



Symmetries in Some Extremal Problems Between Two Parallel Hyperplanes

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Abstract. Let M be a compact hypersurface with boundary $\partial M = \partial D_1 \cup \partial D_2$, $\partial D_1 \subset \Pi_1$, $\partial D_2 \subset \Pi_2$, Π_1 and Π_2 two parallel hyperplanes in \mathbb{R}^{n+1} ($n \geq 2$). Suppose that M is contained in the slab determined by these hyperplanes and that the mean curvature H of M depends only on the distance u to Π_i , $i = 1, 2$ and on ∇u . We prove that these hypersurfaces are symmetric to a perpendicular orthogonal to Π_i , $i = 1, 2$, under different conditions imposed on the boundary of hypersurfaces on the parallel planes: (i) when the angle of contact between M and Π_i , $i = 1, 2$ is constant; (ii) when $\partial u / \partial \eta$ is a non-increasing function of the mean curvature of the boundary, $\partial \eta$ the inward normal; (iii) when $\partial u / \partial \eta$ has a linear dependency on the distance to a fixed point inside the body that hypersurface englobes; (iv) when ∂D_i are symmetric to a perpendicular orthogonal to Π_i , $i = 1, 2$.

1. Introduction

Let Π_1 and Π_2 be parallel hyperplanes in \mathbb{R}^{n+1} ($n \geq 2$). The slab determined by these hyperplanes is the set of \mathbb{R}^{n+1} with boundary $\Pi_1 \cup \Pi_2$. Let Ω be a connected, open and bounded subset of \mathbb{R}^{n+1} , contained in the slab, such that $\partial \Omega = M \cup D_1 \cup D_2$, $D_1 = \overline{\Omega} \cap \Pi_1$, $D_2 = \overline{\Omega} \cap \Pi_2$ and M is a compact hypersurface with boundary $\partial M = \partial D_1 \cup \partial D_2$. Suppose that the mean curvature of M depends only on the distance u to Π_i , $i = 1, 2$ and on ∇u . We will prove that these hypersurfaces are symmetric to a perpendicular orthogonal to Π_i , $i = 1, 2$, under different conditions imposed on the boundary of hypersurfaces on the parallel planes.

In the case $n = 2$, Ω can be interpreted as the interior of a drop of liquid trapped between two parallel planes in the presence of gravity. In this physical problem, there are forces proportional to the area of the free surface, the surface tension and wetting energy proportional to the area on the plates wetted by the drop. We will assume that D_1 and D_2 are nonempty sets. The Euler-Lagrange equation for this problem implies that the mean curvature of the free surface depends on the distance to the planes.

The key analytic tools used in the proofs of major theorems in this present work are presented in the section 2. They will allow us to compare the hypersurface with itself, using a standard procedure, known as Alexandrov symmetrization and generalized by Serrin.

In section 3, we give a full proof of the following theorem:

Theorem 1.1. *Let Π_1 and Π_2 be parallel hyperplanes in \mathbb{R}^{n+1} . Consider an embedded compact connected C^2 hypersurface M contained in the region of \mathbb{R}^{n+1} between Π_1 and Π_2 , with $\partial M \cap \Pi_i \neq \emptyset$, $i = 1, 2$. Suppose that the*

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mean curvature of M depends on the distance u to Π_i , $i = 1, 2$, and on ∇u , and that the angle of contact between M and Π_i , $i = 1, 2$, is constant along ∂M in each one of the two support hyperplanes (maybe with different constants). Then M is rotationally symmetric with respect to a perpendicular orthogonal to Π_i , $i = 1, 2$.

This theorem has been much used in the particular case when the mean curvature depends on the distance to the planes, but it seems that the proof has never been published in full, being instead invariably left to a reader. Here, we give the proof when the mean curvature depends also on ∇u , in order to use it in later results and make the presentation complete. We will use an argument similar to the one that can be found at [7] for only one supporting hyperplane and dependence on u .

In section 4, we consider other symmetry results under different hypothesis on the boundary of hyper-surface.

2. The Touching Principle

We begin with some definitions and analytic lemmas that will be necessary for the proofs of results in the following sections.

Let $w(x) = w(x_1, \dots, x_n)$ be a differentiable function defined on some region of \mathbb{R}^n . We will denote by $w_i = w_i(x_1, \dots, x_n)$ and $w_{ij} = w_{ij}(x_1, \dots, x_n)$ the partial derivative of w with respect to x_i and second partial derivative of w with respect to i and j , respectively. For $a_{ij}(x)$ and $b_i(x)$ being continuous functions in an open set $B \subset \mathbb{R}^n$ satisfying $a_{ij} = a_{ji}$, $1 \leq i, j \leq n$, let $M(w)$ be a linear differential operator on B , defined by

$$M(w) = \sum_{i,j=1}^n a_{i,j}(x)w_{ij} + \sum_{i=1}^n b_i(x)w_i. \tag{1}$$

We say that $M(w)$ is elliptic on B if

$$\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j > 0, \tag{2}$$

for all $x \in B$ and for all $\xi = (\xi_1, \dots, \xi_n) \neq (0, \dots, 0)$.

We say that $M(w)$ is uniformly elliptic on B with ellipticity constant $k > 0$ if

$$\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \geq k|\xi|^2, \tag{3}$$

for all $x \in B$ and for all $\xi = (\xi_1, \dots, \xi_n) \neq (0, \dots, 0)$.

The Hopf’s maximum principle and Hopf’s maximum principle on the boundary are well known results for studying behavior of linear elliptic operators and uniformly elliptic operators, respectively. However, the differential equation that the mean curvature satisfies is not linear, but quasi-linear. The following touching principles are going to play a substitute for our analysis, being a consequence of the mentioned maximum principles.

Lemma 2.1. (Interior touching principle) Let $L(w)$ be an operator

$$L(w) = M(w) + c(x)w, \tag{4}$$

where $M(w)$ is an elliptic operator on B as defined by (1) and $c(x)$ a continuous function on B . If there is a function $w(x) \in C^2(B)$ satisfying

1. $L(w) \geq 0$ on B ,
2. $w(x) \leq 0$ on B ,
3. $w(x_0) = 0$ for a $x_0 \in B$,

then $w(x) \equiv 0$ on B .

Lemma 2.2. (Boundary touching principle) Let B be a region in \mathbb{R}^n such that the boundary of B in a neighborhood of $x_0 \in \partial B$ is of class C^1 . On \bar{B} , we consider an operator $L(w)$ of type (4), where $M(w)$ is a uniformly elliptic operator as defined by (1) on \bar{B} and $c(x)$ is a continuous function on \bar{B} . If there is a function $w(x) \in C^2(B) \cap C^1(\bar{B})$ satisfying

1. $L(w) \geq 0$ on B ,
2. $w(x) \leq 0$ on \bar{B} ,
3. $w(x_0) = 0$,
4. $\frac{\partial w}{\partial \eta}(x_0) = 0$, where ν is the inward normal,

then $w(x) \equiv 0$ on B .

In the cases treated here, the mean curvature H of a hypersurface M is a function of the height u and ∇u . Suppose that M can be locally defined in a non-parametric way by $x_{n+1} = u(x_1, \dots, x_n)$, where u is a smooth function in some bounded region B of \mathbb{R}^n . Hence, in B , u satisfies the nonlinear elliptic differential equation of second order

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = nH(x, u, \nabla u), \tag{5}$$

that can be written as

$$\sum_{i,j=1}^n a_{i,j}(x, u, \nabla u)u_{ij} = \frac{1}{W} \sum_{i=1}^n u_{ii} - \frac{1}{W^3} \sum_{i,j=1}^n u_i u_j u_{ij} = nH(x, u, \nabla u), \tag{6}$$

where $W = \sqrt{1 + |\nabla u|^2}$. As a consequence of (6),

$$\sum_{i,j=1}^n a_{i,j}(x, u, \nabla u)\xi_i \xi_j = \frac{1}{W^3} \left[(1 + |\nabla u|^2)|\xi|^2 - \sum_{i,j=1}^n u_i u_j \xi_i \xi_j \right] \geq \frac{1}{W^3} |\xi|^2. \tag{7}$$

Consider now two hypersurfaces M and \bar{M} given nonparametrically by $x_{n+1} = u(x)$ and $x_{n+1} = \bar{u}(x)$ respectively, with $x = (x_1, \dots, x_n) \in B$. Suppose that u and \bar{u} are C^2 on B and satisfy the same mean curvature equation (5). It is known that the difference $w(x) = u(x) - \bar{u}(x)$ is a solution to a linear partial differential equation $L(w) = 0$, where L is an operator as defined in (4). To see this, we construct between M and \bar{M} a convex family of hypersurfaces $\{u^t\}$, given locally by $u^t(x) = tu(x) + (1 - t)\bar{u}(x) = \bar{u}(x) + tw(x)$, for $t \in [0, 1]$ and $x \in B$. We introduce $H(t) = H(u^t)$, where $H(t)$ is C^1 on $[0, 1]$ and we calculate the integral of $H'(t)$ in the interval $[0, 1]$. By one side, we have

$$n \int_0^1 H'(t) dt = n[H(1) - H(0)] = n[H(u) - H(\bar{u})].$$

Applying the mean value theorem, there exists $\theta \in (0, 1)$ such that $n[H(u) - H(\bar{u})] = n \frac{\partial H}{\partial u}(\bar{u} + \theta w)w$. By the other side, considering the same notation as before, since $u_i^t = \bar{u}_i + tw_i$ and $u_{ij}^t = \bar{u}_{ij} + tw_{ij}$ for $i, j = 1, \dots, n$, we have

$$\begin{aligned} n \int_0^1 H'(t) dt &= \int_0^1 \frac{d}{dt} \left[\sum_{i=1}^n \frac{u_{ii}^t}{(1 + |\nabla u^t|^2)^{1/2}} - \sum_{i,j=1}^n \frac{u_i^t u_j^t u_{ij}^t}{(1 + |\nabla u^t|^2)^{3/2}} \right] dt \\ &= \int_0^1 \left[\sum_{i=1}^n \frac{w_{ii}}{(1 + |\nabla u^t|^2)^{1/2}} - \sum_{i,j=1}^n \frac{w_i u_j^t u_{ij}^t + u_i^t w_j u_{ij}^t + u_i^t u_j^t w_{ij}}{(1 + |\nabla u^t|^2)^{3/2}} \right] dt \\ &\quad + \int_0^1 \left[- \sum_{i=1}^n \frac{u_{ii}^t (u_1^t w_1 + \dots + u_n^t w_n)}{(1 + |\nabla u^t|^2)^{3/2}} + \sum_{i,j=1}^n \frac{3u_i^t u_j^t u_{ij}^t (u_1^t w_1 + \dots + u_n^t w_n)}{(1 + |\nabla u^t|^2)^{5/2}} \right] dt . \end{aligned}$$

This expression is nothing more than a nonlinear partial differential equation of second order in w , with integrals as coefficients. We can write it briefly as

$$\sum_{i,j=1}^n A^{ij}(x, w(x), \nabla w(x))w_{ij} + \sum_{i=1}^n B^i(x, w(x), \nabla w(x))w_i + C(x)w = 0 \tag{8}$$

where

$$A^{ij}(x, w(x), \nabla w(x)) = \int_0^1 a_{ij}(x, \bar{u} + tw, \nabla \bar{u} + \nabla w) dt . \tag{9}$$

It turns out that the differential equation is of type (4), with coefficients A^{ij} independent of the set u^t , thus also independent of w , and $M(w)$ is elliptic on B . The fact that the coefficients B_i are continuous and B is a limited set of \mathbb{R}^n result on the uniform ellipticity of the equation on \bar{B} , with ellipticity constant

$$k \geq \frac{1}{\max((1 + |\nabla u|^2)^{3/2}, (1 + |\nabla \bar{u}|^2)^{3/2})} .$$

3. Proof of Theorem 1

Let L be a line in Π_1 or Π_2 and Π be a hyperplane perpendicular to L and as a consequence also perpendicular to Π_1 and Π_2 . As already mentioned, the hypersurfaces $M \cup D_1 \cup D_2$ compose the boundary of a body Ω , which is compact. So, there exists a hyperplane parallel to Π , that is tangent to M or touches M on its boundary, such that M is entirely on one side of Π . We denote such hyperplane by $\Pi(0)$.

We begin to push the hyperplane $\Pi(0)$ along the line L , "entering" the body Ω . After a displacement of t , we denote by $\Pi(t)$ the obtained hyperplane parallel to $\Pi(0)$. For each $t > 0$ the hyperplane $\Pi(t)$ divides \mathbb{R}^{n+1} in two components: $R(t)$, which contains $\Pi(0)$ and its complement $L(t)$. The portion of M which is in the component $R(t)$, will be denoted by $M(t)$. For small values of t , since $\Pi(0)$ is tangent to M , it is possible to reflect $M(t)$ with respect to $\Pi(t)$ obtaining $\bar{M}(t)$, which is entirely contained in the body Ω without any intersections. We observe that this process of reflection keeps invariant the two hyperplanes Π_1 and Π_2 . It is also valid that the value of mean curvature of $\bar{M}(t)$, since it depends only on the distance to hyperplanes Π_1 and Π_2 agrees to the value of mean curvature of $M(t)$ at points of the same height in both hypersurfaces.

And, since the body Ω does not need to be convex, $\overline{M}(t)$ may have many connected components after reflection process.

We continue this reflection process obtaining $\overline{M}(t)$ till we have a hyperplane $\Pi(t^*)$ where it happens the first contact between the reflected portion of M , $\overline{M}(t^*)$, and the original portion of $M \subset L(t)$, with normals to M and $\overline{M}(t^*)$ pointing to the same direction. This contact may be at an interior point P or on the boundary at a point P in one of the planes $\Pi_i, i = 1, 2$, i.e., one of the two conditions happen:

1. $\overline{M}(t^*)$ will be tangent to M at an interior point P .
2. $\overline{M}(t^*)$ will be tangent to M at a point P in $\partial M \cap \Pi_i, i = 1$ or $i = 2$.

In the first case: P is an interior point of $\overline{M}(t^*)$.

We choose a coordinate system (x_1, \dots, x_n, u) in \mathbb{R}^{n+1} , with the origin at P such that the tangent space to M at P , which coincides with the tangent space to $\overline{M}(t^*)$ at P is $u = 0$. We can also choose the direction of the u axis to be directed to the interior of Ω . This means that in a small neighborhood $B = \{x \in \mathbb{R}^n; |x| < r\}$ of the tangent space at P , M and $\overline{M}(t^*)$ can be represented nonparametrically by two C^2 functions $u = u(x) = u(x_1, \dots, x_n)$ and $\bar{u} = \bar{u}(x) = \bar{u}(x_1, \dots, x_n)$, respectively. Both functions, $u(x)$ and $\bar{u}(x)$ satisfy equation of the mean curvature $H(x, u, \nabla u)$, coincide on the origin and $u(x) \leq \bar{u}(x)$ in this neighborhood. Then, $u(x) \equiv \bar{u}(x)$.

In the second case: P is a boundary point of $\overline{M}(t^*)$.

By hypothesis, the angle of contact is constant along ∂M and Π_i . So, $\overline{M}(t^*)$ will be tangent to M at a point P . As before, we choose a coordinate system (x_1, \dots, x_n, u) in \mathbb{R}^{n+1} , with the origin at P such that the tangent space to M at P , which coincides with the tangent space to $\overline{M}(t^*)$ at P is $u = 0$. We can also choose the direction of the u axis to be directed to the interior of Ω , the direction of x_1 to the interior of M , so that the tangent space to ∂M at P contained in Π_1 or Π_2 is given by $u = x_1 = 0$.

We have M a C^2 hypersurface with boundary, so in a neighborhood of P , M and $\overline{M}(t^*)$ can be represented nonparametrically by C^2 functions $u(x)$ and $\bar{u}(x)$, defined on domains $A_1 \subset \mathbb{R}^n$ and $A_2 \subset \mathbb{R}^n$, respectively, that have the boundary of class C^1 . Observe that, in the case where the angle of contact is $\pi/2$, the neighborhood can be described as $A_1 = A_2 = B = \{x \in \mathbb{R}^n; |x| < r, x_1 > 0\}$. In the cases where the contact angle is different of $\pi/2$, we will have $A_1 \subset A_2$ or $A_2 \subset A_1$. We consider $B = A_1 \cap A_2$. Both functions satisfy:

- At the origin $u = \bar{u}$ and $\frac{\partial u}{\partial \eta} = \frac{\partial \bar{u}}{\partial \eta}$, because M and $\overline{M}(t^*)$ have the same tangent plane at P ;
- On \bar{B} , $u(x)$ and $\bar{u}(x)$ satisfy equation of the mean curvature $H(x, u, \nabla u)$ and $u(x) \leq \bar{u}(x)$ because $\overline{M}(t)$ still lies in the interior of Ω , for $t < t^*$.

Then, $u(x) \equiv \bar{u}(x)$.

To complete the proof, we can observe that, picking up other directions for the line L , we find other hyperplanes of symmetry. The perpendicular orthogonal to $\Pi_i, i = 1, 2$, that we are searching is the intersection of all these hyperplanes of symmetry.

4. Other Boundary Conditions

The method of Alexandrov can be used in other situations.

The first result shows how the symmetry of the boundary in a variational problem with fixed boundary on two parallel hyperplanes implies symmetry of the hypersurface contained in the slab.

Theorem 4.1. Let Π_1 and Π_2 be two parallel hyperplanes in \mathbb{R}^{n+1} . Let Ω be a connected, open and bounded subset of \mathbb{R}^{n+1} ($n \geq 2$), contained in the slab determined by Π_1 and Π_2 , such that $\partial\Omega = M \cup D_1 \cup D_2$, $D_i = \overline{\Omega} \cap \Pi_i \neq \emptyset$, $i = 1, 2$ and M is a compact embedded hypersurface with boundary $\partial M = \partial D_1 \cup \partial D_2$. Suppose D_i are symmetric about a hyperplane α orthogonal to Π_i and that $\partial D_i = \partial D_i^+ \cup \partial D_i^-$, where ∂D_i^+ are graphs of nonnegative C^2 functions f_i defined on domains in $\Pi_i \cap \alpha$, which are positive in the interior of the domains and zero on the border. The part ∂D_i^- is the reflection of ∂D_i^+ in respect to α .

Suppose that M is a hypersurface of class C^2 in its interior and in the portion not touching α . Suppose that the mean curvature $H(x, u, \nabla u)$ of M depends on the distance u to Π_i , $i = 1, 2$, and on ∇u , being a C^1 function on u . Then Ω is symmetric about Π .

Proof. Let L be a line orthogonal to α and contained in Π_1 or Π_2 . Let Π be a hyperplane perpendicular to L and as a consequence also perpendicular to Π_1 and Π_2 and parallel to α . As in the previous theorem, the hypersurfaces $M \cup D_1 \cup D_2$ compose the boundary of a body Ω , which is compact and exists a hyperplane parallel to Π and α , that is tangent to M or touches M on its boundary and M is entirely on one side of Π . We denote such hyperplane by $\Pi(0)$. Using a pushing argument as in the proof of Theorem 1, and keeping the same notations, we find a first contact point P which can be in the interior or on the boundary of M .

What we have to show now is that $\overline{M}(t^*)$ is the hyperplane α . Suppose that this does not happen and let us exam each case:

In the first case, if P is an interior point of $\overline{M}(t^*)$, we can repeat the same argument and obtain that M is symmetric to respect to hyperplane $\overline{M}(t^*)$. In particular, ∂D_i are symmetric to respect to $\overline{M}(t^*) \cap \Pi_i$, which contradicts the assumption made about ∂D_i , $i = 1, 2$.

In the second case, if P is a boundary point of $\overline{M}(t^*)$, since ∂D_1^- is the reflection of ∂D_1^+ in respect to α , we must have again $\overline{M}(t^*)$ and α the same hyperplane.

This proves that the plane of symmetry has to be the same of symmetry of the boundary. □

As a consequence, if ∂D_i are circles with centers in the same perpendicular orthogonal to Π_i (maybe with different radius), then M is rotationally symmetric with respect to this perpendicular.

In the theorem of last section, the condition that the angle of contact between M and Π_i , $i = 1, 2$, is constant along ∂M in each one of the two support hyperplanes, can be written as the fact that $\partial u / \partial \eta$ is constant along the boundary ∂M . We can replace this boundary condition by a more general one and still obtain the same conclusion.

Theorem 4.2. Let Π_1 and Π_2 be two parallel hyperplanes in \mathbb{R}^{n+1} . Consider M an embedded compact connected C^2 hypersurface contained in the region of \mathbb{R}^{n+1} between Π_1 and Π_2 , with $\partial D_i = \partial M \cap \Pi_i \neq \emptyset$, $i = 1, 2$. Suppose that the mean curvature of M depends on the distance u to Π_i , $i = 1, 2$, and on ∇u , and $\partial u / \partial \eta = h_i(H_{0i})$, on ∂D_i , where h_i is a differentiable nonincreasing function of the mean curvature H_{0i} of the boundary ∂D_i . Then M is rotationally symmetric with respect to a perpendicular orthogonal to Π_i , $i = 1, 2$.

Proof We will begin repeating the same steps as in the proof of the first theorem: Let L be a line in Π_1 or Π_2 and Π be a hyperplane perpendicular to L and as a consequence also perpendicular to Π_1 and Π_2 . Exists a hyperplane parallel to Π , that is tangent to M or touches M on its boundary and M is entirely on one side of Π . We denote such hyperplane by $\Pi(0)$. By using the pushing argument we find a first contact point P . If P is an interior point of $\overline{M}(t^*)$, we can prove that this hyperplane is a symmetry plane for M , following the same steps as in section 3.

In the case when P is a boundary point of $\overline{M}(t^*)$, we need a little more care. As before, we choose a coordinate system (x_1, \dots, x_n, u) in \mathbb{R}^{n+1} , with the origin at P such that the tangent space to M at P is $u = 0$. We can also choose the direction of the u axis to be directed to the interior of Ω , the direction of x_1 to the interior of M , so that the tangent space to ∂M at P contained in Π_1 or Π_2 is given by $u = x_1 = 0$.

We have M a C^2 hypersurface with boundary, so in a neighborhood of P , M and $\overline{M}(t^*)$ can be represented nonparametrically by C^2 functions $u(x)$ and $\overline{u}(x)$, defined on a domain $B \subset \mathbb{R}^n$. Both functions satisfy:

- At the origin $u = \overline{u}$;
- On \overline{B} , $u(x)$ and $\overline{u}(x)$ satisfy equation of the mean curvature $H(x, u, \nabla u)$ and $u(x) \leq \overline{u}(x)$ because $\overline{M}(t)$ still lies in the interior of Ω , for $t < t^*$;
- At the origin $\frac{\partial u}{\partial \eta} \leq \frac{\partial \overline{u}}{\partial \eta}$;

Let H_{0i} be the mean curvature of ∂M in P and \overline{H}_0 be the mean curvature of $\partial \overline{M}(t^*)$ in P . Again, since $\overline{M}(t)$ still lies in the interior of Ω , for $t < t^*$, $H_{0i} \leq \overline{H}_0$ in P . Using the hypothesis that h_i is nonincreasing, we have at the origin

$$\frac{\partial u}{\partial \eta} = h_i(H_{0i}) \geq h(\overline{H}_0) = \frac{\partial \overline{u}}{\partial \eta},$$

for $i = 1$ or $i = 2$.

Then, at the origin, $\frac{\partial u}{\partial \eta} = \frac{\partial \overline{u}}{\partial \eta}$ and we can conclude that $u(x) \equiv \overline{u}(x)$.

As usual, we change the direction of L and get to the fact that M is rotationally symmetric with respect to a perpendicular orthogonal to Π_i , $i = 1, 2$.

□

Theorem 4.3. *Let Π_1 and Π_2 be two parallel hyperplanes in \mathbb{R}^{n+1} . Suppose that the origin O of \mathbb{R}^{n+1} is inside the body Ω . Consider M an embedded compact connected C^2 hypersurface contained in the region of \mathbb{R}^{n+1} between Π_1 and Π_2 , with $\partial M \cap \Pi_i \neq \emptyset$, $i = 1, 2$. Suppose that the mean curvature of M depends on the distance u to Π_i , $i = 1, 2$, and on ∇u , and $\partial u / \partial \eta = -cr$, on the boundary of M , with $c > 0$ a constant and r the distance of a point to the origin. Then M is rotationally symmetric with respect to a perpendicular orthogonal to Π_i , $i = 1, 2$, containing the origin.*

Proof Let L be a line passing through the origin O and contained in a parallel plane to Π_1 and Π_2 . Let Π be a hyperplane perpendicular to L and as a consequence also perpendicular to Π_1 and Π_2 . Then there exists a hyperplane parallel to Π that is tangent to M or touches M on its boundary and M is entirely on one side of Π . We denote such hyperplane by $\Pi(0)$. Applying the pushing argument, we find a first touching point P .

If P is an interior point of $\overline{M}(t^*)$, we can prove that this hyperplane is a symmetry plane for M , in a similar way as in section 3. Since on the boundary of M the values of $\partial u / \partial \eta = -cr$ have to coincide, the hyperplane $\Pi(t^*)$ has to be the one with contains the origin O .

In the second case, if P is a boundary point of $\overline{M}(t^*)$, we need again a little more care.

By hypothesis, $\partial u / \partial \eta = -cr$. We choose a coordinate system (x_1, \dots, x_n, u) in \mathbb{R}^{n+1} , with the origin at P such that the tangent space to M at P is $u = 0$. We can also choose the direction of the u axis to be directed to the interior of Ω , the direction of x_1 to the interior of M , so that the tangent space to ∂M at P contained in Π_1 or Π_2 is given by $u = x_1 = 0$.

We have M a C^2 hypersurface with boundary, so in a neighborhood of P , M and $\overline{M}(t^*)$ can be represented nonparametrically by C^2 functions $u(x)$ and $\overline{u}(x)$, defined on a domain $B \subset \mathbb{R}^n$. Both functions satisfy:

- At the origin $u = \overline{u}$;
- On \overline{B} , $u(x)$ and $\overline{u}(x)$ satisfy equation of the mean curvature $H(x, u, \nabla u)$ and $u(x) \leq \overline{u}(x)$ because $\overline{M}(t)$ still lies in the interior of Ω , for $t < t^*$.
- At the origin, $\frac{\partial u}{\partial \eta} = \frac{\partial \overline{u}}{\partial \eta}$, because M and $\overline{M}(t^*)$ have the same tangent plane at P , since its only possible to find one point of contact on the boundary.

Then, the plane of symmetry has to be the one with contains the origin and $u(x) \equiv \bar{u}(x)$.

As usual, we change the direction of L and get to the fact that M is rotationally symmetric with respect to a perpendicular orthogonal to Π_i , $i = 1, 2$, containing the origin. \square

5. Final Conclusions

In [3] and [6], M. Athanassenas and Vogel treated the following free boundary problem: Find the orientable, compact surface, embedded in \mathbb{R}^3 , of least area and enclosing a fixed volume, contained between two parallel planes and subject to the condition that the boundary is constrained to lie on these two parallel planes, studying the stability behavior of solutions. More recent references on this subject are found in [2] and [4].

A work by the present author is under preparation, investigating the stability when we include the influence of gravity, in the same dimension as the previous articles with constant mean curvature.

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