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Caputo Type Fractional Differential Equations with Nonlocal Riemann-Liouville and Erdélyi-Kober Type Integral Boundary Conditions

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Abstract. In this paper, we study nonlocal boundary value problems of nonlinear Caputo fractional differential equations supplemented with different combinations of Riemann-Liouville and Erdélyi-Kober type fractional integral boundary conditions. By applying a variety of tools of fixed point theory, the desired existence and uniqueness results are obtained. Examples illustrating the main results are also constructed.

1. Introduction

Much of the literature on fractional-order boundary problems involve either classical, Riemann-Liouville or Hadamard type integral boundary conditions. Besides aforementioned conditions, there is another type of integral boundary conditions which contain Erdélyi-Kober fractional integral operator (introduced by Arthur Erdélyi and Hermann Kober [1] in 1940). Such operators play an important role in solving single, dual and triple integral equations possessing special functions of mathematical physics in their kernels. For details and applications of the Erdélyi-Kober fractional integrals, for instance, see [2]-[8].

The extensive study of fractional differential equations in recent years owes to widespread applications of the subject in engineering and technical sciences. Important phenomena in finance, electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are well described by differential equations of fractional order. In contrast to classical differential and integral operators, fractional-order operators are nonlocal in nature and accounts for memory and hereditary properties of many phenomena and processes. For examples and recent development of the topic, see [9]-[23] and the references cited therein.

In this paper, we study a new class of boundary value problems of Caputo fractional differential equations:

$$^cD^q x(t) = f(t, x(t)), \quad t \in [0, T], \tag{1}$$

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supplemented with Riemann-Liouville and Erdélyi-Kober fractional integral boundary conditions at the left and right end points of the interval $[0,T]$ respectively, that is,

$$\begin{cases} x(0) = \alpha \dfrac{1}{\Gamma(p)} \displaystyle\int_0^{\zeta} (\zeta - s)^{p-1} ds := \alpha J^p x(\zeta), \\ x(T) = \beta \dfrac{\eta \xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \displaystyle\int_0^{\xi} \dfrac{s^{\eta\gamma+\eta-1} x(s)}{(\xi^\eta - s^\eta)^{1-\delta}} ds := \beta I_\eta^{\gamma,\delta} x(\xi), \quad 0 < \zeta, \xi < T, \end{cases} \tag{2}$$

where ${}^cD^q$ is the Caputo fractional derivative of order $1 < q \leq 2$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, J^p denote Riemann-Liouville fractional integral of order $p > 0$, and $I_\eta^{\gamma,\delta}$ denote Erdélyi-Kober fractional integral of order $\delta > 0, \eta > 0, \gamma \in \mathbb{R}$.

As supplementary problems, we also investigate the existence of solutions for Caputo fractional differential equations (1) equipped with the following combinations of Riemann-Liouville and Erdélyi-Kober fractional integral boundary conditions:

$$\begin{cases} x(0) = \alpha \dfrac{\eta \zeta^{-\eta(\theta+\epsilon)}}{\Gamma(\theta)} \displaystyle\int_0^{\zeta} \dfrac{s^{\eta\gamma+\eta-1} x(s)}{(\zeta^\eta - s^\eta)^{1-\theta}} ds := \alpha I_\sigma^{\epsilon,\theta} x(\zeta), \ \theta, \sigma > 0, \ \epsilon \in \mathbb{R}, \\ x(T) = \beta \dfrac{\eta \xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \displaystyle\int_0^{\xi} \dfrac{s^{\eta\gamma+\eta-1} x(s)}{(\xi^\eta - s^\eta)^{1-\delta}} ds := \beta I_\eta^{\gamma,\delta} x(\xi), \quad 0 < \zeta, \xi < T \end{cases} \tag{3}$$

and

$$\begin{cases} x(0) = \alpha \dfrac{\eta \zeta^{-\eta(\theta+\epsilon)}}{\Gamma(\theta)} \displaystyle\int_0^{\zeta} \dfrac{s^{\eta\gamma+\eta-1} x(s)}{(\zeta^\eta - s^\eta)^{1-\theta}} ds := \alpha I_\sigma^{\epsilon,\theta} x(\zeta), \\ x(T) = \beta \dfrac{1}{\Gamma(\psi)} \displaystyle\int_0^{\xi} (\xi - s)^{\psi-1} ds := \beta J^\psi x(\xi), \ \psi > 0, \quad 0 < \zeta, \xi < T. \end{cases} \tag{4}$$

The paper is organized as follows: In Section 2 we present some useful preliminaries and lemmas. Section 3 deals with the existence and uniqueness results for problem (1)-(2) which are obtained via contraction mapping principle, nonlinear contractions, Krasnoselskii fixed point theorem, Schaefer fixed point theorem, Laray-Schauder nonlinear alternative and Laray-Schauder degree theory. Section 4 contains illustrative examples for the results obtained in Section 3. We provide the outline of the existence results for supplementary problems in Section 5.

2. Preliminaries

This section is devoted to some preliminary concepts of fractional calculus that we need in the sequel [3].

Definition 2.1. *The Riemann-Liouville fractional integral of order q with the lower limit zero for a function f is defined as*

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad t > 0, \quad q > 0,$$

provided the right hand-side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(q) = \int_0^\infty t^{q-1} e^{-t} dt$.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $q > 0$, $n-1 < q < n$, $n \in \mathbb{N}$, is defined as*

$$D_{0+}^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3. *The Caputo derivative of order q for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$^{c}D^{q}_{0+}f(t) = D^{q}\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \quad t > 0, \quad n-1 < q < n.$$

Remark 2.4. *If $f(t) \in C^n[0, \infty)$, then*

$$^{c}D^{q}f(t) = \frac{1}{\Gamma(n-q)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}}ds = I^{n-q}f^{(n)}(t),\ t > 0,\ n-1 < q < n.$$

Definition 2.5. *The Erdélyi-Kober fractional integral of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$I^{\gamma,\delta}_{\eta}f(t) = \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int_0^t \frac{s^{\eta\gamma+\eta-1}f(s)}{(t^\eta - s^\eta)^{1-\delta}}ds,$$

provided the right side is pointwise defined on $\mathbb{R}_+$.

Remark 2.6. *For $\eta = 1$ the above operator is reduced to the Kober operator*

$$I^{\gamma,\delta}_{1}f(t) = \frac{t^{-(\delta+\gamma)}}{\Gamma(\delta)}\int_0^t \frac{s^{\gamma}f(s)}{(t-s)^{1-\delta}}ds,\ \ \gamma,\ \delta > 0,$$

that was introduced for the first time by Kober in [5]. For $\gamma = 0$, the Kober operator is reduced to the Riemann-Liouville fractional integral with a power weight:

$$I^{0,\delta}_{1}f(t) = \frac{t^{-\delta}}{\Gamma(\delta)}\int_0^t \frac{f(s)}{(t-s)^{1-\delta}}ds,\ \delta > 0.$$

Lemma 2.7. *Let $\delta, \eta > 0$ and $\gamma, q \in \mathbb{R}$. Then we have*

$$I^{\gamma,\delta}_{\eta}t^q = \frac{t^q\Gamma(\gamma + (q/\eta) + 1)}{\Gamma(\gamma + (q/\eta) + \delta + 1)}. \tag{5}$$

Proof. By Definition 2.5, we have

$$\begin{aligned} I^{\gamma,\delta}_{\eta}t^q &= \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int_0^t \frac{s^{\eta\gamma+\eta-1}\cdot s^q}{(t^\eta - s^\eta)^{1-\delta}}ds = \frac{t^q}{\Gamma(\delta)}\int_0^1 u^{\gamma+\frac{q}{\eta}}(1-u)^{\delta-1}du \\ &= \frac{t^q}{\Gamma(\delta)}B\left(\gamma + \frac{q}{\eta} + 1, \delta\right) = \frac{t^q\Gamma(\gamma + (q/\eta) + 1)}{\Gamma(\gamma + (q/\eta) + \delta + 1)}, \end{aligned}$$

where we have used the definition of beta function and its property:

$$B(x, y) = \int_0^1 u^{x-1}(1-u)^{y-1}du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ x, y > 0.$$

The proof is complete. □

Lemma 2.8. *For any $y \in AC([0,T],\mathbb{R})$, $x \in C^2([0,T],\mathbb{R})$ is a solution of the linear fractional differential equation*

$$^{c}D^{q}x(t) = y(t), \quad 1 < q \leq 2, \tag{6}$$

supplemented with the boundary conditions (2) *if and only if*

$$x(t) = J^q y(t) + \frac{\alpha}{\Lambda}(v_4 - tv_3)J^{p+q}y(\zeta) + \frac{1}{\Lambda}(v_2 + tv_1)\Big(\beta I_{\eta}^{\gamma,\delta} J^q y(\xi) - J^q y(T)\Big), \tag{7}$$

where $\Lambda = v_1v_4 + v_2v_3 \neq 0$,

$$\begin{aligned} v_1 &= 1 - \alpha\frac{\zeta^p}{\Gamma(p+1)}, \qquad v_2 = \alpha\frac{\zeta^{p+1}}{\Gamma(p+2)}, \\ v_3 &= 1 - \beta\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\delta+1)}, \qquad v_4 = T - \beta\xi\frac{\Gamma(\gamma+1/\eta+1)}{\Gamma(\gamma+1/\eta+\delta+1)}. \end{aligned} \tag{8}$$

Proof. It is well known that the general solution of the fractional differential equation (6) can be written as

$$x(t) = c_0 + c_1 t + J^q y(t), \tag{9}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants. Using the boundary conditions (2) in (9) together with Lemma 2.7, we obtain a system of equations in c_0 and c_1 given by

$$\begin{aligned} \Big(1 - \alpha\frac{\zeta^p}{\Gamma(p+1)}\Big)c_0 - \alpha\frac{\zeta^{p+1}}{\Gamma(p+2)}c_1 &= J^{p+q}y(\zeta) \\ \Big(1 - \beta\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\delta+1)}\Big)c_0 + \left(T - \beta\xi\frac{\Gamma(\gamma+1/\eta+1)}{\Gamma(\gamma+1/\eta+\delta+1)}\right)c_1 &= \beta I_{\eta}^{\gamma,\delta}J^q y(\xi) - J^q y(t). \end{aligned} \tag{10}$$

Solving the system (10), we get

$$c_0 = \frac{1}{\Lambda}\Big\{\alpha v_4 J^{p+q}y(\zeta) + v_2\Big(\beta I_{\eta}^{\gamma,\delta}J^q y(\xi) - J^q y(T)\Big)\Big\},$$

$$c_1 = \frac{1}{\Lambda}\Big\{v_1\Big(\beta I_{\eta}^{\gamma,\delta}J^q y(\xi) - J^q y(T)\Big) - \alpha v_3 J^{p+q}y(\zeta)\Big\},$$

where v_1, v_2, v_3, and v_4 are given by (8). Substituting the values of c_0, c_1 in (9), we obtain (7). Conversely, it can easily be shown by direct computation that the integral solution (7) satisfies the equation (6) and boundary conditions (2). This completes the proof. □

Next we present two more auxiliary results related to the problems (6)-(3), and (6)-(4). We do not provide the proof for these results as it is similar to that of Lemma 2.8.

Lemma 2.9. (Problem with Erdélyi-Kober fractional boundary conditions at both ends) *The solution of problem* (6)-(3) *is equivalent to the integral equation*

$$x(t) = J^q y(t) + \frac{\alpha}{\Lambda_1}(u_4 - tu_3)I_{\sigma}^{\epsilon,\theta}y(\zeta) + \frac{1}{\Lambda_1}(u_2 + tu_1)\Big(\beta I_{\eta}^{\gamma,\delta}J^q f(s,x(s))(\xi) - J^q y(T)\Big),$$

where $\Lambda_1 = u_1u_4 + u_2u_3 \neq 0$,

$$\begin{aligned} u_1 &= 1 - \alpha\frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+\theta+1)}, \qquad u_2 = \alpha\zeta\frac{\Gamma(\epsilon+1/\sigma+1)}{\Gamma(\epsilon+1/\sigma+\theta+1)}, \\ u_3 &= 1 - \beta\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\delta+1)}, \qquad u_4 = T - \beta\xi\frac{\Gamma(\gamma+1/\eta+1)}{\Gamma(\gamma+1/\eta+\delta+1)}. \end{aligned}$$

Lemma 2.10. (Problem with Erdélyi-Kober fractional boundary condition at $t = 0$ and Riemann-Liouville fractional boundary condition at $t = T$) *The integral representation of the solution of problem* (6)-(4) *is given by*

$$x(t) = J^q y(t) + \frac{\alpha}{\Lambda_2}(z_4 - tz_3) I_\sigma^{\epsilon,\theta} y(\zeta) + \frac{1}{\Lambda_2}(z_2 + tz_1)\Big(\beta J^{\phi+q} f(s, x(s))(\xi) - J^q y(T)\Big),$$

where $\Lambda_2 = z_1 z_4 + z_2 z_3 \neq 0$.

$$\begin{aligned} z_1 &= 1 - \beta\frac{\xi^\phi}{\Gamma(\phi+1)}, \qquad z_2 = T - \beta\frac{\xi^{\phi+1}}{\Gamma(\phi+2)}, \\ z_3 &= 1 - \alpha\frac{\Gamma(\epsilon+1)}{\Gamma(\epsilon+\theta+1)}, \qquad z_4 = \alpha\zeta\frac{\Gamma(\epsilon+1/\sigma+1)}{\Gamma(\epsilon+1/\sigma+\theta+1)}. \end{aligned}$$

3. Existence Results

We denote by $\mathcal{U} = C([0,T],\mathbb{R})$ the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,T]\}$. Also by $L^1([0,T],\mathbb{R})$ we denote the Banach space of measurable functions $x : [0,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T |x(t)|dt$.

In view of Lemma 2.8, we define an operator $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ by

$$\begin{aligned} (\mathcal{G}x)(t) &= J^q f(s,x(s))(t) + \frac{\alpha}{\Lambda}(v_4 - tv_3) J^{p+q} f(s,x(s))(\zeta) \\ &+ \frac{1}{\Lambda}(v_2 + tv_1)\Big(\beta I_\eta^{\gamma,\delta} J^q f(s,x(s))(\xi) - J^q f(s,x(s))(T)\Big), \ t \in [0,T]. \end{aligned} \tag{11}$$

In the sequel, we use the following expressions:

$$J^q f(s,x(s))(z) = \frac{1}{\Gamma(q)}\int_0^z (z-s)^{q-1} f(s,x(s))ds, \quad z \in \{t, T\},$$

for $t \in [0,T]$ and

$$I_\eta^{\gamma,\delta} J^q f(s,x(s))(\xi) = \frac{\eta\xi^{-\eta(\delta+\gamma)}}{\Gamma(q)\Gamma(\delta)}\int_0^\xi \int_0^r \frac{r^{\eta\gamma+\eta-1}(r-s)^{q-1}}{(\xi^\eta - r^\eta)^{1-\delta}} f(s,x(s))ds\,dr,$$

where $\xi \in (0,T)$.

For convenience, we set a constant

$$\Psi := \frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|(|v_4| + T|v_3|)\zeta^{p+q}}{|\Lambda|\Gamma(p+q+1)} + \frac{(|v_2| + T|v_1|)}{|\Lambda|}\left(\frac{|\beta|\xi^q\Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)} + \frac{T^q}{\Gamma(q+1)}\right). \tag{12}$$

In the following subsections, we establish the existence and uniqueness of solutions for the boundary value problem (1)-(2) by applying a variety of fixed point theorems.

3.1. Existence and uniqueness result via Banach's fixed point theorem

Theorem 3.1. *Let* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying the Lipschitz condition:*

(H_1) *there exists a positive constant L such that* $|f(t,x) - f(t,y)| \leq L|x - y|$, *for each* $t \in [0,T]$ *and* $x, y \in \mathbb{R}$.

Then the boundary value problem (1)-(2) has a unique solution on $[0,T]$ *if*

$$L\Psi < 1, \tag{13}$$

where Ψ *is defined by (12).*

Proof. Transform problem (1)-(2) into a fixed point problem $x = \mathcal{G}x$, where the operator $\mathcal{G}$ is defined by (11). Observe that the fixed points of the operator $\mathcal{G}$ are solutions of problem (1)-(2). Applying the Banach contraction mapping principle, we shall show that the operator $\mathcal{G}$ has a unique fixed point. Let us fix $\sup_{t\in[0,T]} |f(t,0)| = M < \infty$ and choose

$$r \geq \frac{M\Psi}{1-L\Psi}. \tag{14}$$

To show that $\mathcal{G}B_r \subset B_r$, we take $x \in B_r = \{x \in \mathcal{U} : \|x\| \leq r\}$ so that

$$\begin{aligned}
|(\mathcal{G}x)(t)| &\leq \sup_{t\in[0,T]} \bigg\{ J^q|f(s,x(s))|(t) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}|f(s,x(s))|(\zeta) \\
&\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(\beta I_\eta^{\gamma,\delta} J^q|f(s,x(s))|(\xi) + J^q|f(s,x(s))|(T)\Big)\bigg\} \\
&\leq J^q(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(T) \\
&\quad + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(\zeta) \\
&\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(\beta I_\eta^{\gamma,\delta} J^q(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(\xi) \\
&\quad + J^q(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(T)\Big) \\
&\leq (L\|x\| + M)J^q(1)(T) + (L\|x\| + M)\frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}(1)(\zeta) \\
&\quad + (L\|x\| + M)\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta| I_\eta^{\gamma,\delta} J^q(1)(\xi) + J^q(1)(T)\Big) \\
&\leq (Lr + M)\bigg\{\frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|(|v_4| + T|v_3|)\zeta^{p+q}}{|\Lambda|\Gamma(p+q+1)} \\
&\quad + \frac{(|v_2| + T|v_1|)}{|\Lambda|}\bigg(\frac{|\beta|\xi^q\Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)} + \frac{T^q}{\Gamma(q+1)}\bigg)\bigg\} \\
&\leq (Lr + M)\Psi \leq r,
\end{aligned}$$

which implies that $\mathcal{G}B_r \subset B_r$.

Next, we let $x, y \in \mathcal{U}$. Then, for $t \in [0,T]$, we have

$$\begin{aligned}
|\mathcal{G}x(t) - \mathcal{G}y(t)| &\leq \sup_{t\in[0,T]} \bigg\{ J^q|f(s,x(s)) - f(s,y(s))|(t) \\
&\quad + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}|f(s,x(s)) - f(s,y(s))|(\zeta) \\
&\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(\beta I_\eta^{\gamma,\delta} J^q|f(s,x(s)) - f(s,y(s))|(\xi) \\
&\quad + J^q|f(s,x(s)) - f(s,y(s))|(T)\Big)\bigg\} \\
&\leq L\|x-y\|J^q(1)(T) + L\|x-y\|\frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}(1)(\zeta) \\
&\quad + L\|x-y\|\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta| I_\eta^{\gamma,\delta} J^q(1)(\xi) + J^q(1)(T)\Big) \\
&= L\Psi\|x-y\|,
\end{aligned}$$

which leads to $\|\mathcal{G}x - \mathcal{G}y\| \le L\Psi\|x-y\|$. As $L\Psi < 1$, $\mathcal{G}$ is a contraction. Therefore, we deduce, by Banach's contraction mapping principle, that $\mathcal{G}$ has a fixed point which is the unique solution of problem (1)-(2). The proof is completed. □

3.2. Existence and uniqueness result via nonlinear contractions

Definition 3.2. *Let E be a Banach space and let $\mathcal{F}: E \to E$ be a mapping. $\mathcal{F}$ is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Theta(0) = 0$ and $\Theta(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property:*

$$\|\mathcal{F}x - \mathcal{F}y\| \le \Theta(\|x-y\|), \qquad \forall x, y \in E.$$

Lemma 3.3. *(Boyd and Wong)[26]. Let E be a Banach space and let $\mathcal{F}: E \to E$ be a nonlinear contraction. Then $\mathcal{F}$ has a unique fixed point in E.*

Theorem 3.4. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that the following condition holds:*

(H_2) $|f(t,x) - f(t,y)| \le z(t)\dfrac{|x-y|}{A^* + |x-y|}$, *for $t \in [0,T]$, $x, y \ge 0$, where $z : [0,T] \to \mathbb{R}^+$ is continuous and*

$$A^* := J^q z(T) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}z(T) + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big\{|\beta| I_\eta^{\gamma,\delta} J^q z(\xi) + J^q z(T)\Big\}.$$

Then problem (1)-(2) has a unique solution on $[0,T]$.

Proof. Let $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and nondecreasing function defined by

$$\Theta(\varepsilon) = \frac{A^*\varepsilon}{A^* + \varepsilon}, \qquad \forall \varepsilon \ge 0.$$

Note that Θ satisfies $\Theta(0) = 0$ and $\Theta(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$.

For any $x, y \in \mathcal{U}$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
|\mathcal{G}x(t) - \mathcal{G}y(t)| &\le \sup_{t\in[0,T]} \Bigg\{ J^q|f(s,x(s)) - f(s,y(s))|(t) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}|f(s,x(s)) - f(s,y(s))|(\zeta) \\
&\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta| I_\eta^{\gamma,\delta} J^q|f(s,x(s)) - f(s,y(s))|(\xi) + J^q|f(s,x(s)) - f(s,y(s))|(T)\Big)\Bigg\} \\
&\le J^q\left(z(s)\frac{|x-y|}{A^* + |x-y|}\right)(T) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}\left(z(s)\frac{|x-y|}{A^* + |x-y|}\right)(T) \\
&\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\left\{|\beta| I_\eta^{\gamma,\delta} J^q\left(z(s)\frac{|x-y|}{A^* + |x-y|}\right)(\xi) + J^q\left(z(s)\frac{|x-y|}{A^* + |x-y|}\right)(T)\right\} \\
&\le \frac{\Theta(\|x-y\|)}{A^*}\Big[J^q z(T) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}z(T) \\
&\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big\{|\beta| I_\eta^{\gamma,\delta} J^q z(\xi) + J^q z(T)\Big\}\Big] \\
&= \Theta(\|x-y\|),
\end{aligned}$$

which implies that $\|\mathcal{G}x - \mathcal{G}y\| \le \Theta(\|x-y\|)$. Therefore $\mathcal{G}$ is a nonlinear contraction. Hence, by Lemma 3.3 the operator $\mathcal{G}$ has a unique fixed point which is the unique solution of the problem (1)-(2). This completes the proof. □

3.3. Existence result via Krasnoselskii's fixed point theorem

Lemma 3.5. *(Krasnoselskii's fixed point theorem) [27]. Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that (a) $Ax + Bx \in M$ whenever $x, y \in M$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.6. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (H_1). In addition we assume that:*

(H_3) $|f(t, x)| \le \varphi(t)$, $\forall (t, x) \in [0, T] \times \mathbb{R}$, *and* $\varphi \in C([0, T], \mathbb{R}^+)$.

Then the problem (1)-(2) has at least one solution on $[0, T]$ *provided*

$$L\left\{\frac{|\alpha|(|v_4| + T|v_3|)\zeta^{p+q}}{|\Lambda|\Gamma(p+q+1)} + \frac{(|v_2| + T|v_1|)}{|\Lambda|}\left(\frac{|\beta|\xi^q\Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)} + \frac{T^q}{\Gamma(q+1)}\right)\right\} < 1. \tag{15}$$

Proof. We define the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ by

$$\begin{aligned}
\mathcal{G}_1 x(t) &= J^q f(s, x(s))(t), \quad t \in [0, T], \\
\mathcal{G}_2 x(t) &= \frac{\alpha}{\Lambda}(v_4 - t v_2) J^{p+q} f(s, x(s))(\zeta) \\
&\quad + \frac{1}{\Lambda}(v_2 + t v_1)\Big(\beta I_\eta^{\gamma,\delta} J^q f(s, x(s))(\xi) - J^q f(s, x(s))(T)\Big), \quad t \in [0, T].
\end{aligned}$$

Setting $\sup_{t\in[0,T]} \varphi(t) = \|\varphi\|$ and choosing

$$\rho \ge \|\varphi\|\Psi,$$

where Ψ is defined by (12), we consider $B_\rho = \{x \in \mathcal{U} : \|x\| \le \rho\}$. For any $x, y \in B_\rho$, we have

$$\begin{aligned}
|\mathcal{G}_1 x(t) + \mathcal{G}_2 y(t)| &\le \sup_{t\in[0,T]} \left\{ J^q|f(s, x(s))|(t) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|) J^{p+q}|f(s, x(s))|(\zeta) \right. \\
&\quad \left. + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta| I_\eta^{\gamma,\delta} J^q|f(s, x(s))|(\xi) + J^q|f(s, x(s))|(T)\Big) \right\} \\
&\le \|\varphi\|\left\{ \frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|(|v_4| + T|v_3|)\zeta^{p+q}}{|\Lambda|\Gamma(p+q+1)} \right. \\
&\quad \left. + \frac{(|v_2| + T|v_1|)}{|\Lambda|}\left(\frac{|\beta|\xi^q\Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)} + \frac{T^q}{\Gamma(q+1)}\right)\right\} \\
&= \|\varphi\|\Psi \le \rho.
\end{aligned}$$

This shows that $\mathcal{G}_1 x + \mathcal{G}_2 y \in B_\rho$. It is easy to see using (15) that $\mathcal{G}_2$ is a contraction mapping.

Continuity of f implies that the operator $\mathcal{G}_1$ is continuous. Also, $\mathcal{G}_1$ is uniformly bounded on B_ρ as

$$\|\mathcal{G}_1 x\| \le \frac{T^q}{\Gamma(q+1)}\|\varphi\|.$$

Now we prove the compactness of the operator $\mathcal{G}_1$.

We define $\sup_{(t,x)\in[0,T]\times B_\rho} |f(t, x)| = \bar{f} < \infty$, and consequently, for $t_1, t_2 \in [0, T]$, $t_1 < t_2$, we have

$$\begin{aligned}
|\mathcal{G}_1 x(t_2) - \mathcal{G}_1 x(t_1)| &= \Big| J^q f(s, x(s))(t_2) - J^q f(s, x(s))(t_1) \Big| \\
&\le \frac{\bar{f}}{\Gamma(q)} \left| \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] ds + \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \right| \\
&\le \frac{\bar{f}}{\Gamma(q+1)} [|t_2^q - t_1^q| + 2|t_2 - t_1|^q],
\end{aligned}$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{G}_1$ is equicontinuous. So $\mathcal{G}_1$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{G}_1$ is compact on B_ρ. Thus all the assumptions of Lemma 3.5 are satisfied. So the conclusion of Lemma 3.5 implies that the problem (1)-(2) has at least one solution on $[0, T]$. □

3.4. Existence result via Schaefer fixed point theorem

Lemma 3.7. *[28] Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $V = \{u \in X \mid u = \mu Tu, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in X.*

Theorem 3.8. *Assume that there exists a positive constant L_1 such that $|f(t,x)| \le L_1$ for $t \in [0,T]$, $x \in \mathbb{R}$. Then the boundary value problem (1)-(2) has at least one solution on $[0,T]$.*

Proof. In the first step, we show that the operator $\mathcal{G}$ defined by (11) is completely continuous. Observe that continuity of $\mathcal{G}$ follows from the continuity of f. For a positive constant r, let $B_r = \{x \in \mathcal{U} : \|x\| \le r\}$ be a bounded ball in $\mathcal{U}$. Then for $t \in [0,T]$ we have

$$\begin{aligned}
|\mathcal{G}x(t)| &\le J^q|f(s,x(s))|(t) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}|f(s,x(s))|(\zeta)\\
&\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta| I_\eta^{\gamma,\delta} J^q|f(s,x(s))|(\xi) + J^q|f(s,x(s))|(T)\Big)\\
&\le L_1 J^q(1)(T) + L_1\frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}(1)(\zeta)\\
&\quad + L_1\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta| I_\eta^{\gamma,\delta} J^q(1)(\xi) + J^q(1)(T)\Big)\\
&\le L_1\Bigg\{\frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|(|v_4| + T|v_3|)\zeta^{p+q}}{|\Lambda|\Gamma(p+q+1)}\\
&\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\bigg(\frac{|\beta|\xi^q\Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)} + \frac{T^q}{\Gamma(q+1)}\bigg)\Bigg\}
\end{aligned}$$

and consequently,

$$\|\mathcal{G}x\| \le L_1\Psi.$$

Next we show that *the operator $\mathcal{G}$ maps bounded sets into equicontinuous sets of $\mathcal{U}$*. Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we have

$$\begin{aligned}
|\mathcal{P}x(\tau_2) - \mathcal{P}x(\tau_1)| &\le \Big|J^q f(s,x(s))(\tau_2) - J^q f(s,x(s))(\tau_1)\Big| + \frac{|\alpha||v_2||\tau_2 - \tau_1|}{|\Lambda|}J^{p+q}|f(s,x(s))|(\zeta)\\
&\quad + \frac{|v_1||\tau_2 - \tau_1|}{|\Lambda|}\Big(|\beta| I_\eta^{\gamma,\delta}|f(s,x(s))|(\xi) + J^q|f(s,x(s))|(T)\Big)\\
&\le \frac{L_1}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}]ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1}ds\right|\\
&\quad + \frac{L_1|\alpha||v_2||\tau_2 - \tau_1|}{|\Lambda|}J^{p+q}(\zeta) + \frac{L_1|v_1||\tau_2 - \tau_1|}{|\Lambda|}\Big(|\beta| I_\eta^{\gamma,\delta} p(s)(\xi) + J^q p(T)\Big).
\end{aligned}$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore, by the Arzelá-Ascoli theorem, the operator $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ is completely continuous.

Finally, we consider the set $V = \{x \in \mathcal{U} : x = \mu\mathcal{P}x,\ 0 < \mu < 1\}$ and show that V is bounded. For $x \in V$ and $t \in [0,T]$, we get

$$\begin{aligned}
\|x\| &\le L_1\left\{\frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|(|v_4| + T|v_3|)\zeta^{p+q}}{|\Lambda|\Gamma(p+q+1)} + \frac{(|v_2| + T|v_1|)}{|\Lambda|}\bigg(\frac{|\beta|\xi^q\Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)} + \frac{T^q}{\Gamma(q+1)}\bigg)\right\}\\
&= L_1\Psi.
\end{aligned}$$

Therefore, V is bounded. Hence, by Lemma 3.7, the boundary value problem (1)-(2) has at least one solution on $[0,T]$. □

3.5. Existence result via Leray-Schauder's Nonlinear alternative

Lemma 3.9. *(Nonlinear alternative for single valued maps) [29]. Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $\mathcal{A} : \bar{U} \to C$ is a continuous, compact (that is, $\mathcal{A}(\bar{U})$ is a relatively compact subset of C) map. Then either*

(*i*) *$\mathcal{A}$ has a fixed point in $\bar{U}$, or*

(*ii*) *there is a $x \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $x = \lambda\mathcal{A}(x)$.*

Theorem 3.10. *Assume that*

(H_4) *there exists a continuous nondecreasing function $\Phi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,T],\mathbb{R}^+)$ such that*

$$|f(t,x)| \leq p(t)\Phi(\|x\|) \ \text{for each} \ (t,x) \in [0,T] \times \mathbb{R};$$

(H_5) *there exists a constant $N > 0$ such that*

$$\frac{N}{\Phi(N)\Big(J^q p(s)(T) + A_1 + A_2\Big)} > 1, \tag{16}$$

where

$$\begin{aligned} A_1 &= \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}p(s)(\zeta),\\ A_2 &= \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta| I_\eta^{\gamma,\delta} J^q p(s)(\xi) + J^q p(s)(T)\Big). \end{aligned}$$

Then the boundary value problem (1)-(2) has at least one solution on $[0,T]$.

Proof. We complete the proof in several steps. We first show that the operator $\mathcal{G}$ *defined by (11) maps bounded sets (balls) into bounded sets in $\mathcal{U}$*. For a positive constant r, let $B_r = \{x \in \mathcal{U} : \|x\| \leq r\}$ be a bounded ball in $\mathcal{U}$. Then for $t \in [0,T]$ we have

$$\begin{aligned} |\mathcal{G}x(t)| &\leq J^q|f(s,x(s))|(t) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}|f(s,x(s))|(\zeta)\\ &\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta| I_\eta^{\gamma,\delta} J^q|f(s,x(s))|(\xi) + J^q|f(s,x(s))|(T)\Big)\\ &\leq \Phi(\|x\|)J^q p(s)(T) + \Phi(\|x\|)\frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}p(s)(\zeta)\\ &\quad + \Phi(\|x\|)\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta| I_\eta^{\gamma,\delta} J^q p(s)(\xi) + J^q p(s)(T)\Big), \end{aligned}$$

and consequently,

$$\|\mathcal{G}x\| \leq \Phi(r)\Big\{J^q p(s)(T) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}p(s)(\zeta) + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta| I_\eta^{\gamma,\delta} J^q p(s)(\xi) + J^q p(s)(T)\Big)\Big\}.$$

Next we will show that *the operator $\mathcal{G}$ maps bounded sets into equicontinuous sets of $\mathcal{U}$*. Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we have

$$\begin{aligned} |\mathcal{G}x(\tau_2) - \mathcal{G}x(\tau_1)| &\leq \Big|J^q f(s,x(s))(\tau_2) - J^q f(s,x(s))(\tau_1)\Big| + \frac{|\alpha||v_2||\tau_2 - \tau_1|}{|\Lambda|}J^{p+q}|f(s,x(s))|(T)\\ &\quad + \frac{|v_1||\tau_2 - \tau_1|}{|\Lambda|}\Big(|\beta| I_\eta^{\gamma,\delta}|f(s,x(s))|(\xi) + J^q|f(s,x(s))|(T)\Big)\\ &\leq \frac{\Phi(r)}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}]p(s)ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1}p(s)ds\right|\\ &\quad + \frac{\Phi(r)|\alpha||v_2||\tau_2 - \tau_1|}{|\Lambda|}J^{p+q}p(s)(T) + \frac{\Phi(r)|v_1||\tau_2 - \tau_1|}{|\Lambda|}\Big(|\beta| I_\eta^{\gamma,\delta}p(s)(\xi) + J^q p(s)(T)\Big). \end{aligned}$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore, by the Arzelá-Ascoli theorem, the operator $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ is completely continuous.

Let x be a solution. Then, for $t \in [0,T]$, and following the similar computations as in the first step, we have that

$$|x(t)| \le \Phi(\|x\|)\Big\{J^q p(s)(T) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}p(s)(\zeta) + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta|I_\eta^{\gamma,\delta}J^q p(s)(\xi) + J^q p(s)(T)\Big)\Big\},$$

which leads to

$$\frac{\|x\|}{\Phi(\|x\|)\Big\{J^q p(s)(T) + A_1 + A_2\Big\}} \le 1.$$

In view of (H_5), there exists N such that $\|x\| \ne N$. Let us set

$$\mathcal{X} = \{x \in \mathcal{U} \,:\, \|x\| < N\}.$$

We see that the operator $\mathcal{G} : \overline{\mathcal{X}} \to \mathcal{U}$ is continuous and completely continuous. From the choice of $\mathcal{X}$, there is no $x \in \partial\mathcal{X}$ such that $x = \theta\mathcal{G}x$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathcal{G}$ has a fixed point $x \in \overline{\mathcal{X}}$ which is a solution of problem (1)-(2). This completes the proof. □

3.6. Existence result via Leray-Schauder's Degree Theory

Theorem 3.11. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that*

(H_6) *there exist constants $0 \le \nu < \Psi^{-1}$, and $M > 0$ such that*

$$|f(t,x)| \le \nu|x| + M \ \textit{for all} \ (t,x) \in [0,T] \times \mathbb{R},$$

where Ψ is defined by (12).

Then the boundary value problem (1)-(2) has at least one solution on $[0,T]$.

Proof. In relation to the fixed point problem

$$x = \mathcal{G}x, \tag{17}$$

where the operator $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ is given by (11), we have to show the existence of at least one solution $x \in C[0,T]$ satisfying (17). Set a ball $B_R \subset C[0,T]$ as

$$B_R = \{x \in \mathcal{U} : \max_{t\in[0,T]} |x(t)| < R\},$$

with a constant radius $R > 0$. Hence, we will show that the operator $\mathcal{G} : \overline{B}_R \to C[0,T]$ satisfies a condition

$$x \ne \theta\mathcal{G}x, \quad \forall x \in \partial B_R, \quad \forall \theta \in [0,1]. \tag{18}$$

Set

$$H(\theta,x) = \theta\mathcal{G}x, \quad x \in \mathcal{U}, \quad \theta \in [0,1].$$

As shown in Theorem 3.8 we have that the operator $\mathcal{G}$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map h_θ defined by $h_\theta(x) = x - H(\theta,x) = x - \theta\mathcal{G}x$ is completely continuous. If (18) holds, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follow that

$$\begin{aligned} \deg(h_\theta, B_R, 0) &= \deg(I - \theta\mathcal{G}, B_R, 0) = \deg(h_1, B_R, 0) \\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \ne 0, \quad 0 \in B_R, \end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, we have $h_1(x) = x - \mathcal{G}x = 0$ for at least one $x \in B_R$. Let us assume that $x = \theta\mathcal{G}x$ for some $\theta \in [0,1]$ and for all $t \in [0,T]$ so that

$$
\begin{aligned}
|x(t)| &= |\theta\mathcal{G}x(t)| \\
&\leq J^q|f(s,x(s))|(t) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}|f(s,x(s))|(\zeta) \\
&\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta|I_\eta^{\gamma,\delta}J^q|f(s,x(s))|(\xi) + J^q|f(s,x(s))|(T)\Big) \\
&\leq (\nu|x| + M)J^q p(s)(T) + (\nu|x| + M)\frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)J^{p+q}p(s)(\zeta) \\
&\quad + (\nu|x| + M)\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta|I_\eta^{\gamma,\delta}J^q p(s)(\xi) + J^q p(s)(T)\Big) \\
&= (\nu|x| + M)\Psi,
\end{aligned}
$$

which, on taking the norm $\sup_{t\in[0,T]}|x(t)| = \|x\|$ and solving for $\|x\|$, yields

$$\|x\| \leq \frac{M\Psi}{1 - \nu\Psi}.$$

If $R = \dfrac{M\Psi}{1-\nu\Psi} + 1$, the inequality (18) holds. This completes the proof. □

4. Examples

In this section, we present some examples illustrating our results.

Example 4.1. *Consider the following nonlinear Caputo fractional differential equation with nonlocal Riemann-Liouville and Erdélyi-Kober fractional integral conditions*

$$
\begin{cases}
{}^cD^{3/2}x(t) = \left(\dfrac{3x^2(t) + 4|x(t)|}{1 + |x(t)|}\right)\dfrac{e^{-\cos^2 t}}{8} + \dfrac{1}{5}, & t \in [0,1], \\
x(0) = \dfrac{6}{13}J^{1/2}x\left(\dfrac{1}{4}\right), \quad x(1) = \dfrac{\sqrt{7}}{9}I_{1/6}^{3/4,\sqrt{7}/5}x\left(\dfrac{3}{4}\right).
\end{cases}
\tag{19}
$$

Here $q = 3/2$, $T = 1$, $\alpha = 6/13$, $p = 1/2$, $\zeta = 1/4$, $\beta = \sqrt{7}/9$, $\gamma = 3/4$, $\delta = \sqrt{7}/5$, $\eta = 1/6$, $\xi = 3/4$ and $f(t,x) = ((3x^2 + 4|x|)/(1 + |x|))(e^{-\cos^2 t}/8) + (1/5)$. Using the given data, we find that $v_1 = 0.7396048076$, $v_2 = 0.04339919874$, $v_3 = 0.7655394104$, $v_4 = 0.9241839846$ and $\Psi = 1.653214239$. Since $|f(t,x) - f(t,y)| \leq (1/2)|x-y|$, the condition (H_1) is satisfied with $L = 1/2$. Hence $L\Psi = 0.8266071195 < 1$. Thus, by Theorem 3.1, the problem (19) has a unique solution on $[0,1]$.

Example 4.2. *Consider the following problem of nonlinear Caputo fractional differential equation supplemented with nonlocal Riemann-Liouville and Erdélyi-Kober fractional integral conditions*

$$
\begin{cases}
{}^cD^{5/3}x(t) = \dfrac{\sqrt{t}}{2(t+2)^2}\dfrac{|x(t)|}{|x(t)| + 1} + \dfrac{1}{2}t + \dfrac{2}{3}, & t \in \left[0, \dfrac{3}{2}\right], \\
x(0) = \dfrac{2}{7}J^{3/2}x\left(\dfrac{3}{8}\right), \quad x\left(\dfrac{3}{2}\right) = \dfrac{5}{9}I_{11/4}^{\sqrt{2}/3,1/\sqrt{3}}x\left(\dfrac{9}{8}\right),
\end{cases}
\tag{20}
$$

where $q = 5/3$, $T = 3/2$, $\alpha = 2/7$, $p = 3/2$, $\zeta = 3/8$, $\beta = 5/9$, $\gamma = \sqrt{2}/3$, $\delta = 1/\sqrt{3}$, $\eta = 11/4$, $\xi = 9/8$ and $f(t,x) = (\sqrt{t}|x|)/(2(t+2)^2(|x|+1)) + (1/2)t + (2/3)$. Using the given values, we find that $v_1 = 0.9506436929$, $v_2 = 0.007403446062$, $v_3 = 0.5183793320$, $v_4 = 1.030290752$ and $\Lambda = 0.9832771986$. Choosing $z(t) = \sqrt{t}/8$, we get $A^* = 0.3295079123$. As

$$|f(t,x) - f(t,y)| \leq \frac{\sqrt{t}}{8}\left(\frac{|x-y|}{0.3295079123 + |x-y|}\right),$$

(H_2) is satisfied. Therefore, it follows from Theorem 3.4 that problem (20) has a unique solution on $[0, 3/2]$.

Example 4.3. *Consider the following nonlinear problem*

$$\begin{cases} {}^cD^{7/4}x(t) = \dfrac{e^{-2t}}{3}\sin\left(\dfrac{|x(t)|}{1+|x(t)|}\right) + \dfrac{t^2+3t+1}{2}, & t \in [0,2], \\ x(0) = \dfrac{11}{15}J^{5/2}x\left(\dfrac{1}{2}\right), \quad x(2) = \dfrac{\sqrt{2}}{3}I^{2/7,11/13}_{\sqrt{3}/4}x\left(\dfrac{3}{2}\right). \end{cases} \tag{21}$$

Here $q = 7/4$, $T = 2$, $\alpha = 11/15$, $p = 5/2$, $\zeta = 1/2$, $\beta = \sqrt{2}/3$, $\gamma = 2/7$, $\delta = 11/13$, $\eta = \sqrt{3}/4$, $\xi = 3/2$ and $f(t,x) = (e^{-2t}/3)\sin(|x|/(1+|x|)) + (t^2+3t+1)/2$. Using the given values, we find that $v_1 = 0.9609923104$, $v_2 = 0.005572527089$, $v_3 = 0.6010387134$, $v_4 = 1.756293807$ and $\Lambda = 1.691134148$. Further

$$|f(t,x)| \le \frac{e^{-2t}}{3} + \frac{t^2+3t+1}{2} := \varphi(t).$$

Since $|f(t,x) - f(t,y)| \le (1/3)|x-y|$ with $L = 1/3$, we have

$$L\left\{\frac{|\alpha|(|v_4| + T|v_3|)\zeta^{p+q}}{|\Lambda|\Gamma(p+q+1)} + \frac{(|v_2| + T|v_1|)}{|\Lambda|}\left(\frac{|\beta|\xi^q\Gamma(\gamma+(q/\eta)+1)}{\Gamma(q+1)\Gamma(\gamma+(q/\eta)+\delta+1)} + \frac{T^q}{\Gamma(q+1)}\right)\right\} = 0.8508332020 < 1.$$

Clearly all the assumptions of Theorem 3.6 are satisfied. In consequence, problem (21) has at least one solution on $[0, 2]$.

Example 4.4. *Consider the following nonlinear Caputo fractional differential equation with nonlocal Riemann-Liouville and Erdélyi-Kober fractional integral conditions*

$$\begin{cases} {}^cD^{9/5}x(t) = \tan^{-1}\left(\dfrac{x^2(t)+|x(t)|}{1+|x(t)|}\right)\left(\sin^2 t + 1\right) + \pi, & t \in \left[0, \dfrac{5}{2}\right], \\ x(0) = \dfrac{5}{6}J^{7/2}x\left(\dfrac{5}{8}\right), \quad x\left(\dfrac{5}{2}\right) = \dfrac{3}{4}I^{11/14,3/2}_{2/3}x\left(\dfrac{15}{8}\right), \end{cases} \tag{22}$$

where $q = 9/5$, $T = 5/2$, $\alpha = 5/6$, $p = 7/2$, $\zeta = 5/8$, $\beta = 3/4$, $\gamma = 11/14$, $\delta = 3/2$, $\eta = 2/3$, $\xi = 15/8$ and $f(t,x) = \tan^{-1}((x^2+|x|)/(1+|x|))(\sin^2 t + 1) + \pi$. By the given data, we obtain $v_1 = 0.9861721531$, $v_2 = 0.001920534289$, $v_3 = 0.7641186994$, $v_4 = 2.287156804$ and $\Lambda = 2.256997866 \neq 0$. Also $|f(t,x)| \le 2\pi$. Clearly the hypothesis of Theorem 3.8 is satisfied. Thus problem (22) has at least one solution on $[0, 5/2]$.

Example 4.5. *Consider the following problem*

$$\begin{cases} {}^cD^{11/6}x(t) = \dfrac{(\sqrt{t}+1)}{120}\left(\dfrac{x^2(t)+9|x(t)|}{1+|x(t)|}+1\right), & t \in [0,e], \\ x(0) = \dfrac{3}{8}J^{9/2}x\left(\dfrac{e}{4}\right), \quad x(e) = \dfrac{4}{13}I^{15/19,4/7}_{\sqrt{5}/2}x\left(\dfrac{3e}{4}\right), \end{cases} \tag{23}$$

where $q = 11/6$, $T = e$, $\alpha = 3/8$, $p = 9/2$, $\zeta = e/4$, $\beta = 4/13$, $\gamma = 15/19$, $\delta = 4/7$, $\eta = \sqrt{5}/2$, $\xi = 3e/4$ and $f(t,x) = ((\sqrt{t}+1)/120)((x^2+9|x|)/(1+|x|)+1)$. With the given values, we get $v_1 = 0.7396048076$, $v_2 = 0.0001556329361$, $v_3 = 0.7641186994$, $v_4 = 2.345034116$ and $\Lambda = 1.734517428$. Since $|f(t,x)| \le \frac{(\sqrt{t}+1)}{120}(|x|+10)$, we choose $p(t) = (\sqrt{t}+1)/120$ and $\Phi(|x|) = |x|+10$. Also $A_1 = 6.486485784 \times 10^{-7}$ and $A_2 = 0.07244789774$. Thus there exists a constant $N > 1.495387530$ satisfying (16). Thus all conditions of Theorem 3.10 are satisfied. Hence the problem (23) has at least one solution on $[0, e]$.

Example 4.6. *Consider the following nonlinear Caputo fractional differential equation with nonlocal Riemann-Liouville and Erdélyi-Kober fractional integral conditions*

$$\begin{cases} {}^cD^{13/7}x(t) = \dfrac{(\sin t+1)e^{-t}}{24}\left(\dfrac{x^2(t)}{1+|x(t)|}\right) + \dfrac{3|x(t)|}{1+6|x(t)|} + \dfrac{3}{2}, & t \in [0,\pi], \\ x(0) = \dfrac{\sqrt{3}}{5}J^{11/2}x\left(\dfrac{\pi}{4}\right), \quad x(\pi) = \dfrac{2}{11}I^{\sqrt{2}/5,\sqrt{3}/7}_{\sqrt{5}/9}x\left(\dfrac{3\pi}{4}\right). \end{cases} \tag{24}$$

Here $q = 13/7$, $T = \pi$, $\alpha = \sqrt{3}/5$, $p = 11/2$, $\zeta = \pi/4$, $\beta = 2/11$, $\gamma = \sqrt{2}/5$, $\delta = \sqrt{3}/7$, $\eta = \sqrt{5}/9$, $\xi = 3\pi/4$ and $f(t,x) = (((\sin t + 1)e^{-t})/24)(x^2/(1 + |x|)) + (3|x|/(1 + 6|x|)) + (3/2)$. Form the given data, we get $v_1 = 0.9996813121$, $v_2 = 0.00003850720878$, $v_3 = 0.8155910493$, $v_4 = 2.853050466$, $\Lambda = 2.852172639$ and $\Psi = 10.33403198$. Further, $|f(t,x)| \le \frac{1}{12}|x| + 2$. With $\nu = 1/12 = 0.08333333333 < 0.09676765099 = \Psi^{-1}$ and $M = 2$, Therefore, we deduce from the conclusion of Theorem 3.11 that the problem (24) has at least one solution on $[0, \pi]$.

5. Supplementary Problems

In view of Lemma 2.9, we can define an operator $\mathcal{G}_1 : \mathcal{U} \to \mathcal{U}$ associated with problem (1) and (3) as follows

$$\begin{aligned}(\mathcal{G}_1 x)(t) &= J^q f(s,x(s))(t) + \frac{\alpha}{\Lambda_1}(u_4 - tu_3) I_\sigma^{\epsilon,\theta} f(s,x(s))(\zeta) \\ &\quad + \frac{1}{\Lambda_1}(u_2 + tu_1)\Big(\beta I_\eta^{\gamma,\delta} J^q f(s,x(s))(\xi) - J^q f(s,x(s))(T)\Big). \end{aligned} \tag{25}$$

With the aid of operator $\mathcal{G}_1$ given by (25) and the constant

$$\Psi_1 := \frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|(|u_4| + T|u_2|)\Gamma(\epsilon+1)}{|\Lambda_1|\Gamma(\epsilon+\theta+1)} + \frac{(|u_2| + T|u_1|)}{|\Lambda_1|}\left(\frac{|\beta|\xi^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)} + \frac{T^q}{\Gamma(q+1)}\right), \tag{26}$$

we can establish the existence results for problem (1) and (3) similar to the ones for problem (1)-(2) obtained in Section 3.

In relation to problem (1) and (4), we define an operator $\mathcal{G}_2 : \mathcal{U} \to \mathcal{U}$ by means of Lemma (2.10) as

$$\begin{aligned}(\mathcal{G}_2 x)(t) &= J^q f(s,x(s))(t) + \frac{\alpha}{\Lambda_2}(z_4 - tz_3) I_\sigma^{\epsilon,\theta} f(s,x(s))(\zeta) \\ &\quad + \frac{1}{\Lambda_2}(z_2 + tz_1)\Big(\beta J^{\phi+q} f(s,x(s))(\xi) - J^q f(s,x(s))(T)\Big). \end{aligned} \tag{27}$$

The existence results analog to the ones for problem (1)-(2) established in Section 3 can be obtained for problem (1) and (4) by using the operator $\mathcal{G}_2$ defined by (27) together with the constant

$$\Psi_2 := \frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|(|z_4| + T|z_2|)\Gamma(\epsilon+1)}{|\Lambda_2|\Gamma(\epsilon+\theta+1)} + \frac{(|z_2| + T|z_1|)}{|\Lambda_2|}\left(\frac{|\beta|\xi^{\phi+q}}{\Gamma(\phi+q+1)} + \frac{T^q}{\Gamma(q+1)}\right). \tag{28}$$

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