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The Solution of a Class Functional Equations on Semi-Groups

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Abstract. Let S be a semi-group, and let $\sigma, \tau \in Antihom(S, S)$ satisfy $\tau \circ \tau = \sigma \circ \sigma = id$. We show that any solutions $f : S \to \mathbb{C}$ of the functional equation

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in S,$$

has the form $f = (m + m \circ \sigma \circ \tau)/2$, where m is a multiplicative function on S.

1. Set Up and Notation

Throughout the paper we work in the following framework: S is a semi-group (a set equipped with an associative composition rule $(x, y) \mapsto xy$) and $\sigma, \tau : S \to S$ are two antihomomorphisms (briefly $\sigma, \tau \in Antihom(S, S)$) satisfying $\tau \circ \tau = \sigma \circ \sigma = id$.

For any function $f : S \to \mathbb{C}$ we say that f is σ-even (resp. τ-even) if $f \circ \sigma = f$ (resp. $f \circ \tau = f$), also we use the notation $\check{f}(x) = f(x^{-1})$ in the case S is a group.

We say that a function $m : S \to \mathbb{C}$ is multiplicative, if $m(xy) = m(x)m(y)$ for all $x, y \in S$.

If S is a topological space, then we let $C(S)$ denote the algebra of continuous functions from S into $\mathbb{C}$.

2. Introduction

The classical d'Alembert's functional equation is of the form

$$f(x + y) + f(x - y) = 2f(x)f(y), \quad x, y \in \mathcal{G}, \tag{1}$$

where $(\mathcal{G}, +)$ is a group and $f : \mathcal{G} \to \mathbb{C}$ is the unknown function. It is also called the cosine functional equation since $f = \cos$ satisfies (1) in the real-to-real case. Eq. (1) has a long history going back to d'Alembert [4]. As the name suggests this functional equation was introduced by dAlembert in connection with the composition of forces and plays a central role in determining the sum of two vectors in Euclidean and non-Euclidean geometries [6].

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In the same year Stetkær in [9] obtained the complex valued solution of the following variant of d'Alembert's functional equation

$$f(xy) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in S, \tag{2}$$

where S is a semi-group, σ is an involutive homomorphism of S. The difference between d'Alembert's standard functional equation

$$f(xy) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in S,$$

and the variant (2) is that τ is an antihomomorphism (on a group typically the group inversion). Some information, applications and numerous references concerning (1), (2) and their further generalizations can be found e.g. in [1, 5, 8, 9].

Recently, Chahbi et al. [2] obtained the solution of following functional equation

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in S, \tag{3}$$

where S is a semi-group and σ, τ are two homomorphisms of S such that $\sigma \circ \sigma = \tau \circ \tau = id$.

The natural question that arises is: "What the solution when we replace homomorphism by anti-homomorphism in equation (3)"?

The main purpose of this paper is to study this question by reformulating this equation as:

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in S, \tag{4}$$

where S is a semi-group and $\sigma, \tau \in Antihom(S, S)$ such that $\sigma \circ \sigma = \tau \circ \tau = id$. This equation is a natural generalization of the following new functional equations

$$f(x\sigma(y)) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in S, \tag{5}$$

where $(S, .)$ is a semi-group and $\sigma \in Antihom(S, S)$ such that $\sigma \circ \sigma = id$ and

$$f(xy^{-1}) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in G, \tag{6}$$

$$f(x\sigma(y)) + f(y^{-1}x) = 2f(x)f(y), \quad x, y \in G, \tag{7}$$

$$f(xy) + f(yx) = 2f(x)\check{f}(y), \quad x, y \in G, \tag{8}$$

where $(G, .)$ is a group and $\sigma \in Antihom(G, G)$ such that $\sigma \circ \sigma = id$. In the case $\check{f} = f$ the functional equation (8) known the symmetrized multiplicative Cauchy equation (see for instance [7] or [8, Theorem 3.21]). By elementary methods we find all solutions of (4) on semi-groups in terms of multiplicative functions. Finally, we note that the sine addition law on semi-groups given in [3, 8] is a key ingredient of the proof of our main result (Theorem 3.2).

3. Solution of the Functional Equation (4)

In this section we obtain the solution of the functional equation (4) on semi-groups. The following lemma will be used in the proof of Theorem 3.2.

Lemma 3.1. *Let S be a semi-group and $\sigma \in Antihom(S, S)$ such that $\sigma \circ \sigma = id$.*
If $f : S \to \mathbb{C}$ is a solution of the functional equation

$$f(x\sigma(y)) = f(x)f(y), \quad x, y \in S, \tag{9}$$

then f is a σ-even multiplicative function.

Proof. For all $x, y, z \in S$, we have

$$\begin{aligned} f(x)f(y)f(\sigma(z)) &= f(x)f(yz) = f(x\sigma(yz)) \\ &= f(x\sigma(z)\sigma(y)) = f(x\sigma(z))f(y) = f(x)f(z)f(y), \end{aligned}$$

then f is σ-even. And we have

$$f(xy) = f(x\sigma(\sigma(y))) = f(x)f(\sigma(y)) = f(x)f(y),$$

for all $x, y \in S$. □

Theorem 3.2. *Let S be a semi-group and $\sigma, \tau \in Antihom(S, S)$ such that $\sigma \circ \sigma = \tau \circ \tau = id$ (where id denotes the identity map). The solutions $f : S \to \mathbb{C}$ of (4) are the functions of the form $f = (m + m \circ \sigma \circ \tau)/2$, where $m : S \to \mathbb{C}$ is a multiplicative function such that:*

(i) $m \circ \sigma \circ \tau = m \circ \tau \circ \sigma$, and
(ii) m is σ-even and/or τ-even.

If S is a topological semi-group and $f \in C(S)$, then $m, m \circ \sigma \circ \tau \in C(S)$.

Proof. We use in the proof similar Stetkaer's computations [9]. Let $x, y, z \in S$ be arbitrary. If we replace x by $x\sigma(y)$ and y by z in (4), we get

$$f(x\sigma(zy)) + f(\tau(z)x\sigma(y)) = 2f(x\sigma(y))f(z). \tag{10}$$

On the other hand if we replace x by $\tau(z)x$ in (4), we infer that

$$\begin{aligned} f(\tau(z)x\sigma(y)) + f(\tau(zy)x) &= 2f(\tau(z)x)f(y) \\ &= 2f(y)[2f(x)f(z) - f(x\sigma(z))]. \end{aligned} \tag{11}$$

Replacing y by zy in (4), we obtain

$$f(\tau(zy)x) = 2f(x)f(zy) - f(x\sigma(zy)). \tag{12}$$

It follows from (12) that (11) become

$$f(\tau(z)x\sigma(y)) + 2f(x)f(zy) - f(x\sigma(zy)) = 4f(y)f(x)f(z) - 2f(y)f(x\sigma(z)). \tag{13}$$

Subtracting this from (10) we get after some simplifications that

$$f(x\sigma(zy)) - f(x)f(zy) = f(y)[f(x\sigma(z)) - f(x)f(z)] + f(z)[f(x\sigma(y)) - f(x)f(y)] \tag{14}$$

With the notation $f_x(y) := f(x\sigma(y)) - f(x)f(y)$ we can reformulate (14) to

$$f_a(xy) = f_a(x)f(y) + f_a(y)f(x). \tag{15}$$

This shows that the pair (f_a, f) satisfies the sine addition law for any $a \in S$.

Case 1: If $f_a = 0$ for all $a \in S$, then f satisfies the functional equation (9) by the every definition of f_x. From Lemma 3.1, we see that f is a σ-even multiplicative function. Substituting f into (4), we infer that f is τ-even. This implies that $f = (\varphi + \varphi \circ \sigma \circ \tau)/2$, where $f = \varphi$ is multiplicative.

Case 2: If $f_a \neq 0$ for some $a \in S$ we get from the known solution of the sine addition formula (see for instance [3] or [8, Theorem 4.1]) that there exist two multiplicative functions $\chi_1, \chi_2 : S \to \mathbb{C}$ such that

$$f = \frac{\chi_1 + \chi_2}{2}.$$

If $\chi_1 = \chi_2$, then letting $\eta := \chi_1$, we have $f = \eta$. Substituting $f = \eta$ into (4) we get that

$$\eta \circ \sigma + \eta \circ \tau = 2\eta.$$

So $\eta = \eta \circ \sigma = \eta \circ \tau$ (see for instance [8, Corollary 3.19]). Then f has the desired form.

If $\chi_1 \neq \chi_2$, substituting $f = (\chi_1 + \chi_2)/2$ into (4) we find after a reduction that

$$\chi_1(x)[\chi_1 \circ \sigma(y) + \chi_1 \circ \tau(y) - \chi_1(y) - \chi_2(y)] + \chi_2(x)[\chi_2 \circ \sigma(y) + \chi_2 \circ \tau(y) - \chi_1(y) - \chi_2(y)] = 0$$

for all $x, y \in S$. Since $\chi_1 \neq \chi_2$ we get from the theory of multiplicative functions (see for instance [8, Theorem 3.18]) that both terms are 0, so

$$\begin{cases} \chi_1(x)[\chi_1 \circ \sigma(y) + \chi_1 \circ \tau(y) - \chi_1(y) - \chi_2(y)] = 0 \\ \chi_2(x)[\chi_2 \circ \sigma(y) + \chi_2 \circ \tau(y) - \chi_1(y) - \chi_2(y)] = 0 \end{cases} \tag{16}$$

for all $x, y \in S$. Since $\chi_1 \neq \chi_2$ at least one of χ_1 and χ_2 is not zero.

Subcase 2.1: If $\chi_2 = 0$, then $\chi_1 \neq 0$. From (16), we infer that

$$\chi_1 = \chi_1 \circ \sigma + \chi_1 \circ \tau.$$

Therefore $\chi_1 \circ \sigma = 0$ or $\chi_1 \circ \tau = 0$. In either case $\chi_1 = 0$, because σ and τ are surjective. But that contradicts $\chi_1 \neq 0$. So this subcase is void. The same is true for $\chi_1 = 0$ and $\chi_2 \neq 0$.

Subcase 2.2: $\chi_1 \neq 0$ and $\chi_2 \neq 0$. From (16), we have

$$\chi_1 + \chi_2 = \chi_1 \circ \sigma + \chi_1 \circ \tau = \chi_2 \circ \sigma + \chi_2 \circ \tau.$$

Using $\chi_1 \circ \sigma + \chi_1 \circ \tau = \chi_2 \circ \sigma + \chi_2 \circ \tau$ and the fact that $\chi_1 \neq \chi_2$, we see that $\chi_1 \circ \sigma = \chi_2 \circ \tau$ and $\chi_1 \circ \tau = \chi_2 \circ \sigma$. Thus

$$\chi_2 = \chi_1 \circ \tau \circ \sigma = \chi_1 \circ \sigma \circ \tau.$$

We now use $\chi_1 + \chi_2 = \chi_1 \circ \sigma + \chi_1 \circ \tau$, we get that χ_1 is σ-even or $\chi_1 = \chi_1 \circ \tau$. So we are in the solution stated in the theorem with $m = \chi_1$.

Finally, in view of these cases we deduce that f has the form stated in Theorem 3.2.

The other direction of the proof is trivial to verify. The continuity statement follows from [8, Theorem 3.18 (d)]. □

As immediate consequences of Theorem 3.2, we have the following corollaries.

Corollary 3.3. *Let S be a semi-group and $\sigma \in Antihom(S, S)$ such that $\sigma \circ \sigma = id$ (where id denotes the identity map). The solutions $f : S \to \mathbb{C}$ of the functional equation*

$$f(x\sigma(y)) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in S$$

are the functions of the form $f = m$, where $m : S \to \mathbb{C}$ is a multiplicative such that m is σ-even.

Proof. It suffices to take $\tau(x) = \sigma(x)$ for all $x \in S$ in Theorem 3.2. □

Corollary 3.4. *Let G be a group and $\sigma \in Antihom(G, G)$ such that $\sigma \circ \sigma = id$. The solutions $f : G \to \mathbb{C}$ of (7) are the functions of the form $f = (m + \overline{m \circ \sigma})/2$, where $m : G \to \mathbb{C}$ is a multiplicative function such that m is σ-even and/or $m = \overline{m}$.*

If S is a topological semi-group and $f \in C(G)$, then $m, \overline{m \circ \sigma} \in C(G)$.

Proof. It suffices to take $\tau(x) = x^{-1}$ for all $x \in G$ in Theorem 3.2. □

Corollary 3.5. *Let G be a group and $\sigma \in Antihom(G, G)$ such that $\sigma \circ \sigma = id$. The solutions $f : G \to \mathbb{C}$ of (6) are the functions of the form $f = (m + \overline{m \circ \sigma})/2$, where $m : G \to \mathbb{C}$ is a multiplicative function such that m is σ-even and/or $m = \overline{m}$.*

If S is a topological semi-group and $f \in C(G)$, then $m, \overline{m \circ \sigma} \in C(G)$.

Proof. It suffices to take $\sigma(x) = x^{-1}$ and $\tau(x) = \sigma(x)$ for all $x \in G$ in Theorem 3.2. □

Corollary 3.6. *Let G be a group. The solutions $f : G \to \mathbb{C}$ of (8) are the functions of the form $f = m$, where $m : G \to \mathbb{C}$ is a multiplicative function such that $\overline{m} = m$.*

Proof. It suffices to take $\sigma(x) = \tau(x) = x^{-1}$ for all $x \in G$ in Theorem 3.2. □

4. Some Examples of Possible Applications

In this section we give examples of possible applications of the results obtained in Theorem 3.2.

Definition 4.1. *Let X and Y be non-empty sets. If $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ we define $f \otimes g : X \times Y \to \mathbb{C}$ by the formula*

$$(f \otimes g)(x, y) := f(x)g(y), \quad \text{for all } (x, y) \in X \times Y.$$

Definition 4.2. *An involutive semi-group is a semi-group S together with an unary operation $* : S \to S$, $s \to s^*$ satisfying $(s^*)^* = s$ and $(st)^* = t^*s^*$ for all $s, t \in S$.*

Corollary 4.3. *Let S be an involutive semi-group. The solutions $f : S \times S \times S \to \mathbb{C}$ of the functional equation*

$$f(s_1t_2, s_2t_1, s_3t_3) + f(t_1s_1, t_3s_2, t_2s_3) = 2f(s_1, s_2, s_3)f(t_1, t_2, t_3), \tag{17}$$

for $s_1, s_2, s_3, t_1, t_2, t_3 \in S$, are the functions of the form

$$f(s_1, s_2, s_3) = m(s_1s_2s_3), \quad s_1, s_2, s_3 \in S,$$

where m is a multiplicative function on S such that $m(s^*) = \lambda m(s)$ for all $s \in S$.

If S is a topological semi-group and $f \in C(S \times S \times S)$, then $m \in C(S)$.

Proof. Let $f : S \times S \times S \to \mathbb{C}$ be a solution of (17). Then f solves (4) with S is an involutive semi-group and σ, τ are defined as follow

$$\sigma(s_1, s_2, s_3) := (s_2^*, s_1^*, s_3^*) \text{ and } \tau(s_1, s_2, s_3) := (s_1^*, s_3^*, s_2^*).$$

So, from Theorem 3.2, we read that f has the form

$$f = \frac{m + m \circ \sigma \circ \tau}{2},$$

where $m : S \to \mathbb{C}$ is a multiplicative function such that:

1. $m \circ \sigma \circ \tau = m \circ \tau \circ \sigma$, and
2. m is σ-even and/or τ-even.

Assume first $m = 0$. Hence $f = 0$.
Assume next $m \neq 0$. Using Lemma 3.2 in [9], we see that there exist three multiplicative functions m_1, m_2, m_3 on S such that $m = m_1 \otimes m_2 \otimes m_3$. The condition (1) becomes

$$m_1(s_3)m_2(s_1)m_3(s_2) = m_1(s_2)m_2(s_3)m_3(s_1), \quad \text{for all } s_1, s_2, s_3 \in S.$$

Since $m \neq 0$, we have $m_1, m_2, m_3 \neq 0$. Then there exists a $\lambda \in \mathbb{C}\backslash\{0\}$ such that $m_1 = \lambda m_2$, but m_1 and m_2 are multiplicative, so $\lambda = 1$ and hence $m_1 = m_2$. Similarly, we can get $m_2 = m_3$. Thus $m_1 = m_2 = m_3$.

Similarly, by the condition (2), there exists a $\lambda \in \mathbb{C}\backslash\{0\}$ such that $m(s^*) = \lambda m(s)$, as m is a multiplicative $\lambda = 1$. This implies that m is σ-even and τ-even. Consequently, we get the correct form of f.

Conversely, simple computations prove that the formula above for f define a solution of (17).

The continuity statement follows from Theorem 3.2 and Lemma 3.2 in [9]. □

Example 4.4. *For a non-abelian example of a monoid, consider the set of complex* 2×2 *matrices under matrix multiplication* $S = M(2, \mathbb{C})$, *and take as anti-homomorphisms*

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = J\begin{pmatrix} a & c \\ b & d \end{pmatrix}J^{-1} = \begin{pmatrix} a & -ic \\ ib & d \end{pmatrix}$$

where $J = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, *and*

$$\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

We indicate here the corresponding continuous solutions of (4). We write $Re(\lambda)$ *for the real part of the complex number* λ.

The continuous non-zero multiplicative functions on S are (see [3, Example 5.6]): $\chi = 1$, *or else*

$$\chi(X) = \begin{cases} |\det(X)|^{\lambda-n}(\det(X))^n & \text{when} \quad \det(X) \neq 0 \\ 0 & \text{when} \quad \det(X) = 0 \end{cases}$$

where $\lambda \in \mathbb{C}$ *with* $Re(\lambda) > 0$ *and* $n \in \mathbb{Z}$. *Simple computations show that*

$$\sigma \circ \tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -ib \\ ic & d \end{pmatrix}$$

and

$$\tau \circ \sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & ib \\ -ic & d \end{pmatrix}.$$

Therefore, any continuous multiplicative function m on $M(2, \mathbb{C})$ *satisfies* $m \circ \sigma = m$ *and* $m \circ \sigma \circ \tau = m \circ \tau \circ \sigma$.

In conclusion, using Theorem 3.2, the non-zero continuous solutions $f : M(2, \mathbb{C}) \to \mathbb{C}$ *of (4) are:*

(1) $f = 1$; *and*

(2) $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = |ad - bc|^{\lambda-n}(ad - bc)^n$, *for all* $a, b, c, d \in \mathbb{C}$, *where* λ *is a complex number such that* $Re(\lambda) > 0$ *and* $n \in \mathbb{Z}$.

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