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Fundamental Properties of Statistical Convergence and Lacunary Statistical Convergence on Time Scales

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Abstract. In this paper, we first obtain a Tauberian condition for statistical convergence on time scales. We also find necessary and sufficient conditions for the equivalence of statistical convergence and lacunary statistical convergence on time scales. Some significant applications are also presented.

1. Introduction

Discrete and continuous cases, sometimes in nature or in some engineering problems, can occur at the same time. Then, time scale calculus is often used in order to solve these problems. Actually, the main idea of time scale calculus, which was introduced by Hilger [11], is to unify such discrete and continuous cases. Although time scales have important applications in many areas of mathematics, their usage in the summability theory just begins with our recent papers [14, 15].

We introduced and systematically investigated the notions of statistical convergence and lacunary statistical convergence on time scales in [14] and [15], respectively. In this paper, we continue our works on these concepts. More precisely, in the second section, we obtain a Tauberian condition for statistical convergence of functions defined on time scales. In the third section, we find necessary and sufficient conditions for the equivalence of statistical convergence and lacunary statistical convergence on time scales. Furthermore, throughout the paper, we discuss some important special cases of our results and state some open problems on this area.

Now we recall the concepts used in the present paper.

A time scale is any closed nonempty subset of real numbers. The function

$$\sigma : \mathbb{T} \to \mathbb{T},\ \sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

is called the forward jump operator. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t$$

By $[a, b]_{\mathbb{T}}$, we denote the intervals in $\mathbb{T}$, i.e., $[a, b] \cap \mathbb{T}$, where $[a, b]$ is the usual real interval. In this paper, we also use the Lebesgue Δ-measure μ_Δ introduced by Guseinov [10]. It is known that if $a, b \in \mathbb{T}$ and $a \le b$, then

$$\mu_\Delta([a, b)_{\mathbb{T}}) = b - a \text{ and } \mu_\Delta((a, b)_{\mathbb{T}}) = b - \sigma(a),$$

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and that if $a, b \in \mathbb{T} \setminus \max \mathbb{T}$ and $a \leq b$, then

$$\mu_\Delta\left((a,b]_{\mathbb{T}}\right) = \sigma(b) - \sigma(a) \text{ and } \mu_\Delta\left([a,b]_{\mathbb{T}}\right) = \sigma(b) - a.$$

Throughout the paper we study on a time scale $\mathbb{T}$ such that

$$\inf \mathbb{T} = t_0 \ (t_0 > 0) \text{ and } \sup \mathbb{T} = \infty.$$

Δ-derivative of a function $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}$ is denoted by $f^\Delta(t)$ and is defined to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. If $\mathbb{T} = \mathbb{N}$, then it reduces to the forward difference operator; if $\mathbb{T} = \mathbb{R}$, then we get the usual derivative; if $\mathbb{T} = q^{\mathbb{N}}$, then it turns out to be the concept of q-derivative.

Now let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Using the above terminology, in [14] we introduced the notion of statistical convergence of f on $\mathbb{T}$ (see also [13]). Recall that f is said to be statistically convergent to a number L if, for every $\varepsilon > 0$,

$$\lim_{t\to\infty} \frac{\mu_\Delta\left\{s \in [t_0, t]_{\mathbb{T}} : \left|f(s) - L\right| \geq \varepsilon\right\}}{\mu_\Delta\left([t_0, t]_{\mathbb{T}}\right)} = 0. \tag{1}$$

By $S_{\mathbb{T}}$ we denote the set of all statistically convergent functions. If $\mathbb{T} = \mathbb{N}$, then (1) reduces to the concept of statistical convergence of number sequence, which was first introduced by Fast [6] (see also [8]); the case of $\mathbb{T} = [a, \infty)$, $a > 0$, was studied by Móricz [12]. Finally, if $\mathbb{T} = q^{\mathbb{N}}$, $q > 1$, then we get the notion of q-statistical convergence introduced by Aktuğlu and Bekar [1].

The convergence method in (1) can also be defined with respect to the density on time scales as in the following way. For a Δ-measurable subset Ω of $\mathbb{T}$, the density of Ω over the time scale $\mathbb{T}$ is defined to be the number

$$\delta_{\mathbb{T}}(\Omega) := \lim_{t\to\infty} \frac{\mu_\Delta\{s \in [t_0, t]_{\mathbb{T}} : s \in \Omega\}}{\mu_\Delta\left([t_0, t]_{\mathbb{T}}\right)}$$

provided that the above limit exists. Then, (1) is equivalent to

$$\delta_{\mathbb{T}}\left(\left\{t \in \mathbb{T} : \left|f(s) - L\right| \geq \varepsilon\right\}\right) = 0 \text{ for every } \varepsilon > 0.$$

In [14], we also defined the notion of strongly p-Cesàro summability on time scales ($p > 0$). f is called strongly p-Cesàro summable to a number L if

$$\lim_{t\to\infty} \frac{1}{\mu_\Delta\left([t_0, t]_{\mathbb{T}}\right)} \int_{[t_0,t]_{\mathbb{T}}} \left|f(s) - L\right|^p \Delta s = 0, \tag{2}$$

where we use the Lebesgue Δ-integral on time scales introduced by Cabada and Vivero [4] (see also [2]). Let $N_{\mathbb{T}}^p$ denote the set of all strongly p-Cesàro summable functions on $\mathbb{T}$. We proved in [14] that

$$N_{\mathbb{T}}^p \cap C_b(\mathbb{T}) = S_{\mathbb{T}} \cap C_b(\mathbb{T}),$$

where $C_b(\mathbb{T})$ is the set of all bounded functions on $\mathbb{T}$.

In [15], we study the notion of lacunary statistical convergence on time scales as follows. Let $\mathbb{T}$ be a time scale including the lacunary sequence $\theta = (k_r)$, where by a lacunary sequence we mean the increasing sequence for which $\sigma(k_r) - \sigma(k_{r-1}) \to \infty$ as $r \to \infty$ (with $k_0 = 0$). Then, a Δ-measurable function f is lacunary statistically convergent to a number L if

$$\lim_{r\to\infty} \frac{\mu_\Delta\left\{s \in (k_{r-1}, k_r]_{\mathbb{T}} : \left|f(s) - L\right| \geq \epsilon\right\}}{\mu_\Delta\left((k_{r-1}, k_r]_{\mathbb{T}}\right)} = 0. \tag{3}$$

The set of all lacunary convergent functions is denoted by $S_{\theta-\mathbb{T}}$.

Finally, we defined in [15] that a Δ-measurable function f is strongly lacunary Cesàro summable to L if

$$\lim_{t\to\infty}\frac{1}{\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)}\int_{(k_{r-1},k_r]_{\mathbb{T}}}\left|f(s)-L\right|\Delta s=0. \tag{4}$$

In this case, by $N_{\theta-\mathbb{T}}$ we denote the set of all strongly lacunary Cesàro summable functions on $\mathbb{T}$. Then, it is known from [15] that

$$N_{\theta-\mathbb{T}}\cap C_b(\mathbb{T})=S_{\theta-\mathbb{T}}\cap C_b(\mathbb{T}).$$

Observe that the discrete cases of (1)−(4) are well-known in theory of sequence spaces.

2. Tauberian Conditions for Statistical Convergence on Time Scales

In this section, we obtain the following Tauberian condition for statistical convergence on time scales. As stated before, we assume that $\mathbb{T}$ is a time scale such that $\inf\mathbb{T}=t_0>0$ and $\sup\mathbb{T}=\infty$.

Theorem 2.1. *Let $\mathbb{T}$ be a time scale for which the graininess function μ is nondecreasing on $\mathbb{T}$, and let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable and Δ-differentiable function on $\mathbb{T}$. Assume that*

$$st_{\mathbb{T}}-\lim_{t\to\infty}f(t)=L. \tag{5}$$

If

$$\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)\left|f^{\Delta}(t)\right|\leq B \tag{6}$$

holds for some $B>0$ and for every $t\in\mathbb{T}$, then we have

$$\lim_{t\to\infty}f(t)=L. \tag{7}$$

Proof. Since $st_{\mathbb{T}}-\lim_{t\to\infty}f(t)=L$, we get from Theorem 3.9 in [14] that there exists a Δ-measurable set $\Omega\subset\mathbb{T}$ with $\delta_{\mathbb{T}}(\Omega)=1$ such that $\lim_{t\to\infty}f|_{\Omega}(t)=L$. Let $g:\mathbb{T}\to R$ be a Δ-measurable function such that $f|_{\Omega}=g|_{\Omega}$, i.e., $f(t)=g(t)$ for all $t\in\Omega$. Then, we may write that $\{t\in\mathbb{T}:f(t)\neq g(t)\}\subset\mathbb{T}\backslash\Omega$. Since $\delta_{\mathbb{T}}(\mathbb{T}\backslash\Omega)=0$, we get $\delta_{\mathbb{T}}\{t\in\mathbb{T}:f(t)\neq g(t)\}=0$, which implies that

$$\lim_{t\to\infty}\frac{\mu_\Delta\left(\{s\in[t_0,t]_{\mathbb{T}}:f(s)\neq g(s)\}\right)}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)}=0. \tag{8}$$

Also, it is clear that

$$\lim_{t\to\infty}g|_{\Omega}(t)=L. \tag{9}$$

Now, for sufficiently large $t\in\mathbb{T}$, let

$$u(t):=\max\left\{s\in[t_0,t]_{\mathbb{T}}:f(s)=g(s)\right\}.$$

Observe that $u(t)\in\Omega$ due to $f=g$ on Ω. Since $\delta_{\mathbb{T}}(\Omega)=1$, the set

$$\{s\in[t_0,t]_{\mathbb{T}}:f(s)=g(s)\}$$

is nonempty for sufficiently large $t\in\mathbb{T}$. Then, we claim that

$$\lim_{t\to\infty}\frac{\mu_\Delta\left((u(t),t]_{\mathbb{T}}\right)}{\mu_\Delta\left([t_0,u(t)]_{\mathbb{T}}\right)}=\lim_{t\to\infty}\frac{\sigma(t)-\sigma(u(t))}{\sigma(u(t))-t_0}=0. \tag{10}$$

Indeed, if $\frac{\mu_\Delta\left((u(t),t]_{\mathbb{T}}\right)}{\mu_\Delta\left([t_0,u(t)]_{\mathbb{T}}\right)} > \varepsilon_0$ for some $\varepsilon_0 > 0$ and for sufficiently large t, then

$$\begin{aligned}\frac{\mu_\Delta\left(\{s\in[t_0,t]_{\mathbb{T}}: f(s)\neq g(s)\}\right)}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)} &\geq \frac{\mu_\Delta\left((u(t),t]_{\mathbb{T}}\right)}{\mu_\Delta\left([t_0,u(t)]_{\mathbb{T}}\right)+\mu_\Delta\left((u(t),t]_{\mathbb{T}}\right)} \\ &> \frac{\varepsilon_0}{1+\varepsilon_0} > 0,\end{aligned}$$

which contradicts with (8). On the other hand, by using (6), we get from the fundamental theorem of calculus on time scales (see [3]) that

$$\begin{aligned}\left|f(t)-g\left(u(t)\right)\right| &= \left|f(t)-f\left(u(t)\right)\right| = \left|\int_{u(t)}^{t} f^{\Delta}(s)\Delta s\right| \\ &\leq \int_{u(t)}^{t}\left|f^{\Delta}(s)\right|\Delta s = \int_{[u(t),t)_{\mathbb{T}}}\left|f^{\Delta}(s)\right|\Delta s \\ &\leq B\frac{\mu_\Delta\left([u(t),t)_{\mathbb{T}}\right)}{\mu_\Delta\left([t_0,u\left(t\right)]_{\mathbb{T}}\right)}.\end{aligned}$$

Thus, we get

$$\left|f(t)-g\left(u(t)\right)\right| \leq B\frac{t-u(t)}{\sigma(u(t))-t_0} \tag{11}$$

for all sufficiently large $t\in\mathbb{T}$. Since the graininess function μ is nondecreasing on $\mathbb{T}$, we find that

$$\frac{t-u(t)}{\sigma(u(t))-t_0} \leq \frac{\sigma(t)-\sigma(u(t))}{\sigma(u(t))-t_0}$$

Then, combining the last inequality with (10) we get

$$\lim_{t\to\infty}\frac{t-u(t)}{\sigma(u(t))-t_0} = 0,$$

Thus, the right hand side of (11) tends to 0 as $t\to\infty$. Also, by (9), $\lim_{t\to\infty} g(u(t)) = L$, and hence we conclude that $\lim_{t\to\infty} f(t) = L$, which completes the proof. □

Now we focus on some special cases of Theorem 2.1.

Case I. Take $\mathbb{T} = \mathbb{N}$ in Theorem 2.1. In this case, $t_0 = 1$, and replacing t with n, observe that $\mu(n) = 1$ for every $n \in \mathbb{N}$. Setting $x_n = f(n)$, we see that $f^{\Delta}(n) = \Delta(x_n)$, where Δ denotes the usual forward difference operator. In this case, since $\mu_{\Delta}([1,n]_{\mathbb{N}}) = n$, condition (6) turns out to be $|\Delta x_n| = O\left(\frac{1}{n}\right)$. Hence, Theorem 2.1 reduces to Theorem 3 in [8].

Case II. Take $\mathbb{T} = [a,\infty)$, $a > 0$, in Theorem 2.1. Check that $\mu(t) = 0$ and $f^{\Delta}(t) = f'(t)$. So, we immediately get that $\mu_{\Delta}\left([a,t]_{[a,\infty)}\right) = t - a$. It follows that condition (6) becomes $(t-a)\left|f'(t)\right| \leq B$ for every $t \geq a$. Thus, we obtain the following Tauberian result.

Corollary 2.2. *Let $f : [a,\infty) \to \mathbb{R}$ be a differentiable function. Assume that $st_{[a,\infty)} - \lim_{t\to\infty} f(t) = L$. If $(t-a)\left|f'(t)\right| \leq B$ holds for some $B > 0$ and for every $t \geq a$, then we have* (7).

We know from [14] that $st_{[a,\infty)} - \lim_{t\to\infty} f(t) = L$ is equivalent to the following: for every $\varepsilon > 0$,

$$\lim_{t\to\infty} \frac{m\left(\left\{s \in [a,t] : \left|f(s) - L\right| \geq \varepsilon\right\}\right)}{t-a} = 0, \tag{12}$$

where $m(B)$ denotes the classical Lebesgue measure of the set B. We should note that the definition in (12) was also introduced by Móricz [12] without using any time scale.

Notice that, in Corollary 2.2, the Tauberian condition $(t-a)\left|f'(t)\right| \leq B$ can be replaced with the stronger condition $\left|f'(t)\right| = O(\frac{1}{t})$.

Case III. Take $\mathbb{T} = q^{\mathbb{N}}$, $q > 1$. Then, replacing t with q^n $(n \in \mathbb{N})$, if $n \leq m$, then $\mu(q^n) = q^{n+1} - q^n = q^n(q-1) \leq q^m(q-1) = \mu(q^m)$, which gives that the graininess function μ is nondecreasing. Also, in this case, $f^{\Delta}(q^n) = D^q f(q^n) = \frac{f(q^{n+1}) - f(q^n)}{q^n(q-1)}$, which is known as q-derivative of f. Also since $t_0 = q$, condition (6) becomes $\left|D^q f(q^n)\right| = O\left(\frac{1}{q(q^n-1)}\right)$. Then, we get the next result at once, which is also new in the literature.

Corollary 2.3. *Let $f : q^{\mathbb{N}} \to \mathbb{R}$ $(q > 1)$ be a q-differentiable function. Assume that $st_{q^{\mathbb{N}}} - \lim_{t\to\infty} f(t) = L$. If $\left|D^q f(q^n)\right| = O\left(\frac{1}{q(q^n-1)}\right)$ holds, then we have* (7).

We show in [14] that $st_{q^{\mathbb{N}}} - \lim_{t\to\infty} f(t) = L$ is equivalent to

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} q^{k-1} \chi_{K(\varepsilon)}(q^k)}{[n]_q} = 0 \text{ for every } \varepsilon > 0, \tag{13}$$

where $K(\varepsilon) := \left\{q^k \in [q,q^n]_{q^{\mathbb{N}}} : \left|f(q^k) - L\right| \geq \varepsilon\right\}$ and $[n]_q$ denotes the q-integers given by $[n]_q = 1 + q + ... + q^{n-1} = \frac{q^n-1}{q-1}$. Recall that the definition in (13) was also presented by Aktuğlu and Bekar [1] without using any time scale.

As we can see from the above special cases, graininess functions of many well-known time scales are already nondecreasing. However, there are time scales that do not satisfy this condition. For example, if

$$\mathbb{T} = \bigcup_{n=1}^{\infty} [2n, 2n+1], \tag{14}$$

then

$$\mu(t)=\begin{cases}0, & \text{if } t\in\bigcup_{n=1}^{\infty}[2n,2n+1)\\ 1, & \text{if } t\in\bigcup_{n=1}^{\infty}\{2n+1\}.\end{cases}$$

Hence, at this stage, it is an open problem whether Theorem 2.1 is valid or not without the nondecreasing condition of μ. However, following a similar method used in the proof of Theorem 2.1, one can prove the next result, where a different monotonicity condition and a different Tauberian condition are used.

Theorem 2.4. *Let $\mathbb{T}$ be a time scale for which the function $h:\mathbb{T}\to\mathbb{T}$, $h(t)=\dfrac{\sigma(t)-t_0}{t}$, is nondecreasing on $\mathbb{T}$, and let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable and Δ-differentiable function on $\mathbb{T}$. Assume that* (5) *holds. If*

$$\left|f^{\Delta}(t)\right|=O\left(\frac{1}{t}\right),$$

then we have (7).

Observe again that, for many time scales, such as, $\mathbb{N}$, $[a,\infty)$ $(a>0)$ and $q^{\mathbb{N}}$ $(q>1)$, the functions h in Theorem 2.4 are nondecreasing. However, of course, there are time scales which do not satisfy this condition. For example, consider again the time scale $\mathbb{T}$ defined in (14). In this case, check that

$$h(t)=\begin{cases}\dfrac{t-2}{t}, & \text{if } t\in\bigcup_{n=1}^{\infty}[2n,2n+1)\\ \dfrac{2n}{2n+1}, & \text{if } t=2n+1,\ n\in\mathbb{N}.\end{cases}$$

3. Conditions for the Equivalence of Statistical Convergence and Lacunary Statistical Convergence

In this section, studying inclusions between $S_{\mathbb{T}}$ and $S_{\theta-\mathbb{T}}$, we obtain a characterization for the equivalence of $S_{\mathbb{T}}$ and $S_{\theta-\mathbb{T}}$.

We first need the following two lemmas. In this section, we assume again that $\mathbb{T}$ is any time scale such that $\inf\mathbb{T}=t_0>0$ and $\sup\mathbb{T}=\infty$.

Lemma 3.1. *Let $\mathbb{T}$ be a time scale including a lacunary sequence $\theta=(k_r)$. Then, we have*

$$S_{\mathbb{T}}\subset S_{\theta-\mathbb{T}}\Leftrightarrow\liminf_{r\to\infty}\frac{\sigma(k_r)}{\sigma(k_{r-1})}>1.$$

Proof. Sufficiency. Suppose that $\liminf_{r\to\infty}\dfrac{\sigma(k_r)}{\sigma(k_{r-1})}>1$. Then, for sufficiently large r, we get

$$\frac{\sigma(k_r)}{\sigma(k_{r-1})}\geq 1+\delta$$

for some $\delta>0$, and hence

$$\frac{\sigma(k_r)-\sigma(k_{r-1})}{\sigma(k_r)}\geq\frac{\delta}{1+\delta}. \tag{15}$$

Now let $f \in S_{\mathbb{T}}$ with the limit L.Then, from (15), we may write, for every $\varepsilon > 0$, that

$$\begin{aligned} & \frac{\mu_\Delta\left(\left\{s \in [t_0, k_r]_{\mathbb{T}} : \left|f(s) - L\right| \geqslant \varepsilon\right\}\right)}{\mu_\Delta\left([t_0, k_r]_{\mathbb{T}}\right)} \\ \geq\ & \frac{\mu_\Delta\left(\left\{s \in (k_{r-1}, k_r]_{\mathbb{T}} : \left|f(s) - L\right| \geqslant \varepsilon\right\}\right)}{\sigma(k_r) - t_0} \\ \geq\ & \frac{\sigma(k_r) - \sigma(k_{r-1})}{\sigma(k_r)} \frac{\mu_\Delta\left(\left\{s \in (k_{r-1}, k_r]_{\mathbb{T}} : \left|f(s) - L\right| \geqslant \varepsilon\right\}\right)}{\sigma(k_r) - \sigma(k_{r-1})} \\ \geq\ & \frac{\delta}{1+\delta} \frac{\mu_\Delta\left(\left\{s \in (k_{r-1}, k_r]_{\mathbb{T}} : \left|f(s) - L\right| \geqslant \varepsilon\right\}\right)}{\sigma(k_r) - \sigma(k_{r-1})}. \end{aligned}$$

Since $st_{\mathbb{T}} - \lim_{t \to \infty} f(t) = L$, the left hand side of the last inequality tends to 0 as $r \to \infty$, which yields that

$$\lim_{r \to \infty} \frac{\mu_\Delta\left(\left\{s \in (k_{r-1}, k_r]_{\mathbb{T}} : \left|f(s) - L\right| \geqslant \varepsilon\right\}\right)}{\mu_\Delta\left((k_{r-1}, k_r]_{\mathbb{T}}\right)} = 0.$$

Thus, the proof of sufficiency of the lemma is completed.

Necessity. Conversely, suppose that $\liminf_{r \to \infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} = 1$. Then, as in [7], we can select a subsequence $\left(k_{r(j)}\right)$ of the lacunary sequence $\theta = (k_r)$ such that

$$\frac{\sigma(k_{r(j)}) - t_0}{\sigma(k_{r(j)-1}) - t_0} < 1 + \frac{1}{j} \tag{16}$$

and

$$\frac{\sigma(k_{r(j)-1}) - t_0}{\sigma(k_{r(j-1)}) - t_0} > j, \quad \text{where } \ r(j) > r(j-1) + 1. \tag{17}$$

Now define a Δ-measurable function $f : \mathbb{T} \to \mathbb{R}$ by

$$f(s) = \begin{cases} 1, & s \in (k_{r(j)-1}, k_{r(j)}]_{\mathbb{T}} \text{ for } j = 2, 3, ... \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Then, we claim that $f \notin N_{\theta-\mathbb{T}}$. Indeed, if $r = r(j)$, then, for any real L, we have

$$\begin{aligned} \frac{1}{\mu_\Delta\left((k_{r-1}, k_r]_{\mathbb{T}}\right)} \int_{(k_{r-1}, k_r]_{\mathbb{T}}} \left|f(s) - L\right| \Delta s &= \frac{1}{\mu_\Delta\left((k_{r(j)-1}, k_{r(j)}]_{\mathbb{T}}\right)} \int_{(k_{r(j)-1}, k_{r(j)}]_{\mathbb{T}}} |1 - L| \, \Delta s \\ &= |1 - L| \, . \end{aligned}$$

Also, if $r \neq r(j)$, then we get

$$\begin{aligned} \frac{1}{\mu_\Delta\left((k_{r-1}, k_r]_{\mathbb{T}}\right)} \int_{(k_{r-1}, k_r]_{\mathbb{T}}} \left|f(s) - L\right| \Delta s &= \frac{1}{\mu_\Delta\left((k_{r-1}, k_r]_{\mathbb{T}}\right)} \int_{(k_{r-1}, k_r]_{\mathbb{T}}} |L| \, \Delta s \\ &= |L| \, . \end{aligned}$$

Since $|1 - L| \neq |L|$ for any real L, we see that $f \notin N_{\theta-\mathbb{T}}$. Since f is bounded, it follows from Corollary 1 in [15] that $f \notin S_{\theta-\mathbb{T}}$. Now, we show that the function f defined by (18) is strongly Cesàro summable to

0, i.e., $f \in N_{\mathbb{T}}$ with the limit 0. Indeed, for any sufficiently large $t \in \mathbb{T}$, we can find a unique j for which $k_{r(j)-1} < t \leq k_{r(j+1)-1}$. Then, we may write from (16) and (17) that

$$\begin{aligned}
\frac{1}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)}\int_{[t_0,t]_{\mathbb{T}}} \left|f(s)\right|\Delta s &\leq \frac{1}{\mu_\Delta\left([t_0,k_{r(j)-1}]_{\mathbb{T}}\right)}\int_{[t_0,k_{r(j-1)}]_{\mathbb{T}}} \Delta s + \frac{1}{\mu_\Delta\left([t_0,k_{r(j)-1}]_{\mathbb{T}}\right)}\int_{(k_{r(j)-1},k_{r(j)}]_{\mathbb{T}}} \Delta s \\
&= \frac{\sigma\left(k_{r(j-1)}\right)-t_0}{\sigma(k_{r(j)-1})-t_0} + \frac{\sigma\left(k_{r(j)}\right)-\sigma\left(k_{r(j)-1}\right)}{\sigma(k_{r(j)-1})-t_0} \\
&< \frac{1}{j} + \frac{\sigma\left(k_{r(j)}\right)-t_0}{\sigma(k_{r(j)-1})-t_0} - 1 \\
&< \frac{1}{j} + \frac{1}{j} = \frac{2}{j} \to 0 \ \ (\text{as } j \to \infty).
\end{aligned}$$

This means that $f \in N_{\mathbb{T}}$ with the limit 0. Since f is bounded, Theorem 3.16 of [14] (for $p = 1$ and $L = 0$) implies that $f \in S_{\mathbb{T}}$, which gives $S_{\mathbb{T}} \not\subseteq S_{\theta-\mathbb{T}}$. □

Lemma 3.2. *Let $\mathbb{T}$ be a time scale including a lacunary sequence $\theta = (k_r)$ such that $\mu(t) \leq Mt$ for some $M \geq 0$ and for every $t \in \mathbb{T}$. Then, we have*

$$S_{\theta-\mathbb{T}} \subset S_{\mathbb{T}} \Leftrightarrow \limsup_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} < \infty.$$

Proof. Sufficiency. Assume first that $\limsup_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} < \infty$ holds. Hence, we obtain that $\limsup_{r\to\infty} \frac{\sigma(k_r)-t_0}{\sigma(k_{r-1})-t_0} < \infty$, which gives, for some $K > 0$, that

$$\frac{\sigma(k_r)-t_0}{\sigma(k_{r-1})-t_0} \leq K \ \text{ for all } r \in \mathbb{N}. \tag{19}$$

Now let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function belonging to $S_{\theta-\mathbb{T}}$. Then, there exists a number L such that

$$\lim_{r\to\infty} \frac{U_r}{\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)} = 0 \ \text{ for any } \varepsilon > 0,$$

where

$$U_r := U_r(\varepsilon) = \mu_\Delta\left(\left\{s \in (k_{r-1},k_r]_{\mathbb{T}} : \left|f(s)-L\right| \geq \varepsilon\right\}\right).$$

This means that there exists a natural number $r_0 = r_0(\varepsilon)$ such that

$$\frac{U_r}{\sigma(k_r)-\sigma(k_{r-1})} < \varepsilon \text{ for all } r > r_0. \tag{20}$$

For a given $t \in \mathbb{T}$, we may find an interval $(k_{r-1},k_r]_{\mathbb{T}}$ including t, i.e., $t \in (k_{r-1},k_r]_{\mathbb{T}}$. Letting $B =$

$\max\{U_1, U_2, ..., U_{r_0}\}$, for sufficiently large r's, we get

$$\begin{aligned}
\frac{\mu_\Delta\left(\left\{s\in[t_0,t]_{\mathbb{T}}:\left|f(s)-L\right|\geqslant\varepsilon\right\}\right)}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)} &\leq \frac{\mu_\Delta\left(\left\{s\in[t_0,k_r]_{\mathbb{T}}:\left|f(s)-L\right|\geqslant\varepsilon\right\}\right)}{\mu_\Delta\left([t_0,k_{r-1}]_{\mathbb{T}}\right)} \\
&\leq \frac{U_1+U_2+...+U_{r_0}+U_{r_0+1}+...+U_r}{\sigma(k_{r-1})-t_0} \\
&\leq \frac{r_0B}{\sigma(k_{r-1})-t_0}+\frac{1}{\sigma(k_{r-1})-t_0}\left\{\frac{\left(\sigma(k_{r_0+1})-\sigma(k_{r_0})\right)U_{r_0+1}}{\sigma(k_{r_0+1})-\sigma(k_{r_0})}\right. \\
&\quad \left.+...+\frac{\left(\sigma(k_r)-\sigma(k_{r-1})\right)U_r}{\sigma(k_r)-\sigma(k_{r-1})}\right\} \\
&\leq \frac{r_0B}{\sigma(k_{r-1})-t_0}+\varepsilon\frac{\sigma(k_r)-\sigma(k_{r_0})}{\sigma(k_{r-1})-t_0} \\
&\leq \frac{r_0B}{\sigma(k_{r-1})-t_0}+\varepsilon\frac{\sigma(k_r)-t_0}{\sigma(k_{r-1})-t_0} \\
&\leq \frac{r_0B}{\sigma(k_{r-1})-t_0}+\varepsilon K.
\end{aligned}$$

Taking limit as $r\to\infty$ on the both sides of the last inequality, we see that

$$\lim_{t\to\infty}\frac{\mu_\Delta\left(\left\{s\in[t_0,t]_{\mathbb{T}}:\left|f(s)-L\right|\geqslant\varepsilon\right\}\right)}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)}=0,$$

which proves the sufficiency of the lemma.

Necessity. Conversely, suppose that $\limsup_{r\to\infty}\frac{\sigma(k_r)}{\sigma(k_{r-1})}=\infty$. By hypothesis, we see that $t\leq\sigma(t)\leq(M+1)t$ for all $t\geq t_0$ with $t\in\mathbb{T}$. Then, one can get

$$\frac{k_r}{\sigma(k_{r-1})}=\frac{\sigma(k_r)}{\sigma(k_{r-1})}\frac{k_r}{\sigma(k_r)}\geq\frac{1}{(M+1)}\frac{\sigma(k_r)}{\sigma(k_{r-1})},$$

which gives that

$$\limsup_{r\to\infty}\frac{k_r}{\sigma(k_{r-1})}=\infty.$$

Then, we can select a subsequence $\left(k_{r(j)}\right)$ of the lacunary sequence $\theta=(k_r)$ such that

$$\frac{k_{r(j)}}{\sigma(k_{r(j)-1})}>j. \tag{21}$$

Now define a Δ-measurable function $f:\mathbb{T}\to\mathbb{R}$ by

$$f(s)=\begin{cases}1, & s\in(k_{r(j)-1},2\sigma(k_{r(j)-1}))_{\mathbb{T}} \text{ for some } j=1,2,...\\ 0, & \text{otherwise.}\end{cases} \tag{22}$$

Then, we claim that $f\in N_{\theta-\mathbb{T}}$ with the limit 0. Indeed, letting

$$\tau_r:=\frac{1}{\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)}\int\limits_{(k_{r-1},k_r]_{\mathbb{T}}}\left|f(s)\right|\Delta s,$$

if $r \neq r(j)$, then we immediately see that $\tau_r = 0$. If $r = r(j)$, then we get from (21) and (22) that

$$\begin{aligned}\tau_{r(j)} &= \frac{1}{\mu_\Delta\left((k_{r(j)-1}, k_{r(j)}]_{\mathbb{T}}\right)} \int_{(k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_{\mathbb{T}}} \Delta s \\ &= \frac{\mu_\Delta\left((k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_{\mathbb{T}}\right)}{\sigma(k_{r(j)}) - \sigma(k_{r(j)-1})}.\end{aligned}$$

Here, there are two possible cases: $2\sigma(k_{r(j)-1}) \in \mathbb{T}$ and $2\sigma(k_{r(j)-1}) \notin \mathbb{T}$. Now, if $2\sigma(k_{r(j)-1}) \in \mathbb{T}$, then

$$\mu_\Delta\left((k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_{\mathbb{T}}\right) = \sigma(k_{r(j)-1}).$$

On the other hand, if $2\sigma(k_{r(j)-1}) \notin \mathbb{T}$, then we may write that

$$(k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_{\mathbb{T}} = (k_{r(j)-1}, \alpha_j]_{\mathbb{T}},$$

where

$$\alpha_j := \max\left\{s \in \mathbb{T} : s < 2\sigma(k_{r(j)-1})\right\}. \tag{23}$$

Hence, we get from the hypothesis that

$$\begin{aligned}\mu_\Delta\left((k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_{\mathbb{T}}\right) &= \mu_\Delta\left((k_{r(j)-1}, \alpha_j]_{\mathbb{T}}\right) \\ &= \sigma(\alpha_j) - \sigma\left(k_{r(j)-1}\right) \\ &\leq (M+1)\alpha_j - \sigma\left(k_{r(j)-1}\right) \\ &\leq 2(M+1)\sigma(k_{r(j)-1}) - \sigma\left(k_{r(j)-1}\right) \\ &= (2M+1)\sigma\left(k_{r(j)-1}\right).\end{aligned}$$

As a result, if $2\sigma(k_{r(j)-1}) \in \mathbb{T}$, then

$$\begin{aligned}\tau_{r(j)} &= \frac{\sigma(k_{r(j)-1})}{\sigma(k_{r(j)}) - \sigma(k_{r(j)-1})} \\ &\leq \frac{\sigma(k_{r(j)-1})}{k_{r(j)} - \sigma(k_{r(j)-1})} < \frac{1}{j-1} \to 0 \text{ (as } j \to \infty),\end{aligned}$$

or if $2\sigma(k_{r(j)-1}) \notin \mathbb{T}$, then

$$\tau_{r(j)} \leq \frac{(2M+1)\sigma\left(k_{r(j)-1}\right)}{\sigma(k_{r(j)}) - \sigma(k_{r(j)-1})} < \frac{2M+1}{j-1} \to 0 \text{ (as } j \to \infty).$$

Thus, $f \in N_{\theta-\mathbb{T}}$. Since f is bounded, $f \in S_{\theta-\mathbb{T}}$. Now, we will show that the function f defined by (22) is not strongly Cesàro summable to neither 1 nor 0, i.e., $f \notin N_{\mathbb{T}}$. Indeed, we may write from (21) that

$$\begin{aligned}\frac{1}{\mu_\Delta\left(\left[t_0, k_{r(j)}\right]_{\mathbb{T}}\right)} \int_{[t_0, k_{r(j)}]_{\mathbb{T}}} \left|f(s) - 1\right| \Delta s &\geq \frac{1}{\sigma(k_{r(j)}) - t_0} \int_{[2\sigma(k_{r(j)-1}), k_{r(j)}]_{\mathbb{T}}} \Delta s \\ &\geq \frac{\mu_\Delta\left([2\sigma(k_{r(j)-1}), k_{r(j)}]_{\mathbb{T}}\right)}{\sigma(k_{r(j)})}.\end{aligned}$$

Here, if $2\sigma(k_{r(j)-1}) \in \mathbb{T}$, then

$$\begin{aligned}\frac{1}{\mu_\Delta\left(\left[t_0,k_{r(j)}\right]_{\mathbb{T}}\right)}\int_{[t_0,k_{r(j)}]_{\mathbb{T}}}\left|f(s)-1\right|\Delta s &\geq \frac{\sigma(k_{r(j)})-2\sigma(k_{r(j)-1})}{\sigma(k_{r(j)})}\\ &= 1-\frac{2\sigma(k_{r(j)-1})}{\sigma(k_{r(j)})}\\ &\geq 1-\frac{2\sigma(k_{r(j)-1})}{k_{r(j)}}\\ &> 1-\frac{2}{j}\to 1 \quad (\text{as } j\to\infty).\end{aligned}$$

Also, if $2\sigma(k_{r(j)-1}) \notin \mathbb{T}$, then we may write that

$$[2\sigma(k_{r(j)-1}), k_{r(j)}]_{\mathbb{T}} = (\alpha_j, k_{r(j)}]_{\mathbb{T}},$$

where α_j is given by (23). In this case, we get

$$\begin{aligned}\frac{1}{\mu_\Delta\left(\left[t_0,k_{r(j)}\right]_{\mathbb{T}}\right)}\int_{[t_0,k_{r(j)}]_{\mathbb{T}}}\left|f(s)-1\right|\Delta s &\geq \frac{\sigma(k_{r(j)})-\sigma(\alpha_j)}{\sigma(k_{r(j)})}\\ &\geq \frac{\sigma(k_{r(j)})-(M+1)\alpha_j}{\sigma(k_{r(j)})}\\ &\geq \frac{\sigma(k_{r(j)})-2(M+1)\sigma(k_{r(j)-1})}{\sigma(k_{r(j)})}\\ &= 1-2(M+1)\frac{\sigma(k_{r(j)-1})}{\sigma(k_{r(j)})}\\ &> 1-\frac{2(M+1)}{j}\to 1 \quad (\text{as } j\to\infty).\end{aligned}$$

From (22), we also get

$$\begin{aligned}&\frac{1}{\mu_\Delta\left(\left[t_0,2\sigma(k_{r(j)-1})\right]_{\mathbb{T}}\right)}\int_{[t_0,2\sigma(k_{r(j)-1})]_{\mathbb{T}}}\left|f(s)\right|\Delta s\\ \geq\ &\frac{1}{\mu_\Delta\left(\left[t_0,2\sigma(k_{r(j)-1})\right]_{\mathbb{T}}\right)}\int_{(k_{r(j)-1},2\sigma(k_{r(j)-1}))_{\mathbb{T}}}\Delta s\\ =\ &\frac{\mu_\Delta\left((k_{r(j)-1},2\sigma(k_{r(j)-1}))_{\mathbb{T}}\right)}{\mu_\Delta\left(\left[t_0,2\sigma(k_{r(j)-1})\right]_{\mathbb{T}}\right)}.\end{aligned}$$

If $2\sigma(k_{r(j)-1}) \in \mathbb{T}$, then

$$\begin{aligned}\frac{1}{\mu_\Delta\left(\left[t_0,2\sigma(k_{r(j)-1})\right]_{\mathbb{T}}\right)}\int_{[t_0,2\sigma(k_{r(j)-1})]_{\mathbb{T}}}\left|f(s)\right|\Delta s &= \frac{\sigma(k_{r(j)-1})}{\sigma\left(2\sigma(k_{r(j)-1})\right)-t_0}\\ &\geq \frac{\sigma(k_{r(j)-1})}{2(M+1)\sigma(k_{r(j)-1})}\\ &= \frac{1}{2(M+1)}.\end{aligned}$$

Also, if $2\sigma(k_{r(j)-1}) \notin \mathbb{T}$, one can write that

$$(k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_{\mathbb{T}} = (k_{r(j)-1}, \beta_j)_{\mathbb{T}}$$

and

$$\left[t_0, 2\sigma(k_{r(j)-1})\right]_{\mathbb{T}} = \left[t_0, \alpha_j\right]_{\mathbb{T}},$$

where α_j is the same as in (23), and

$$\beta_j := \min\left\{s \in \mathbb{T} : s > 2\sigma(k_{r(j)-1})\right\}.$$

These facts yield that

$$\begin{aligned} \frac{1}{\mu_\Delta\left(\left[t_0, 2\sigma(k_{r(j)-1})\right]_{\mathbb{T}}\right)} \int_{[t_0, 2\sigma(k_{r(j)-1})]_{\mathbb{T}}} \left|f(s)\right| \Delta s &= \frac{\mu_\Delta\left((k_{r(j)-1}, \beta_j)_{\mathbb{T}}\right)}{\mu_\Delta\left(\left[t_0, \alpha_j\right]_{\mathbb{T}}\right)} \\ &= \frac{\beta_j - \sigma\left(k_{r(j)-1}\right)}{\sigma\left(\alpha_j\right) - t_0} \\ &\geq \frac{2\sigma(k_{r(j)-1}) - \sigma\left(k_{r(j)-1}\right)}{(M+1)\alpha_j} \\ &\geq \frac{\sigma\left(k_{r(j)-1}\right)}{2(M+1)\sigma\left(k_{r(j)-1}\right)} \\ &= \frac{1}{2(M+1)}. \end{aligned}$$

Thus, we see that $f \notin N_{\mathbb{T}}$. Since f is bounded, we also get $f \notin S_{\mathbb{T}}$. Therefore, we find $S_{\theta-\mathbb{T}} \nsubseteq S_{\mathbb{T}}$, which is a contradiction. □

Now, combining Lemmas 3.1 and 3.2, we get the following result.

Theorem 3.3. *Let $\mathbb{T}$ be a time scale including a lacunary sequence $\theta = (k_r)$ such that $\mu(t) \leq Mt$ for some $M \geq 0$ and for every $t \in \mathbb{T}$. Then, we have*

$$S_{\theta-\mathbb{T}} = S_{\mathbb{T}} \Leftrightarrow 1 < \liminf_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} \leq \limsup_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} < \infty. \tag{24}$$

Notice that we need the restriction $\mu(t) \leq Mt$ in Theorem 3.3 due to only of the necessity part of Lemma 3.2. Actually, many time scales, such as $\mathbb{N}$, $[a, \infty)$ $(a > 0)$ and $q^{\mathbb{N}}$ $(q > 1)$, satisfy this condition. However, for example, $\mathbb{T} = 2^{\mathbb{N}^2} = \left\{2^{n^2} : n \in \mathbb{N}\right\}$ does not satisfy this condition. Thus, an open problems arises: is Theorem 3.3 valid without this restriction?

Finally, we give some special cases of Theorem 3.3.

If we take $\mathbb{T} = \mathbb{N}$ in Theorem 3.3, then, the right-hand side of (24) turns out to be

$$1 < \liminf_{r\to\infty} \frac{k_r + 1}{k_{r-1} + 1} \leq \limsup_{r\to\infty} \frac{k_r + 1}{k_{r-1} + 1} < \infty,$$

which is equivalent to

$$1 < \liminf_{r\to\infty} \frac{k_r}{k_{r-1}} \leq \limsup_{r\to\infty} \frac{k_r}{k_{r-1}} < \infty.$$

In this case, we immediately get Theorem 4 in [9].

Also, if we take $\mathbb{T} = [a, \infty)$, $a > 0$, in Theorem 3.3, then we obtain the following characterization.

Corollary 3.4. *Let $\theta = (k_r) \subset [a, \infty)$, $a > 0$, be a lacunary sequence. Then, we have*

$$S_{\theta-[a,\infty)} = S_{[a,\infty)} \Leftrightarrow 1 < \liminf_{r\to\infty} \frac{k_r}{k_{r-1}} \leq \limsup_{r\to\infty} \frac{k_r}{k_{r-1}} < \infty.$$

Finally, if we take $\mathbb{T} = q^{\mathbb{N}}$, $q > 1$, in Theorem 3.3, then, we get the next result at once.

Corollary 3.5. *Let $\theta = (q^{k_r}) \subset q^{\mathbb{N}}$, $q > 1$, be a lacunary sequence. then we have*

$$S_{\theta-q^{\mathbb{N}}} = S_{q^{\mathbb{N}}} \Leftrightarrow 1 < \liminf_{r\to\infty} q^{k_r-k_{r-1}} \leq \limsup_{r\to\infty} q^{k_r-k_{r-1}} < \infty.$$

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