(\bigcup_{i\in\omega} V'_i, \mathcal{V}\right) : n \in \omega\right\} \in \mathcal{C} \Rightarrow \left\{St\left(\bigcup_{i\in\omega} V'_i, \mathcal{V}\right)\right\} \in \mathcal{C} \Rightarrow St\left(\bigcup_{i\in\omega} V'_i, \mathcal{V}\right) = X,$$

i.e. $\{X\} \in \mathcal{C}$. □

Theorem 2.19. *If X^k have the property $^*\mathcal{U}_1(\mathcal{O},\mathcal{O})$ for any finite k, then X has the property $^*\mathcal{U}_{fin}(\mathcal{O},\Omega)$.*

Proof. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X and let $\omega = N_1 \bigcup N_2 \bigcup N_3 \bigcup ....$ be a countable partition of ω into countable subsets. For each $k \in \omega$ and each $m \in N_k$, let $\mathcal{W}_m = \{U_1 \times U_2 \times ... \times U_k : U_1, U_2, U_3, ..., U_k \in \mathcal{U}_m\}$. Then $\{\mathcal{W}_m : m \in N_k\}$ is a sequence of open covers of X^k.

Since $^*\mathcal{U}_1(\mathcal{O},\mathcal{O})$ holds for X^k, we can choose a sequence $\{H_m : m \in N_k\}$ such that for each m, $H_m \in \mathcal{W}_m$ and $\{St(\bigcup_{i\in N_k} H_i, \mathcal{W}_m) : m \in N_k\}$ is an open cover of X^k. For every $m \in N_k$ and $H_m \in \mathcal{W}_m$. Let, $H_m = U_1(H_m) \times U_2(H_m) \times U_3(H_m) \times ... \times U_k(H_m)$, where $U_i(H_m) \in \mathcal{U}_m$ for $i \leq k$.

Let $F = \{x_1, x_2, x_3, ...x_s\}$ be a finite subset of X. Then $(x_1, x_2, x_3, ..., x_s) \in X^s$, so there exists $n \in N_s$ such that $(x_1, x_2, x_3, ..., x_s) \in St(\bigcup_{i\in N_s} H_i, \mathcal{W}_n)$, where $H_i \in \mathcal{W}_i$ and $i \in N_s$. So, there exists a $W \in \mathcal{W}_n$ such that $(x_1, x_2, x_3, ..., x_s) \in W$ and $W \bigcap(\bigcup_{i\in N_s} H_i) \neq \emptyset$. Let $W = U_1(W) \times U_2(W) \times ... \times U_s(W)$. $U_i(W) \in \mathcal{U}_n$, $i \leq s$.

Thus $x_1 \in U_1(W), x_2 \in U_2(W), ..., x_s \in U_s(W)$ and $(U_1(W) \times U_2(W) \times ... \times U_s(W)) \bigcap(\bigcup_{i\in N_s} H_i) \neq \emptyset$. i.e. $(U_1(W) \times U_2(W) \times ... \times U_s(W)) \bigcap(\bigcup_{i\in N_s}(U_1(H_i) \times U_2(H_i) \times U_3(H_i) \times ... \times U_s(H_i))) \neq \emptyset$ which implies $(U_1(W) \times U_2(W) \times ... \times U_s(W)) \bigcap((\bigcup_{i\in N_s} U_1(H_i)) \times (\bigcup_{i\in N_s} U_2(H_i)) \times (\bigcup_{i\in N_s} U_3(H_i))... \times (\bigcup_{i\in N_s} U_s(H_i))) \neq \emptyset$.

Thus, for each $j \leq s$, $U_j(W) \bigcap(\bigcup_{i\in N_s} U_j(H_i)) \neq \emptyset$. Hence, for each $j \leq s$, $U_j(W) \bigcap(\bigcup_{i\in N_s}(\bigcup_{j=1}^{s} U_j(H_i))) \neq \emptyset$.

The set $\{U_1(W), U_2(W), ..., U_s(W)\} = \mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $j \leq s$, $x_j \in U_j(W) \subset St((\bigcup_{i\in N_s}(\bigcup_{j=1}^{s} U_j(H_i))), \mathcal{U}_n)$, i.e. for each $j \leq s$, $x_j \in U_j(W) \subset St((\bigcup_{i\in N_s}(\bigcup \mathcal{V}_i)), \mathcal{U}_n)$. Thus $F \subset St((\bigcup_{i\in N_s}(\bigcup \mathcal{V}_i)), \mathcal{U}_n)$, i.e. $F \subset St((\bigcup_{i\in\omega}(\bigcup \mathcal{V}_i)), \mathcal{U}_n)$.

For each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ satisfying: for each finite set $F \subset X$ there is an n such that $F \subset St((\bigcup_{i\in\omega}(\bigcup \mathcal{V}_i)), \mathcal{U}_n)$ and $\{St(\bigcup_{i\in\omega}(\bigcup \mathcal{V}_i), \mathcal{U}_n) : n \in \omega\} \in \Omega$. This implies that X satisfies ${}^*\mathcal{U}_{fin}(\mathcal{O}, \Omega)$. □

Note 2.20. For a finite collection of open covers $\{\mathcal{U}_i : i = 1, 2, 3, ..., n\}$ we define $\bigcap\{\mathcal{U}_i : i = 1, 2, 3, ..., n\} = \{U_1 \cap U_2 \cap U_3 \cap ... \cap U_n : U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, U_3 \in \mathcal{U}_3, ..., U_n \in \mathcal{U}_n\}$.

Theorem 2.21. *If a space has the property ${}^*\mathcal{U}_1(\mathcal{O}, \Gamma)$, then it has the property ${}^*\mathcal{U}_1(\mathcal{O}, \mathcal{O}^{gp})$.*

Proof. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. We construct new open covers follows.

$$\mathcal{V}_n = \bigcap \left\{ \mathcal{U}_i : \frac{n(n+1)}{2} \leq i < \frac{(n+1)(n+2)}{2} \right\}, \text{ for each } n \in \omega.$$

So, $\{\mathcal{V}_n : n \in \omega\}$ is also a sequence of open covers of X. Since X has the property ${}^*\mathcal{U}_1(\mathcal{O}, \Gamma)$, we can find a sequence $\{W_n : n \in \omega\}$ such that $W_n \in \mathcal{V}_n$ for each $n \in \omega$ and every $x \in X$ belongs to all but finitely many members of $\{St(\bigcup_{i\in\omega} W_i, \mathcal{V}_n) : n \in \omega\}$.

For each $i \in \omega$, $W_i \subset U_j$, for some $U_j \in \mathcal{U}_j$ with $\frac{i(i+1)}{2} \leq j < \frac{(i+1)(i+2)}{2}$. We consider the set of non-negative integers $n_0 < n_1 < ... < n_p < ...$ defined by $n_p = \frac{p(p+1)}{2}$.

If $x \in X$ belongs to $St(\bigcup_{i\in\omega} W_i, \mathcal{V}_k)$ for some $k \in \omega$, then x belongs to $St(\bigcup_{i\in\omega} W_i, \mathcal{U}_l)$, for each l such that $n_k \leq l < n_{k+1}$. i.e. $x \in \bigcup_{n_k\leq l<n_{k+1}} St(\bigcup_{i\in\omega} W_i, \mathcal{U}_l)$.

So, for each $x \in X$, we have $x \in \bigcup_{n_k\leq l<n_{k+1}} St(\bigcup_{i\in\omega} W_i, \mathcal{U}_l)$ for all but infinitely many $k \in \omega$. $\bigcup_{i\in\omega} W_i \subset \bigcup_{i\in\omega} U_i$. So, for each $x \in X$, we have $x \in \bigcup_{n_k\leq l<n_{k+1}} St(\bigcup_{i\in\omega} U_i, \mathcal{U}_l)$ for all but infinitely many $k \in \omega$. Thus the cover $\{St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n) : n \in \omega\}$ is groupable. □

3. Topological Games Related to ${}^*\mathcal{U}_1(\mathcal{A}, \mathcal{B})$ and ${}^*\mathcal{U}_{fin}(\mathcal{A}, \mathcal{B})$

In this section, we introduce two topological games which are naturally associated with the selection principles introduced in Section 2. The game related to the selection principle ${}^*\mathcal{U}_1(\mathcal{A}, \mathcal{B})$ is denoted by ${}^*G_1(\mathcal{A}, \mathcal{B})$ and the game related to the selection principle ${}^*\mathcal{U}_{fin}(\mathcal{A}, \mathcal{B})$ is denoted by ${}^*G_{fin}(\mathcal{A}, \mathcal{B})$.

Two games, say P and P', are equivalent if: ONE has a winning strategy in P if, and only if, ONE has a winning strategy in P', and TWO has a winning strategy in P if, and only if, TWO has a winning strategy in P' [17].

Two games, P and P', are dual if: ONE has a winning strategy in P if, and only if, TWO has a winning strategy in P', and TWO has a winning strategy in P if, and only if, ONE has a winning strategy in P' [17].

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of a families of subsets of a set X.

Definition 3.1. ${}^*G_1(\mathcal{A}, \mathcal{B})$ denotes an infinitely long game for two players, ONE and TWO, who play a round for each non-negative integer. In the n-th round ONE chooses $\mathcal{U}_n \in \mathcal{A}$, TWO responds by choosing an element $U_n \in \mathcal{U}_n$. The play $\{\mathcal{U}_0, U_0, \mathcal{U}_1, U_1, ..., \mathcal{U}_n, U_n, ...\}$ is won by TWO if $\{St(\bigcup_{i\in\omega} U_i, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$; otherwise, ONE wins.

Definition 3.2. ${}^*G_{fin}(\mathcal{A}, \mathcal{B})$ denotes an infinitely long game for two players, ONE and TWO, who play a round for each non-negative integer. In the n-th round ONE chooses $\mathcal{U}_n \in \mathcal{A}$, TWO responds by choosing $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$. The play $\{\mathcal{U}_0, \mathcal{V}_0, \mathcal{U}_1, \mathcal{V}_1, ..., \mathcal{U}_n, \mathcal{V}_n, ...\}$ is won by TWO if $\{St(\bigcup_{i\in\omega}(\bigcup \mathcal{V}_i), \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$; otherwise, ONE wins.

Proposition 3.3. *If for a space X, TWO has a winning strategy in the game $G_1^*(\mathcal{O}, \mathcal{O})$, then TWO has a winning strategy in ${}^*G_1(\mathcal{O}, \mathcal{O})$.*

Proof. Suppose TWO has a winning strategy σ in the game ${}^*G_1(\mathcal{O},\mathcal{O})$. Use σ to define then a strategy φ for TWO in the game $G_1^*(\mathcal{O},\mathcal{O})$ on X. Suppose that first move of ONE in the game ${}^*G_1(\mathcal{O},\mathcal{O})$ is an open cover $\mathcal{U}_1$ of X. If TWO responds in $G_1^*(\mathcal{O},\mathcal{O})$ by $\sigma(\mathcal{U}_1) = U_1 \in \mathcal{U}_1$, then TWO plays $\varphi(\mathcal{U}_1) = \sigma(\mathcal{U}_1) = U_1$. Assume that then ONE plays $\mathcal{U}_2 \in \mathcal{O}$ in the game $G_1^*(\mathcal{O},\mathcal{O})$, and TWO responds by $\sigma(\mathcal{U}_1,\mathcal{U}_2) = U_2$, then TWO plays $\varphi(\mathcal{U}_1,\mathcal{U}_2) = U_2$. And so on.

As σ is a winning strategy for TWO in the game $G_1^*(\mathcal{O},\mathcal{O})$, consider a σ-play

$$\mathcal{U}_1, \sigma(\mathcal{U}_1); \mathcal{U}_2, \sigma(\mathcal{U}_1,\mathcal{U}_2), \ldots$$

won by TWO, i.e.

$$\bigcup_{i\in\omega} St\,(U_i,\mathcal{U}_i) = X.$$

By the definition of φ, the φ-play

$$\mathcal{U}_1, \varphi(\mathcal{U}_1); \mathcal{U}_2, \varphi(\mathcal{U}_1,\mathcal{U}_2), \ldots$$

is won by TWO, since

$$\bigcup_{i\in\omega} St\left(\bigcup_{i\in\omega} U_i, \mathcal{U}_i\right) \supset \bigcup_{i\in\omega} St\,(U_i,\mathcal{U}_i).$$

□

Now we show that there exists a space X in which TWO has a winning strategy in the game ${}^*G_1(\mathcal{O},\mathcal{O})$, but no winning strategy in $G_1^*(\mathcal{O},\mathcal{O})$.

Example 3.4. Consider the space X constructed in Example 2.10. This space is Lindelöf because the real line $\mathbb{R}$ with upper limit topology is a hereditarily Lindeöf space. Suppose ONE and TWO are playing the game ${}^*G_1(\mathcal{O},\mathcal{O})$ and ONE chooses $\mathcal{U}_0$ for the 0-th innings. Clearly there exists $\mathcal{W} \in [\mathcal{U}_0]^{\leq\omega}$ such that $X = \bigcup\mathcal{W}$. Suppose $\mathcal{W} = \{W_0, W_1, W_2, \ldots.\}$. TWO, according to his strategy σ, responds by choosing $\sigma(\mathcal{U}_0) = W_0 \in \mathcal{W} \subset \mathcal{U}_0$. After that for n-th innings ($n \in \omega \setminus \{0\}$), whenever ONE chooses $\mathcal{U}_n \in \mathcal{O}$, TWO responds by choosing a $U_n \in \mathcal{U}_n$ such that $U_n \cap W_n \neq \emptyset$. We observe that $W_n \subset St(U_n,\mathcal{U}_0)$ for each $n \in \omega$. So, $W_n \subset St(\bigcup_{i\in\omega} U_i,\mathcal{U}_0)$ for each $n \in \omega$. Therefore, $X = \bigcup\{\mathcal{W}\} \subset St(\bigcup_{i\in\omega} U_i,\mathcal{U}_0)$,. which means that σ is a winning strategy for TWO in the game ${}^*G_1(\mathcal{O},\mathcal{O})$.

Now suppose ONE and TWO are playing the game $G_1^*(\mathcal{O},\mathcal{O})$ on the same space X. If ONE chooses $\mathcal{U}_n = \left\{\left(0,\frac{1}{2^n}\right],\left(\frac{1}{2^n},\frac{2}{2^n}\right],\left(\frac{2}{2^n},\frac{3}{2^n}\right],\ldots,\left(\frac{3.2^2-1}{2^n},\frac{3.2^n}{2^n}\right]\right\} \in \mathcal{O}$ for each inning (i.e. for each $n \in \omega$), then for any choice $U_n \in \mathcal{U}_n$ by TWO, $\{St(U_n,\mathcal{U}_n) : n \in \omega\} \notin \mathcal{O}$. So, TWO does not have a winning strategy in $G_1^*(\mathcal{O},\mathcal{O})$.

From the above results we conclude that the games ${}^*G_1(\mathcal{O},\mathcal{O})$ and $G_1^*(\mathcal{O},\mathcal{O})$ are neither equivalent nor dual to each other, which intern reflects the significance of our study.

Proposition 3.5. *If for a space X, TWO has a winning strategy in ${}^*G_1(\mathcal{A},\mathcal{B})$, then TWO has a winning strategy in ${}^*G_{fin}(\mathcal{A},\mathcal{B})$.*

Proposition 3.6. *If for a space X, TWO has no winning strategy in ${}^*G_{fin}(\mathcal{A},\mathcal{B})$, then TWO has no winning strategy in ${}^*G_1(\mathcal{A},\mathcal{B})$.*

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