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Relation-Theoretic Metrical Coincidence Theorems

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Abstract. In this article, we generalize some frequently used metrical notions such as: completeness, closedness, continuity, g-continuity and compatibility to relation-theoretic setting and utilize these relatively weaker notions to prove our results on the existence and uniqueness of coincidence points involving a pair of mappings defined on a metric space endowed with an arbitrary binary relation. Particularly, under universal relation our results deduce the classical coincidence point theorems of Goebel, Jungck and others. Furthermore, our results generalize, modify, unify and extend several well-known results of the existing literature.

1. Introduction

In recent years, various fixed and coincidence point theorems are proved in metric spaces equipped with different types of binary relations, *e.g.*, partial order (see Ran and Reurings [1], Nieto and Rodríguez-López [2] and Nieto and Rodríguez-López [3]), preorder (see Turinici [4], Roldán and Karapinar [5], Roldán-López-de-Hierro and Shahzad [6]), transitive relation (see Ben-El-Mechaiekh [7], Shahzad *et al*. [8]), finitely transitive relation (see Berzig and Karapinar [9], Berzig *et al*. [10]), tolerance (see Turinici [11, 12]), strict order (see Ghods *et al*. [13]), symmetric closure (see Samet and Turinici [14], Berzig [15]) and arbitrary binary relation (see Alam and Imdad [16], Roldán-López-de-Hierro [17], Roldán-López-de-Hierro and Shahzad [18], Shahzad *et al*. [19], Khan *et al*. [20], Ayari *et al*. [21]). In the present context, the contraction condition remains relatively weaker than usual contraction as it is required to hold merely for those elements which are related in the underlying relation.

The aim of this paper is to prove some existence and uniqueness results on coincidence points in metric spaces endowed with arbitrary binary relation for linear contractions. In proving our results, we use some relation-theoretic notions such as: $\mathcal{R}$-completeness, $\mathcal{R}$-closedness, $\mathcal{R}$-continuity, $(g, \mathcal{R})$-continuity, $\mathcal{R}$-compatibility, $\mathcal{R}$-connected sets *etc*. In this course, we also observe that our results combine the idea contained in Karapinar *et al*. [22] as the set M (utilized by Karapinar *et al*. [22]) being subset of X^2 is, in fact, a binary relation on X. As consequences of our newly proved results, we deduce several other established metrical coincidence point theorems. Finally, we furnish some illustrative examples to demonstrate our results.

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2. Preliminaries

Throughout this manuscript, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Q}$ and $\mathbb{R}$ denote the sets of natural numbers, whole numbers (*i.e.*, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), rational numbers and real numbers respectively . For the sake of completeness, firstly we recall some known relevant definitions.

Definition 2.1. *[23, 24] Let X be a nonempty set and f and g two self-mappings on X. Then*

(i) an element $x \in X$ is called a coincidence point of f and g if

$$gx = fx,$$

(ii) if $x \in X$ is a coincidence point of f and g and $\overline{x} \in X$ such that $\overline{x} = gx = fx$, then $\overline{x}$ is called a point of coincidence of f and g,

(iii) if $x \in X$ is a coincidence point of f and g such that $x = gx = fx$, then x is called a common fixed point of f and g,

(iv) f and g are called commuting if

$$g(fx) = f(gx) \quad \forall\, x \in X \text{ and}$$

(v) f and g are called weakly compatible if f and g commute at their coincidence points, i.e., for any $x \in X$,

$$gx = fx \Rightarrow g(fx) = f(gx).$$

Definition 2.2. *[25–27] Let (X, d) be a metric space and f and g two self-mappings on X. Then*

(i) f and g are called weakly commuting if for all $x \in X$,

$$d(g(fx), f(gx)) \leq d(gx, fx) \text{ and}$$

(ii) f and g are called compatible if

$$\lim_{n\to\infty} d(g(fx_n), f(gx_n)) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n,$$

(iii) f is called g-continuous at some $x \in X$ if for all sequences $\{x_n\} \subset X$,

$$gx_n \xrightarrow{d} gx \Rightarrow fx_n \xrightarrow{d} fx.$$

Moreover, f is called g-continuous if it is g-continuous at each point of X.

3. Relation-Theoretic Notions and Auxiliary Results

In this section, to make our exposition self-contained, we give some definitions and basic results related to our main results.

Definition 3.1. *[28] Let X be a nonempty set. A subset $\mathcal{R}$ of X^2 is called a binary relation on X.*

Notice that for each pair $x, y \in X$, one of the following holds:

(i) $(x, y) \in \mathcal{R}$; means that "x is $\mathcal{R}$-related to y" or "x relates to y under $\mathcal{R}$". Sometimes, we write $x\mathcal{R}y$ instead of $(x, y) \in \mathcal{R}$.

(ii) $(x, y) \notin \mathcal{R}$; means that "$x$ is not $\mathcal{R}$-related to y" or "x doesn't relate to y under $\mathcal{R}$".

Trivially, X^2 and $\emptyset$ are binary relations on X, which are respectively called the universal relation (or full relation) and empty relation.

Throughout this paper, $\mathcal{R}$ stands for a nonempty binary relation but for the sake of simplicity, we often write 'binary relation' instead of 'nonempty binary relation'.

Definition 3.2. *[16] Let $\mathcal{R}$ be a binary relation on a nonempty set X and $x, y \in X$. We say that x and y are $\mathcal{R}$-comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$.*

One can predict the following.

Proposition 3.3. *If (X, d) is a metric space, $\mathcal{R}$ is a binary relation on X, f and g are two self-mappings on X and $\alpha \in [0, 1)$, then the following contractivity conditions are equivalent:*
(I) $d(fx, fy) \leq \alpha d(gx, gy) \;\; \forall\, x, y \in X$ with $(gx, gy) \in \mathcal{R}$,
(II) $d(fx, fy) \leq \alpha d(gx, gy) \;\; \forall\, x, y \in X$ with $[gx, gy] \in \mathcal{R}$.

Definition 3.4. *[28, 29] A binary relation $\mathcal{R}$ on a nonempty set X is called*
- reflexive *if $(x, x) \in \mathcal{R} \;\; \forall x \in X$,*
- symmetric *if whenever $(x, y) \in \mathcal{R}$ then $(y, x) \in \mathcal{R}$,*
- antisymmetric *if whenever $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ then $x = y$,*
- transitive *if whenever $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ then $(x, z) \in \mathcal{R}$,*
- complete *or* connected *or* dichotomous *if $[x, y] \in \mathcal{R} \;\; \forall\, x, y \in X$,*
- weakly complete *or* weakly connected *or* trichotomous *if $[x, y] \in \mathcal{R}$ or $x = y \;\; \forall\, x, y \in X$.*

Definition 3.5. *[14, 28–32] A binary relation $\mathcal{R}$ defined on a nonempty set X is called*
- amorphous *if $\mathcal{R}$ has no specific property at all,*
- strict order *or* sharp order *if $\mathcal{R}$ is irreflexive and transitive,*
- near-order *if $\mathcal{R}$ is antisymmetric and transitive,*
- pseudo-order *if $\mathcal{R}$ is reflexive and antisymmetric,*
- quasi-order *or* preorder *if $\mathcal{R}$ is reflexive and transitive,*
- partial order *if $\mathcal{R}$ is reflexive, antisymmetric and transitive,*
- tolerance *if $\mathcal{R}$ is reflexive and symmetric,*
- equivalence *if $\mathcal{R}$ is reflexive, symmetric and transitive.*

Remark 3.6. *Clearly, universal relation X^2 on a nonempty set X remains a complete equivalence relation.*

Definition 3.7. *[28] Let X be a nonempty set and $\mathcal{R}$ a binary relation on X.*

(1) The inverse or transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} := \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}$.
(2) The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (i.e. $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$). Indeed, $\mathcal{R}^s$ is the smallest symmetric relation on X containing $\mathcal{R}$.

Proposition 3.8. *[16] For a binary relation $\mathcal{R}$ on a nonempty set X,*

$$(x, y) \in \mathcal{R}^s \iff [x, y] \in \mathcal{R}.$$

Definition 3.9. *[33] Let X be a nonempty set, $E \subseteq X$ and $\mathcal{R}$ a binary relation on X. Then, the restriction of $\mathcal{R}$ to E, denoted by $\mathcal{R}|_E$, is defined to be the set $\mathcal{R} \cap E^2$ (i.e. $\mathcal{R}|_E := \mathcal{R} \cap E^2$). Indeed, $\mathcal{R}|_E$ is a relation on E induced by $\mathcal{R}$.*

Definition 3.10. *[16] Let X be a nonempty set and $\mathcal{R}$ a binary relation on X. A sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-preserving if*

$$(x_n, x_{n+1}) \in \mathcal{R} \;\; \forall\, n \in \mathbb{N}_0.$$

Shahzad *et al.* [19] and Roldán-López-de-Hierro and Shahzad [18] used the term "$\mathcal{R}$-nondecreasing" instead of "$\mathcal{R}$-preserving".

Motivated by the notion of F-invariant subset of X^6 (for the mapping $F : X^3 \to X$) due to Charoensawan [48] and the notion of F-closed subset of X^4 (for the mapping $F : X^2 \to X$) due to Kutbi *et al.* [49], Alam and Imdad [16] introduced the following.

Definition 3.11. *[16] Let X be a nonempty set and f a self-mapping on X. A binary relation $\mathcal{R}$ on X is called f-closed if for all $x, y \in X$,*

$$(x, y) \in \mathcal{R} \Rightarrow (fx, fy) \in \mathcal{R}.$$

Here it can be pointed out that Definition 3.11 is equivalent to saying that f is $\mathcal{R}$-nondecreasing as defined by Shahzad *et al*. [8].

Definition 3.12. *Let X be a nonempty set and f and g two self-mappings on X. A binary relation $\mathcal{R}$ on X is called (f, g)-closed if for all $x, y \in X$,*

$$(gx, gy) \in \mathcal{R} \Rightarrow (fx, fy) \in \mathcal{R}.$$

Notice that under the restriction $g = I$, the identity mapping on X, Definition 3.12 reduces to Definition 3.11.

Here it can be pointed out that Definition 3.12 is equivalent to saying that f is $(g, \mathcal{R})$-nondecreasing as defined by Roldán-López-de-Hierro and Shahzad [6].

Proposition 3.13. *Let X be a nonempty set, $\mathcal{R}$ a binary relation on X and f and g two self-mappings on X. If $\mathcal{R}$ is (f, g)-closed, then $\mathcal{R}^s$ is also (f, g)-closed.*

In the following lines, we introduce relation-theoretic variants of the metrical notions: completeness, closedness, continuity, g-continuity and compatibility.

Definition 3.14. *Let (X, d) be a metric space and $\mathcal{R}$ a binary relation on X. We say that (X, d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges.*

Here it can be pointed out that instead of saying that (X, d) is $\mathcal{R}$-complete, Shahzad *et al*. [19] and Roldán-López-de-Hierro and Shahzad [18] said that (X, d) is $\mathcal{R}$-nondecreasing-complete.

Remark 3.15. *Every complete metric space is $\mathcal{R}$-complete, for any binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$-completeness coincides with usual completeness.*

Definition 3.16. *Let (X, d) be a metric space and $\mathcal{R}$ a binary relation on X. A subset E of X is called $\mathcal{R}$-closed if every $\mathcal{R}$-preserving convergent sequence in E converges to a point of E.*

Here it can be pointed out that instead of saying that E is $\mathcal{R}$-closed, Roldán-López-de-Hierro and Shahzad [6] said that E is $(d, \preccurlyeq)$-nondecreasing-closed in case of preorder $\preccurlyeq$ and E is $\overline{\mathrm{O}}$-closed defined by Alam *et al*. [47] in case of partial order $\preceq$.

Remark 3.17. *Every closed subset of a metric space is $\mathcal{R}$-closed, for any binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$-closedness coincides with usual closedness.*

Proposition 3.18. *An $\mathcal{R}$-complete subspace of a metric space is $\mathcal{R}$-closed.*

Proof. Let (X, d) be a metric space. Suppose that Y is an $\mathcal{R}$-complete subspace of X. Take an $\mathcal{R}$-preserving sequence $\{x_n\} \subset Y$ such that $x_n \xrightarrow{d} x \in X$. As each convergent sequence is Cauchy, $\{x_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence in Y. Hence, $\mathcal{R}$-completeness of Y implies that the limit of $\{x_n\}$ must lie in Y, *i.e.*, $x \in Y$. Therefore, Y is $\mathcal{R}$-closed. □

Proposition 3.19. *An $\mathcal{R}$-closed subspace of an $\mathcal{R}$-complete metric space is $\mathcal{R}$-complete.*

Proof. Let (X, d) be an $\mathcal{R}$-complete metric space. Suppose that Y is $\mathcal{R}$-closed subspace of X. Let $\{x_n\}$ be an $\mathcal{R}$-preserving Cauchy sequence in Y. As X is $\mathcal{R}$-complete, $\exists\ x \in X$ such that $x_n \xrightarrow{d} x$ and so $\{x_n\}$ is an $\mathcal{R}$-preserving sequence converging to x. Hence, $\mathcal{R}$-closedness of Y implies that $x \in Y$. Therefore, Y is $\mathcal{R}$-complete. □

Definition 3.20. *Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X and $x \in X$. A mapping $f : X \to X$ is called $\mathcal{R}$-continuous at x if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, we have $fx_n \xrightarrow{d} fx$. f is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of X.*

Here it can be pointed out that instead of saying that f is $\mathcal{R}$-continuous, Roldán-López-de-Hierro *et al.* [17], Shahzad *et al.* [19] and Roldán-López-de-Hierro and Shahzad [18] said that f is $(d, \mathcal{R})$-nondecreasing-continuous.

Remark 3.21. *Every continuous mapping is $\mathcal{R}$-continuous, for any binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$-continuity coincides with usual continuity.*

Definition 3.22. *Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X, g a self-mapping on X and $x \in X$. A mapping $f : X \to X$ is called $(g, \mathcal{R})$-continuous at x if for any sequence $\{x_n\}$ such that $\{gx_n\}$ is $\mathcal{R}$-preserving and $gx_n \xrightarrow{d} gx$, we have $fx_n \xrightarrow{d} fx$. Moreover, f is called $(g, \mathcal{R})$-continuous if it is $(g, \mathcal{R})$-continuous at each point of X.*

Notice that under the restriction $g = I$, the identity mapping on X, Definition 3.22 reduces to Definition 3.20.

Remark 3.23. *Every g-continuous mapping is $(g, \mathcal{R})$-continuous, for any binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $(g, \mathcal{R})$-continuity coincides with usual g-continuity.*

Definition 3.24. *Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X and f and g two self-mappings on X. We say that f and g are $\mathcal{R}$-compatible if for any sequence $\{x_n\} \subset X$ such that $\{fx_n\}$ and $\{gx_n\}$ are $\mathcal{R}$-preserving and $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n$, we have*

$$\lim_{n\to\infty} d(g(fx_n), f(gx_n)) = 0.$$

Remark 3.25. *In a metric space (X, d) endowed with a binary relation $\mathcal{R}$, commutativity $\Rightarrow$ weak commutativity $\Rightarrow$ compatibility $\Rightarrow$$\mathcal{R}$-compatibility $\Rightarrow$ weak compatibility. Particularly, under the universal relation the notion of $\mathcal{R}$-compatibility coincides with usual compatibility.*

The following notion is a generalization of 'd-self-closedness of partial order relation $(\preceq)$' defined by Turinici [4].

Definition 3.26. *[16] Let (X, d) be a metric space. A binary relation $\mathcal{R}$ on X is called d-self-closed if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, there exists a subsequence $\{x_{n_k}\}$* of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R} \ \ \forall\, k \in \mathbb{N}_0$.

Here it can be pointed out that the notion of d-self-closedness of $\mathcal{R}$ is relatively weaker than the notion of $\mathcal{R}$-nondecreasing regularity of (X, d) defined by Roldán-López-de-Hierro *et al.* [17] and Roldán-López-de-Hierro and Shahzad [18].

Definition 3.27. *Let (X, d) be a metric space and g a self-mapping on X. A binary relation $\mathcal{R}$ on X is called (g, d)-self-closed if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[gx_{n_k}, gx] \in \mathcal{R} \ \ \forall\, k \in \mathbb{N}_0$.*

Notice that under the restriction $g = I$, the identity mapping on X, Definition 3.27 reduces to Definition 3.26.

Definition 3.28. *[14] Let X be a nonempty set and $\mathcal{R}$ a binary relation on X. A subset E of X is called $\mathcal{R}$-directed if for each pair $x, y \in E$, there exists $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.*

Definition 3.29. *[33] Let X be a nonempty set and $\mathcal{R}$ a binary relation on X. For $x, y \in X$, a path of length k (where k is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{z_0, z_1, z_2, ..., z_k\} \subset X$ satisfying the following conditions:*
(i) $z_0 = x$ and $z_k = y$,
(ii) $(z_i, z_{i+1}) \in \mathcal{R}$ *for each* i $(0 \leq i \leq k - 1)$.

Notice that a path of length k involves $k+1$ elements of X, although they are not necessarily distinct.

Definition 3.30. *Let X be a nonempty set and $\mathcal{R}$ a binary relation on X. A subset E of X is called $\mathcal{R}$-connected if for each pair $x, y \in E$, there exists a path in $\mathcal{R}$ from x to y.*

Given a binary relation $\mathcal{R}$ and two self-mappings f and g defined on a nonempty set X, we use the following notations:

(i) $C(f,g) := \{x \in X : gx = fx\}$, *i.e.*, the set of all coincidence points of f and g,
(ii) $\overline{C}(f,g) := \{\overline{x} \in X : \overline{x} = gx = fx,\ x \in X\}$, *i.e.*, the set of all points of coincidence of f and g,
(iii) $X(f,\mathcal{R}) := \{x \in X : (x, fx) \in \mathcal{R}\}$,
(iv) $X(f,g,\mathcal{R}) := \{x \in X : (gx, fx) \in \mathcal{R}\}$.

The main result of Alam and Imdad [16] which is indeed the relation-theoretic version of Banach contraction principle runs as follows:

Theorem 3.31. *[16] Let (X,d) be a complete metric space, $\mathcal{R}$ a binary relation on X and f a self-mapping on X. Suppose that the following conditions hold:*
(i) $\mathcal{R}$ is f-closed,
(ii) either f is continuous or $\mathcal{R}$ is d-self-closed,
(iii) $X(f,\mathcal{R})$ is nonempty,
(iv) there exists $\alpha \in [0,1)$ such that
$$d(fx, fy) \leq \alpha d(x,y) \quad \forall\, x, y \in X \text{ with } (x,y) \in \mathcal{R}.$$
Then f has a fixed point. Moreover, if
(v) X is $\mathcal{R}^s$-connected,
then f has a unique fixed point.

Finally, we record the following known results, which are needed in the proof of our main results.

Lemma 3.32. *[34] Let X be a nonempty set and g a self-mapping on X. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one.*

Lemma 3.33. *[35] Let X be a nonempty set and f and g two self-mappings on X. If f and g are weakly compatible, then every point of coincidence of f and g is also a coincidence point of f and g.*

4. Main Results

Now, we are equipped to prove our main result on the existence of coincidence points which runs as follows:

Theorem 4.1. *Let (X,d) be a metric space, $\mathcal{R}$ a binary relation on X and Y an $\mathcal{R}$-complete subspace of X. Let f and g be two self-mappings on X. Suppose that the following conditions hold:*
(a) $f(X) \subseteq g(X) \cap Y$,
(b) $\mathcal{R}$ is (f,g)-closed,
(c) $X(f,g,\mathcal{R})$ is nonempty,
(d) there exists $\alpha \in [0,1)$ such that
$$d(fx, fy) \leq \alpha d(gx, gy) \quad \forall\, x, y \in X \text{ with } (gx, gy) \in \mathcal{R},$$
(e) (e1) f and g are $\mathcal{R}$-compatible,
(e2) g is $\mathcal{R}$-continuous,
(e3) either f is $\mathcal{R}$-continuous or $\mathcal{R}$ is (g,d)-self-closed,
or alternately,
(e′) (e′1) $Y \subseteq g(X)$,
(e′2) either f is $(g,\mathcal{R})$-continuous or f and g are continuous or $\mathcal{R}|_Y$ is d-self-closed.
Then f and g have a coincidence point.

Proof. Assumption (a) is equivalent to saying that $f(X) \subseteq g(X)$ and $f(X) \subseteq Y$. In view of assumption (c), let x_0 be an arbitrary element of $X(f, g, \mathcal{R})$, then $(gx_0, fx_0) \in \mathcal{R}$. If $gx_0 = fx_0$, then x_0 is a coincidence point of f and g and hence we are done. Otherwise, if $gx_0 \neq fx_0$, then from $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Again from $f(X) \subseteq g(X)$, we can choose $x_2 \in X$ such that $gx_2 = fx_1$. Continuing this process, we construct a sequence $\{x_n\} \subset X$ (of joint iterates) such that

$$gx_{n+1} = fx_n \quad \forall\, n \in \mathbb{N}_0. \tag{1}$$

Now, we claim that $\{gx_n\}$ is $\mathcal{R}$-preserving sequence, *i.e.*,

$$(gx_n, gx_{n+1}) \in \mathcal{R} \quad \forall\, n \in \mathbb{N}_0. \tag{2}$$

We prove this claim by mathematical induction. On using equation (1) (with $n = 0$) and the fact that $x_0 \in X(f, g, \mathcal{R})$, we have

$$(gx_0, gx_1) \in \mathcal{R},$$

which shows that (2) holds for $n = 0$. Suppose that (2) holds for $n = r > 0$, *i.e.*,

$$(gx_r, gx_{r+1}) \in \mathcal{R}.$$

As $\mathcal{R}$ is (f, g)-closed, we have

$$(fx_r, fx_{r+1}) \in \mathcal{R},$$

which, on using (1), yields that

$$(gx_{r+1}, gx_{r+2}) \in \mathcal{R},$$

i.e., (2) holds for $n = r + 1$. Hence, by induction, (2) holds for all $n \in \mathbb{N}_0$.
In view of (1) and (2), the sequence $\{fx_n\}$ is also an $\mathcal{R}$-preserving, *i.e.*,

$$(fx_n, fx_{n+1}) \in \mathcal{R} \quad \forall\, n \in \mathbb{N}_0. \tag{3}$$

On using (1), (2) and assumption (d), we obtain

$$d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n) \leq \alpha d(gx_{n-1}, gx_n) \quad \forall\, n \in \mathbb{N}.$$

By induction, we have

$$d(gx_n, gx_{n+1}) \leq \alpha d(gx_{n-1}, gx_n) \leq \alpha^2 d(gx_{n-2}, gx_{n-1}) \leq \cdots \leq \alpha^n d(gx_0, gx_1) \quad \forall\, n \in \mathbb{N}$$

so that

$$d(gx_n, gx_{n+1}) \leq \alpha^n d(gx_0, gx_1) \quad \forall\, n \in \mathbb{N}. \tag{4}$$

For $n < m$, using (4), we obtain

$$\begin{aligned} d(gx_n, gx_m) &\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m) \\ &\leq (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}) d(gx_0, gx_1) \\ &= \frac{\alpha^n - \alpha^m}{1-\alpha} d(gx_0, gx_1) \\ &\leq \frac{\alpha^n}{1-\alpha} d(gx_0, gx_1) \\ &\to 0 \text{ as } m, n \to \infty. \end{aligned}$$

Therefore $\{gx_n\}$ is a Cauchy sequence.

In view of (1), $\{gx_n\} \subset f(X) \subseteq Y$ so that $\{gx_n\}$ is $\mathcal{R}$-preserving Cauchy sequence in Y. As Y is $\mathcal{R}$-complete, there exists $z \in Y$ such that

$$\lim_{n\to\infty} gx_n = z. \tag{5}$$

On using (1) and (5), we obtain

$$\lim_{n\to\infty} fx_n = z. \tag{6}$$

Now, we use assumptions (e) and (e') to accomplish the proof. Assume that (e) holds. Using (2), (5) and assumption ($e2$) (*i.e.* $\mathcal{R}$-continuity of g), we have

$$\lim_{n\to\infty} g(gx_n) = g(\lim_{n\to\infty} gx_n) = gz. \tag{7}$$

Now, using (3), (6) and assumption ($e2$) (*i.e.* $\mathcal{R}$-continuity of g), we have

$$\lim_{n\to\infty} g(fx_n) = g(\lim_{n\to\infty} fx_n) = gz. \tag{8}$$

As $\{fx_n\}$ and $\{gx_n\}$ are $\mathcal{R}$-preserving (due to (2) and (3)) and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ (due to (5) and (6)), on using assumption ($e1$)(*i.e.* $\mathcal{R}$-compatibility of f and g), we obtain

$$\lim_{n\to\infty} d(g(fx_n), f(gx_n)) = 0. \tag{9}$$

Now, we show that z is a coincidence point of f and g. To accomplish this, we use assumption (e). Firstly, suppose that f is $\mathcal{R}$-continuous. On using (2), (5) and $\mathcal{R}$-continuity of f, we obtain

$$\lim_{n\to\infty} f(gx_n) = f(\lim_{n\to\infty} gx_n) = fz. \tag{10}$$

On using (8), (9), (10) and continuity of d, we obtain

$$\begin{aligned} d(gz, fz) &= d(\lim_{n\to\infty} g(fx_n), \lim_{n\to\infty} f(gx_n)) \\ &= \lim_{n\to\infty} d(g(fx_n), f(gx_n)) \\ &= 0 \end{aligned}$$

so that

$$gz = fz.$$

Hence we are done. Alternately, suppose that $\mathcal{R}$ is (g, d)-self-closed. As $\{gx_n\}$ is $\mathcal{R}$-preserving (due to (2)) and $gx_n \xrightarrow{d} z$ (due to (5)), by using (g, d)-self-closedness of $\mathcal{R}$, there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$[ggx_{n_k}, gz] \in \mathcal{R} \ \ \forall\, k \in \mathbb{N}_0. \tag{11}$$

Since $gx_{n_k} \xrightarrow{d} z$, so equations (5)-(9) also hold for $\{x_{n_k}\}$ instead of $\{x_n\}$. On using (11), assumption (d) and Proposition 3.3, we obtain

$$d(f(gx_{n_k}), fz) \le \alpha d(g(gx_{n_k}), gz) \ \ \forall\, k \in \mathbb{N}_0. \tag{12}$$

On using triangular inequality, (7), (8), (9) and (12), we get

$$\begin{aligned} d(gz, fz) &\le d(gz, g(fx_{n_k}) + d(g(fx_{n_k}), f(gx_{n_k}) + d(f(gx_{n_k}), fz) \\ &\le d(gz, g(fx_{n_k})) + d(g(fx_{n_k}), f(gx_{n_k})) + \alpha d(g(gx_{n_k}), gz) \\ &\to 0 \text{ as } k \to \infty \end{aligned}$$

so that

$$gz = fz.$$

Thus $z \in X$ is a coincidence point of f and g and hence we are done.

Now, assume that assumption (e') holds. In view of the assumption ($e'1$) (*i.e.*, $Y \subseteq g(X)$), we can find some $u \in X$ such that $z = gu$. Hence, (5) and (6) respectively reduce to

$$\lim_{n\to\infty} gx_n = gu. \tag{13}$$

$$\lim_{n\to\infty} fx_n = gu. \tag{14}$$

Now, we show that u is a coincidence point of f and g. To accomplish this, we use assumption $(e'2)$. Firstly, suppose that f is $(g, \mathcal{R})$-continuous, then using (2) and (13), we get

$$\lim_{n\to\infty} fx_n = fu. \tag{15}$$

On using (14) and (15), we get

$$gu = fu.$$

Hence, we are done.
Secondly, suppose that f and g are continuous. In view of Lemma 3.32, there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one. Now, define $T : g(E) \to g(X)$ by

$$T(ga) = fa \ \ \forall\ ga \in g(E) \text{ where } a \in E. \tag{16}$$

As $g : E \to X$ is one-to-one and $f(X) \subseteq g(X)$, T is well defined. As f and g are continuous, so, is T. Using the fact $g(X) = g(E)$, assumptions (a) and $(e'1)$ respectively, give rise $f(X) \subseteq g(E) \cap Y$ and $Y \subseteq g(E)$, which ensures that (without loss of generality), we can construct $\{x_n\}_{n=1}^{\infty} \subset E$ satisfying (1) and we can choose $u \in E$. On using (13), (14), (16) and continuity of T, we get

$$fu = T(gu) = T(\lim_{n\to\infty} gx_n) = \lim_{n\to\infty} T(gx_n) = \lim_{n\to\infty} fx_n = gu.$$

Thus $u \in X$ is a coincidence point of f and g and hence we are done. Finally, suppose that $\mathcal{R}|_Y$ is d-self-closed. As $\{gx_n\}$ is $\mathcal{R}|_Y$-preserving (due to (2)) and $gx_n \xrightarrow{d} gu \in Y$ (due to (13)), using d-self-closedness of $\mathcal{R}|_Y$, there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$[gx_{n_k}, gu] \in \mathcal{R}|_Y \ \ \forall\ k \in \mathbb{N}_0. \tag{17}$$

On using (13), (17), assumption (d) and Proposition 3.3, we obtain

$$\begin{aligned} d(fx_{n_k}, fu) &\leq \alpha d(gx_{n_k}, gu) \\ &\to 0 \text{ as } k \to \infty \end{aligned}$$

so that

$$\lim_{k\to\infty} fx_{n_k} = fu. \tag{18}$$

Using (14) and (18), we get

$$gu = fu.$$

Thus, we are done. This completes the proof. □

Now, as a consequence, we particularize Theorem 4.1 by assuming the $\mathcal{R}$-completeness of whole space X.

Corollary 4.2. *Let X be a nonempty set equipped with a binary relation $\mathcal{R}$ and a metric d such that (X, d) is an $\mathcal{R}$-complete metric space. Let f and g be two self-mappings on X. Suppose that the following conditions hold:*
(*a*) $f(X) \subseteq g(X)$,
(*b*) $\mathcal{R}$ *is* (f, g)*-closed,*
(*c*) $X(f, g, \mathcal{R})$ *is nonempty,*
(*d*) *there exists* $\alpha \in [0, 1)$ *such that*
$$d(fx, fy) \leq \alpha d(gx, gy) \quad \forall\, x, y \in X \text{ with } (gx, gy) \in \mathcal{R},$$
(*e*) (*e*1) *f and g are $\mathcal{R}$-compatible,*
(*e*2) *g is $\mathcal{R}$-continuous,*
(*e*3) *either f is $\mathcal{R}$-continuous or $\mathcal{R}$ is (g, d)-self-closed,*
or alternately,
(*e*′) (*e*′1) *there exists an $\mathcal{R}$-closed subspace Y of X such that $f(X) \subseteq Y \subseteq g(X)$,*
(*e*′2) *either f is $(g, \mathcal{R})$-continuous or f and g are continuous or $\mathcal{R}|_Y$ is d-self-closed.*
Then f and g have a coincidence point.

Proof. The result corresponding to part (*e*) follows easily on setting $Y = X$ in Theorem 4.1, while the same (result) in the presence of part (*e*′) follows using Proposition 3.19. □

Remark 4.3. *If g is onto in Corollary 4.2, then we can drop assumption (a) as in this case it trivially holds. Also, we can remove assumption (e′1) as it trivially holds for $Y = g(X) = X$ using Proposition 3.18. Whenever, f is onto, In view of assumption (a), g must be onto and hence again same conclusion is immediate.*

On using Remarks 3.15, 3.17, 3.21, 3.23 and 3.25, we obtain the more natural version of Theorem 4.1 in the form of the following consequence.

Corollary 4.4. *Theorem 4.1 (also Corollary 4.2) remains true if the usual metrical terms namely: completeness, closedness, compatibility (or commutativity/weak commutativity), continuity and g-continuity are used instead of their respective $\mathcal{R}$-analogues.*

Now, we present certain results enunciating the uniqueness of a point of coincidence, coincidence point and common fixed point corresponding to Theorem 4.1.

Theorem 4.5. *In addition to the hypotheses of Theorem 4.1, suppose that the following condition holds:*
(u_1): *$f(X)$ is $\mathcal{R}|^s_{g(X)}$-connected.*
Then f and g have a unique point of coincidence.

Proof. In view of Theorem 4.1, $\overline{C}(f, g) \neq \emptyset$. Take $\overline{x}, \overline{y} \in \overline{C}(f, g)$, then $\exists\, x, y \in X$ such that
$$\overline{x} = gx = fx \text{ and } \overline{y} = gy = fy. \tag{19}$$

Now, we show that $\overline{x} = \overline{y}$. As $fx, fy \in f(X) \subseteq g(X)$, by assumption ($u_1$), there exists a path (say $\{gz_0, gz_1, gz_2, ..., gz_k\}$) of some finite length k in $\mathcal{R}|^s_{g(X)}$ from fx to fy (where $z_0, z_1, z_2, ..., z_k \in X$). In view of (19), without loss of generality, we may choose $z_0 = x$ and $z_k = y$. Thus, we have
$$[gz_i, gz_{i+1}] \in \mathcal{R}|_{g(X)} \text{ for each } i\ (0 \leq i \leq k-1). \tag{20}$$

Define the constant sequences $z^0_n = x$ and $z^k_n = y$, then using (19), we have $gz^0_{n+1} = fz^0_n = \overline{x}$ and $gz^k_{n+1} = fz^k_n = \overline{y}\ \ \forall\, n \in \mathbb{N}_0$. Put $z^1_0 = z_1, z^2_0 = z_2, ..., z^{k-1}_0 = z_{k-1}$. Since $f(X) \subseteq g(X)$, on the lines similar to that of Theorem 4.1, we can define sequences $\{z^1_n\}, \{z^2_n\}, ..., \{z^{k-1}_n\}$ in X such that $gz^1_{n+1} = fz^1_n, gz^2_{n+1} = fz^2_n, ..., gz^{k-1}_{n+1} = fz^{k-1}_n\ \forall\, n \in \mathbb{N}_0$. Hence, we have
$$gz^i_{n+1} = fz^i_n \quad \forall\, n \in \mathbb{N}_0 \text{ and for each } i\ (0 \leq i \leq k). \tag{21}$$

Now, we claim that

$$[gz_n^i, gz_n^{i+1}] \in \mathcal{R} \ \ \forall\, n \in \mathbb{N}_0 \text{ and for each } i \ (0 \le i \le k-1). \tag{22}$$

We prove this fact by the method of mathematical induction. It follows from (20) that (22) holds for $n = 0$. Suppose that (22) holds for $n = r > 0$, *i.e.*,

$$[gz_r^i, gz_r^{i+1}] \in \mathcal{R} \ \text{ for each } i \ (0 \le i \le k-1).$$

As $\mathcal{R}$ is (f, g)-closed, using Proposition 3.13, we obtain

$$[fz_r^i, fz_r^{i+1}] \in \mathcal{R} \ \text{ for each } i \ (0 \le i \le k-1),$$

which on using (22), gives rise

$$[gz_{r+1}^i, gz_{r+1}^{i+1}] \in \mathcal{R} \ \text{ for each } i \ (0 \le i \le k-1).$$

It follows that (22) holds for $n = r+1$. Thus, by induction, (22) holds for all $n \in \mathbb{N}_0$. Now for all $n \in \mathbb{N}_0$ and for each i $(0 \le i \le k-1)$, define $t_n^i =: d(gz_n^i, gz_n^{i+1})$. Then, we claim that

$$\lim_{n\to\infty} t_n^i = 0. \tag{23}$$

On using (21), (22), assumption (*d*) and Proposition 3.3, for each i $(0 \le i \le k-1)$ and for all $n \in \mathbb{N}_0$, we obtain

$$\begin{array}{lll} t_{n+1}^i & = & d(gz_{n+1}^i, gz_{n+1}^{i+1}) \\ & = & d(fz_n^i, fz_n^{i+1}) \\ & \le & \alpha d(gz_n^i, z_n^{i+1}) \\ & = & \alpha t_n^i. \end{array}$$

By induction, we have

$$t_{n+1}^i \le \alpha t_n^i \le \alpha^2 t_{n-1}^i \le ... \le \alpha^{n+1} t_0^i$$

so that

$$t_{n+1}^i \le \alpha^{n+1} t_0^i,$$

yielding thereby

$$\lim_{n\to\infty} t_n^i = 0 \text{ for each } i \ (0 \le i \le k-1).$$

Thus, (23) is proved for each i $(0 \le i \le k-1)$. On using triangular inequality and (23), we obtain

$$d(\overline{x}, \overline{y}) \le t_n^0 + t_n^1 + \cdots + t_n^{k-1} \to 0 \ \ as \ \ n \to \infty.$$

Therefore, $\overline{x} = \overline{y}$, which concludes the proof □

Corollary 4.6. *Theorem 4.5 remains true if we replace the condition* (u_1) *by one of the following conditions:*
(u_1') $\mathcal{R}|_{f(X)}$ *is complete,*
(u_1'') $f(X)$ *is* $\mathcal{R}|^s_{g(X)}$*-directed.*

Proof. If (u_1') holds, then for each $u, v \in f(X)$, $[u, v] \in \mathcal{R}|_{f(X)} \subseteq \mathcal{R}|_{g(X)}$ (in view of assumption $f(X) \subseteq g(X)$), which amounts to say that $\{u, v\}$ is a path of length 1 in $\mathcal{R}|^s_{g(X)}$ from u to v. Hence $f(X)$ is $\mathcal{R}|^s_{g(X)}$-connected consequently Theorem 4.5 gives rise the conclusion.
Otherwise, if (u_1'') holds then for each $u, v \in f(X)$, $\exists\, w \in g(X)$ such that $[u, w] \in \mathcal{R}|_{g(X)}$ and $[v, w] \in \mathcal{R}|_{g(X)}$ (In view of assumption $f(X) \subseteq g(X)$), which amounts to say that $\{u, w, v\}$ is a path of length 2 in $\mathcal{R}|^s_{g(X)}$ from u to v. Hence $f(X)$ is $\mathcal{R}|^s_{g(X)}$-connected and again by Theorem 4.5 conclusion is immediate. □

Theorem 4.7. *In addition to the hypotheses of Theorem 4.5, suppose that the following condition holds:*
(u_2)*: one of f and g is one-to-one.*
Then f and g have a unique coincidence point.

Proof. In view of Theorem 4.1, $C(f,g) \neq \emptyset$. Take $x, y \in C(f,g)$, then in view of Theorem 4.5, we have

$$gx = fx = fy = gy.$$

As f or g is one-to-one, we have $x = y$. □

Theorem 4.8. *In addition to the hypotheses embodied in condition (e') of Theorem 4.5, suppose that the following condition holds:*
$(e'3)$*: f and g are weakly compatible.*
Then f and g have a unique common fixed point.

Proof. In view of Remark 3.25 and assumption $(e'3)$, the mappings f and g are weakly compatible. Take $x \in C(f,g)$ and denote $gx = fx = \overline{x}$. Then in view of Lemma 2, $\overline{x} \in C(f,g)$. It follows from Theorem 4.5 with $y = \overline{x}$ that $gx = g\overline{x}$, *i.e.*, $\overline{x} = g\overline{x}$, which yields that

$$\overline{x} = g\overline{x} = f\overline{x}.$$

Hence, $\overline{x}$ is a common fixed point of f and g. To prove uniqueness, assume that x^* is another common fixed point of f and g. Then again from Theorem 4.5, we have

$$x^* = gx^* = g\overline{x} = \overline{x}.$$

Hence, we are done. □

Corollary 4.9. *Let (X,d) be a metric space, $\mathcal{R}$ a binary relation on X and f a self-mapping on X. Let Y be an $\mathcal{R}$-complete subspace of X such that $f(X) \subseteq Y$. Suppose that the following conditions hold:*
(i) $\mathcal{R}$ is f-closed,
(ii) either f is $\mathcal{R}$-continuous or $\mathcal{R}|_Y$ is d-self-closed,
(iii) $X(f,\mathcal{R})$ is nonempty,
(iv) there exists $\alpha \in [0,1)$ such that
$$d(fx, fy) \leq \alpha d(x,y) \quad \forall\, x, y \in X \text{ with } (x,y) \in \mathcal{R}.$$
Then f has a fixed point. Moreover, if
(v) $f(X)$ is $\mathcal{R}^s$-connected,
then f has a unique fixed point.

Notice that Corollary 4.9 is an improvement of Theorem 3.31 in the following respects:

- Usual notions of completeness and continuity are not necessary. Alternately, they can be replaced by their respective $\mathcal{R}$-analogues.
- $\mathcal{R}$-completeness of whole space X and d-self-closedness of whole relation $\mathcal{R}$ are not necessary as they can be respectively replaced by $\mathcal{R}$-completeness of any subspace Y and d-self-closedness of $\mathcal{R}|_Y$, where$f(X) \subseteq Y \subseteq X$.
- For the uniqueness part, $\mathcal{R}^s$-connectedness of whole space X is not required. Only it suffices to take the same merely (*i.e.*, $\mathcal{R}^s$-connectedness) of the subset $f(X)$ of X.

Corollary 4.10. *Corollary 4.9 remains true if we replace assumption (v) by one of the following conditions:*
(v)′ $\mathcal{R}|_{f(X)}$ is complete,
(v)′′ $f(X)$ is $\mathcal{R}^s$-directed.

5. Some Consequences

In this section, we derive several results of the existing literature as consequences of our newly proved results presented in the earlier sections.

5.1. Coincidence theorems in abstract metric spaces

Under the universal relation (*i.e.* $\mathcal{R} = X^2$), Theorems 4.1, 4.5, 4.7 and 4.8 reduce to the following coincidence point theorems:

Corollary 5.1. *Let (X, d) be a metric space and Y a complete subspace of X. Let f and g be two self-mappings on X. Suppose that the following conditions hold:*
(a) $f(X) \subseteq g(X) \cap Y$,
(b) *there exists* $\alpha \in [0, 1)$ *such that*

$$d(fx, fy) \leq \alpha d(gx, gy) \quad \forall\, x, y \in X \text{ with } (gx, gy) \in \mathcal{R},$$

(c) $(c1)$ *f and g are compatible,*
$(c2)$ *g is continuous,*
or alternately,
(c') $Y \subseteq g(X)$.
Then f and g have a unique point of coincidence.

Corollary 5.2. *In addition to the hypotheses of Corollary 5.1, if one of f and g is one-to-one, then f and g have a unique coincidence point.*

Observe that Corollary 5.1 (corresponding to conditions (a), (b) and (e)) ensures the uniqueness of the common fixed point of the maps f and g. For the alternate hypothesis, we have the following.

Corollary 5.3. *In addition to the hypothesis of Corollary 5.1 (corresponding to conditions (a), (b) and (e')), if f and g are weakly compatible, then f and g have a unique common fixed point.*

Notice that Corollaries 5.1, 5.2 and 5.3 improve the well-known coincidence theorems of Goebel [36] and Jungck [26].

5.2. Coincidence theorems under (f, g)-closed sets

Samet and Vetro [37] introduced the notion of F-invariant sets and utilized the same to prove some coupled fixed point results for generalized linear contractions on metric spaces without any partial order. Recently, Kutbi *et al*. [38] weakened the notion of F-invariant sets by introducing the notion of F-closed sets. Most recently, Karapinar *et al*. [22] proved some unidimensional versions of earlier coupled fixed point results involving F-closed sets. To describe such results, we need to recall the following notions:

Definition 5.4. *[22] Let $f, g : X \to X$ be two mappings and $M \subseteq X^2$ a subset. We say that M is:*

(i) *(f, g)-closed if $(fx, fy) \in M$ for all $x, y \in X$ whenever $(gx, gy) \in M$,*
(ii) *(f, g)-compatible if $fx = fy$ for all $x, y \in X$ whenever $gx = gy$.*

Definition 5.5. *[22]. We say that a subset M of X^2 is transitive if $(x, y), (y, z) \in M$ implies that $(x, z) \in M$.*

Definition 5.6. *[22]. Let (X, d) be a metric space and $M \subseteq X^2$ a subset. We say that (X, d, M) is regular if for all sequence $\{x_n\} \subseteq X$ such that $x_n \xrightarrow{d} x$ and $(x_n, x_{n+1}) \in M$ for all n, we have $(x_n, x) \in M$ for all n.*

Definition 5.7. *[22] Let (X, d) be a metric space, $M \subseteq X^2$ a subset and $x \in X$. A mapping $f : X \to X$ is said to be M-continuous at x if for all sequence $\{x_n\} \subseteq X$ such that $x_n \xrightarrow{d} x$ and $(x_n, x_{n+1}) \in M$ for all n, we have $fx_n \xrightarrow{d} fx$. Moreover, f is called M-continuous if it is M-continuous at each $x \in X$.*

The following notion is introduced in order to improve the commutativity condition of the pair of mappings f and g, which is inspired by the notion of O-compatibility of Luong and Thuan [39] in ordered metric spaces.

Definition 5.8. *[22] Let (X,d) be a metric space and $M\subseteq X^2$. Two mappings $f,g : X \to X$ are said to be M-compatible if*

$$\lim_{n\to\infty} d(g(fx_n), f(gx_n)) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $(gx_n, gx_{n+1}) \in M$ for all n and $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n \in X$.

Notice that Karapinar *et al*. [22] (inspired by the notion of O-compatibility in [39]) preferred to call"*(O,M)*-compatible" instead of "M-compatible". Here the symbol "O" has no pertinence as Luong and Thuan [39] used the term "O-compatible" due to available partial ordering on the underlying metric space (*i.e*. O means order relation). But in above context, Karapinar *et al*. [22] used a nonempty subset M without partial ordering, so it is appropriate to use the term "M-compatible".

Here, it can be pointed out that the involved set M being a subset of X^2 is indeed a binary relation on X. Therefore, the concept of (f,g)-closed subset of X^2 can be interpreted as (f,g)-closed binary relation on X. Obviously, Definitions 3.24 and 3.26 are weaker than Definitions 5.8 and 5.6 respectively. Taking $\mathcal{R} = M$ in Corollary 4.2, we get an improved version of the following result of Karapinar *et al*. [22].

Corollary 5.9. *(see Corollary 34, [22]) Let (X,d) be a complete metric space, $f,g : X \to X$ be two mappings and $M\subseteq X^2$ be a subset such that*

(i) $f(X) \subseteq g(X)$,

(ii) M is (f,g)-compatible and (f,g)-closed,

(iii) there exists $x_0 \in X$ such that $(gx_0, fx_0) \in M$,

(iv) there exits $\alpha \in [0,1)$ such that

$$d(fx,fy) \leq \alpha d(gx,gy) \ \ \forall\, x,y \in X \text{ with } (gx,gy) \in M.$$

Also assume that, at least, one of the following conditions holds:

(a) f and g are M-continuous and M-compatible,

(b) f and g are continuous and commuting,

(c) (X,d,M) is regular and $g(X)$ is closed. Then f and g have, at least, a coincidence point.

Observe that (f,g)-compatibility of M (see assumption (ii)) is unnecessary.

5.3. Coincidence theorems in ordered metric spaces via increasing mappings

Indeed the present trend was initiated by Turinici [40, 41], Ran and Reurings [1] and Nieto and Rodríguez-López [2] which was later generalized by many authors (e.g. [35, 42, 43]). In this subsection as well as in succeeding subsection, X denotes a nonempty set endowed with a partial order $\preceq$. In what follows, we write $\succeq := \preceq^{-1}$ and $\prec\succ := \preceq^s$. On the lines of O'Regan and Petruşel [42], the triple $(X,d,\preceq)$ is called ordered metric space wherein X denotes a nonempty set endowed with a metric d and a partial order $\preceq$.

Definition 5.10. *[43]. Let $(X,\preceq)$ be an ordered set and f and g two self-mappings on X. We say that f is g-increasing if for any $x,y \in X$, $gx \preceq gy \Rightarrow fx \preceq fy$.*

Remark 5.11. *It is clear that f is g-increasing iff $\preceq$ is (f,g)-closed.*

Definition 5.12. *[35] Given a mapping $g : X \to X$, we say that an ordered metric space $(X,d,\preceq)$ has g-ICU (increasing-convergence-upper bound) property if g-image of every increasing sequence $\{x_n\}$ in X such that $x_n \xrightarrow{d} x$, is bounded above by g-image of its limit (as an upper bound), i.e., $gx_n \preceq gx \ \ \forall\, n \in \mathbb{N}_0$.*

Notice that under the restriction $g = I$, the identity mapping on X, Definition 5.12 transforms to the notion of *ICU* property.

Remark 5.13. *It is clear that if $(X, d, \preceq)$ has ICU property (resp. g-ICU property), then $\preceq$ is d-self-closed (resp. (g, d)-self-closed).*

On taking $\mathcal{R} =\preceq$ in Corollary 4.4 and using Remarks 5.11 and 5.13, we obtain the following result, which is an improved version of Corollary 4.6 of Alam *et al*. [35].

Corollary 5.14. *Let $(X, d, \preceq)$ be an ordered metric space and Y a complete subspace of X. Let f and g be two self-mappings on X. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X) \cap Y$,
(b) *f is g-increasing,*
(c) *there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$,*
(d) *there exists $\alpha \in [0, 1)$ such that*

$$d(fx, fy) \leq \alpha d(gx, gy) \quad \forall\, x, y \in X \text{ with } gx \preceq gy,$$

(e) $(e1)$ *f and g are compatible,*
$(e2)$ *g is continuous,*
$(e3)$ *either f is continuous or $(Y, d, \preceq)$ has g-ICU property,*
or alternately
(e') $(e'1)$ $Y \subseteq g(X)$,
$(e'2)$ *either f is g-continuous or f and g are continuous or $(Y, d, \preceq)$ has ICU property.*

Then f and g have a coincidence point.

5.4. Coincidence points in ordered metric spaces via comparable mappings

The core results involving comparable mappings are contained in Nieto and Rodríguez-López [3], Turinici [11, 12], Dorić *et al*. [44] and Alam and Imdad [45].

Definition 5.15. *[45]. Let $(X, \preceq)$ be an ordered set and f and g two self-mappings on X. We say that f is g-comparable if for any $x, y \in X$,*

$$gx \prec\succ gy \Rightarrow fx \prec\succ fy.$$

Remark 5.16. *It is clear that f is g-comparable iff $\prec\succ$ is (f, g)-closed.*

Definition 5.17. *[46]. Let $(X, \preceq)$ be an ordered set and $\{x_n\} \subset X$.*

(i) the sequence $\{x_n\}$ is said to be termwise bounded if there is an element $z \in X$ such that each term of $\{x_n\}$ is comparable with z, i.e.,

$$x_n \prec\succ z \qquad \forall\, n \in \mathbb{N}_0$$

so that z is a c-bound of $\{x_n\}$ and

(ii) the sequence $\{x_n\}$ is said to termwise monotone if consecutive terms of $\{x_n\}$ are comparable, i.e.,

$$x_n \prec\succ x_{n+1} \;\; \forall\, n \in \mathbb{N}_0.$$

Remark 5.18. *Clearly, $\{x_n\}$ is termwise monotone iff it is $\prec\succ$-preserving.*

Definition 5.19. *[46] Given a mapping $g : X \to X$, we say that an ordered metric space $(X, d, \preceq)$ has g-TCC (termwise monotone-convergence-c-bound) property if every termwise monotone sequence $\{x_n\}$ in X such that $x_n \xrightarrow{d} x$ has a subsequence, whose g-image is termwise bounded by g-image of limit (of the sequence) as a c-bound, i.e., $gx_{n_k} \prec\succ gx \;\forall\, k \in \mathbb{N}_0$.*

Notice that under the restriction $g = I$, the identity mapping on X, Definition 5.19 transforms to the notion of *TCC* property.

Remark 5.20. *Clearly, $(X, d, \preceq)$ has TCC property (resp. g-TCC property) iff $\prec\succ$ is d-self-closed (resp. (g, d)-self-closed).*

On taking $\mathcal{R} = <>$ in Corollary 4.4 and using Remarks 5.16 and 5.20, we obtain the following result, which is an improved version of Theorem 3.7 of Alam and Imdad [45].

Corollary 5.21. *Let $(X, d, \preceq)$ be an ordered metric space and Y a complete subspace of X. Let f and g be two self-mappings on X. Suppose that the following conditions hold:*

(a) *$f(X) \subseteq g(X) \cap Y$,*
(b) *f is g-comparable,*
(c) *there exists $x_0 \in X$ such that $gx_0 <> fx_0$,*
(d) *there exists $\alpha \in [0, 1)$ such that*

$$d(fx, fy) \leq \alpha d(gx, gy) \quad \forall\ x, y \in X \text{ with } gx <> gy,$$

(e) ($e1$) *f and g are compatible,*
($e2$) *g is continuous,*
($e3$) *either f is continuous or $(Y, d, \preceq)$ has g-TCC property,*
or alternately
(e') ($e'1$) *$Y \subseteq g(X)$,*
($e'2$) *either f is g-continuous or f and g are continuous or $(Y, d, \preceq)$ has TCC property.*

Then f and g have a coincidence point.

5.5. Coincidence theorems under symmetric closure of a binary relation

The origin of such results can be traced back to Samet and Turinici [14] which is also pursued in Berzig [15]. In this context, $\mathcal{R}$ stands for an arbitrary binary relation on a nonempty set X and $\mathcal{S} := \mathcal{R}^s$.

Definition 5.22. *[15]. Let f and g be two self-mappings on X. We say that f is g-comparative if for any $x, y \in X$,*

$$(gx, gy) \in \mathcal{S} \Rightarrow (fx, fy) \in \mathcal{S}.$$

Remark 5.23. *It is clear that f is g-comparative iff $\mathcal{S}$ is (f, g)-closed.*

Definition 5.24. *[14]. We say that $(X, d, \mathcal{S})$ is regular if the following condition holds: if the sequence $\{x_n\}$ in X and the point $x \in X$ are such that*

$$(x_n, x_{n+1}) \in \mathcal{S} \text{ for all } n \text{ and } \lim_{n\to\infty} d(x_n, x) = 0,$$

then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in \mathcal{S}$ for all k.

Remark 5.25. *Clearly, $(X, d, \mathcal{S})$ is regular iff $\mathcal{S}$ is d-self-closed.*

Taking the symmetric closure $\mathcal{S}$ of an arbitrary relation $\mathcal{R}$ in Corollary 4.4 and using Remarks 5.23 and 5.25, we obtain an improved version of the following result of Berzig [15].

Corollary 5.26. *(see Corollary 4.5, [15]) Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X and f and g two self-mappings on X. Suppose that the following conditions hold:*

(a) *$f(X) \subseteq g(X)$,*
(b) *f is g-comparative,*
(c) *there exists $x_0 \in X$ such that $(gx_0, fx_0) \in \mathcal{S}$,*
(d) *there exists $\alpha \in [0, 1)$ such that*

$$d(fx, fy) \leq \alpha d(gx, gy) \quad \forall\ x, y \in X \text{ with } (gx, gy) \in \mathcal{S},$$

(e) *(X, d) is complete and $g(X)$ is closed,*
(f) *$(X, d, \mathcal{S})$ is regular.*

Then f and g have a coincidence point.

6. Examples

In this section, we provide two examples establishing the utility of Theorems 4.1, 4.5, 4.7 and 4.8.

Example 6.1. *Consider* $X = \mathbb{R}$ *equipped with usual metric and also define a binary relation* $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : |x| - |y| \geq 0\}$*. Then* (X, d) *is an* $\mathcal{R}$*-complete metric space. Consider the mappings* $f, g : X \to X$ *defined by* $fx = \frac{x^2}{3}$ and $gx = \frac{x^2}{2}$ $\forall\, x \in X$*. Clearly,* $\mathcal{R}$ *is* (f, g)*-closed. Now, for* $x, y \in X$ *with* $(gx, gy) \in \mathcal{R}$*, we have*

$$d(fx, fy) = \left|\frac{x^2}{3} - \frac{y^2}{3}\right| = \frac{2}{3}\left|\frac{x^2}{2} - \frac{y^2}{2}\right| = \frac{2}{3}d(gx, gy) < \frac{3}{4}d(gx, gy).$$

Thus, f *and* g *satisfy assumption* (d) *of Theorem 4.1 with* $\alpha = \frac{3}{4}$*. By a routine calculation, one can verify all the conditions mentioned in* (e) *of Theorem 4.1. Hence all the conditions of Theorem 4.1 are satisfied for* $Y = X$*, which guarantees that* f *and* g *have a coincidence point in* X*. Moreover, observe that* (u_1) *holds and henceforth in view of Theorem 4.5,* f *and* g *have a unique point of coincidence (namely:* $\overline{x} = 0$*), which remains also a unique common fixed point (in view of Theorem 4.8).*

Observe that the underlying binary relation $\mathcal{R}$ *is a preorder which is not antisymmetric and henceforth not a partial order. Thus, in all, our results are genuine extension of several corresponding results proved under partial ordering.*

Example 6.2. *Consider* $X = \mathbb{R}$ *equipped with usual metric and also define a binary relation* $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \in \mathbb{Q}\}$*. Consider the mappings* $f, g : X \to X$ *defined by* $fx = 1$ and $gx = x^2 - 3$ $\forall\, x \in X$*. Clearly,* $\mathcal{R}$ *is* (f, g)*-closed. Now, for* $x, y \in X$ *with* $(gx, gy) \in \mathcal{R}$*, we have*

$$d(fx, fy) = |1 - 1| = 0 \leq \alpha|x^2 - y^2| = \alpha d(gx, gy).$$

Thus, f *and* g *satisfy assumption* (d) *of Theorem 4.1 for any arbitrary* $\alpha \in [0, 1)$*. Also, the mappings* f *and* g *are not* $\mathcal{R}$*-compatible and hence* (e) *does not hold. But the subspace* $Y := g(X) = [-3, \infty)$ *is* $\mathcal{R}$*-complete and* f *and* g *are continuous, i.e., all the conditions mentioned in* (e') *are satisfied. Hence, in view of Theorem 4.1,* f *and* g *have a coincidence point in* X*. Further, in this example* (u_1) *holds and henceforth, in view of Theorem 4.5,* f *and* g *have a unique point of coincidence (namely:* $\overline{x} = 1$*). Notice that neither* f *nor* g *is one-one, i.e.,* (u_2) *does not hold and hence, we can not apply Theorem 4.7, which guarantees the uniqueness of coincidence point. Observe that, in the present example, there are two coincidence points (namely: x=2 and x=-2). Also,* f *and* g *are not weakly compatible , i.e.,* $(e'3)$ *does not hold and hence, we can not apply Theorem 4.8, which ensures the uniqueness of common fixed point. Notice that there is no common fixed point of* f *and* g*.*

Observe that the underlying binary relation $\mathcal{R}$ *is a transitive relation. Indeed,* $\mathcal{R}$ *is non-reflexive, non-irreflexive, non-symmetric as well as non-antisymmetric and hence it is not a preorder, partial order, near order, strict order, tolerance or equivalence and also never turns out to be a symmetric closure of any binary relation.*

Here, it can be point out that corresponding results of the existence literature cannot be used in the context of present example, which substantiate the utility of our newly proved coincidence theorems over corresponding several relevant results.

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