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# Libera Operator on Mixed Norm Spaces $H_\nu^{p,q,\alpha}$ when $0 < p < 1$

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**Abstract.** Results from [7] on Libera operator acting on mixed norm spaces $H_\nu^{p,q,\alpha}$, for $1 \le p \le \infty$, are extended to the case $0 < p < 1$.

## 1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the space of all functions holomorphic in the unit disk $\mathbb{D}$ of the complex plane endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$. The dual of $\mathcal{H}(\mathbb{D})$ is equal $\mathcal{H}(\overline{\mathbb{D}})$, where $g \in \mathcal{H}(\overline{\mathbb{D}})$ means that $g$ is holomorphic in a neighborhood of $\overline{\mathbb{D}}$ (depending on $g$). The duality pairing is given by

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)},$$

where $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n \in \mathcal{H}(\mathbb{D})$ and $g(z) = \sum_{n=0}^{\infty} \widehat{g}(n)z^n \in \mathcal{H}(\overline{\mathbb{D}})$.

It is easy to see that the Libera operator defined by

$$\overline{\mathcal{L}}g(z) = \sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty} \frac{\widehat{g}(k)}{k+1}\right)z^n = \int_0^1 g(t + (1-t)z)dt,$$

$g(z) = \sum_{n=0}^{\infty} \widehat{g}(n)z^n \in \mathcal{H}(\overline{\mathbb{D}})$ maps $\mathcal{H}(\overline{\mathbb{D}})$ into $\mathcal{H}(\overline{\mathbb{D}})$.

We denote by $\mathcal{L}$ the operator

$$\mathcal{L}g(z) = \int_0^1 g(t + (1-t)z)dt,$$

$g(z) = \sum_{n=0}^{\infty} \widehat{g}(n)z^n \in \mathcal{H}(\mathbb{D})$, whenever the integral converges uniformly on compact subsets of $\mathbb{D}$. Uniform convergence means that the limit

$$\lim_{r\to 1^-} \int_0^r g(t + (1-t)z)dt$$

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is uniform with respect to $z$ in any compact subset of $\mathbb{D}$. This hypothesis guarantees that $\mathcal{L}g$ is holomorphic function in $\mathbb{D}$. We also call $\mathcal{L}$ the Libera operator, since $\mathcal{L} = \overline{\mathcal{L}}$ on $\mathcal{H}(\overline{\mathbb{D}})$.

A function $f \in \mathcal{H}(\mathbb{D})$ is said to belong to the mixed norm space $H^{p,q,\alpha}$, $0 < p \leq \infty$, $0 < q \leq \infty$, $0 < \alpha < \infty$, if

$$\|f\|_{H^{p,q,\alpha}} = \|f\|_{p,q,\alpha} = \left(\int_0^1 M_p^q(r,f)(1-r)^{q\alpha-1}dr\right)^{\frac{1}{q}} < \infty,\ 0 < q < \infty,$$

$$\|f\|_{H^{p,\infty,\alpha}} = \|f\|_{p,\infty,\alpha} = \sup_{0\leq r<1}(1-r)^{\alpha}M_p(r,f) < \infty.$$

Here, as usual,

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{it})\right|^p dt\right)^{\frac{1}{p}},\ 0 < p < \infty;$$

$$M_\infty(r,f) = \sup_{0\leq t<2\pi}\left|f(re^{it})\right|.$$

The space $H^{p,\infty,\alpha}$ is specific in that the set $\mathcal{P}$, of all analytic polynomials, is not dense in it. The closure of $\mathcal{P}$ in $H^{p,\infty,\alpha}$ coincides with the "little oh" space

$$h^{p,\infty,\alpha} = \{f \in H^{p,\infty,\alpha} : M_p(r,f) = o\left((1-r)^{-\alpha}\right), r \to 1\}.$$

Hardy space $H^p$ is defined as follows

$$H^p = \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_{H^p} = \|f\|_p = \sup_{0\leq r<1} M_p(r,f) < \infty\}.$$

For $t \in \mathbb{R}$ we write $D^t$ for the sequence $\{(n+1)^t\}$, for all $n \geq 0$. If $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ is a sequence and $X$ is a sequence space (by identifying holomorphic function $f(z) = \sum_{n=0}^{\infty}\widehat{f}(n)z^n$ with the sequence $\{\widehat{f}(n)\}_{n=0}^{\infty}$ we may consider the spaces of holomorphic functions as sequence spaces) we write

$$\lambda X = \{\{\lambda_n x_n\} : \{x_n\} \in X\}.$$

If $x = \{x_n\}$ and $\lambda = \{\lambda_n\}$ we also write $\lambda * x = \{\lambda_n x_n\}$. For example, $\{a_n\} \in D^1 l^1$ if and only if $\sum_{n=0}^{\infty}\frac{|a_n|}{n+1} < \infty$. The space $D^t H^{p,q,\alpha}$, for $t \neq 0$, will be denoted by $H_{-t}^{p,q,\alpha}$. Similarly $D^t h^{p,\infty,\alpha} = h_{-t}^{p,\infty,\alpha}$.

Among the spaces $H_s^{p,q,\alpha}$, $s > 0$, the spaces $H_{1+s}^{p,q,1}$ are of independent interest, and are known as Besov spaces, for $0 < q < \infty$, and as Lipschitz spaces when $q = \infty$.

We note that in [6] the spaces of functions $f \in \mathcal{H}(\mathbb{D})$ such that $D^n f \in H^{p,q,n-\alpha}$, $\alpha \in \mathbb{R}$ (equivalently $f^{(n)} \in H^{p,q,n-\alpha}$), for some (any) nonnegative integer $n$ such that $n - \alpha > 0$ are called Besov spaces and they are denoted by $\mathcal{B}_\alpha^{p,q}$. Comparing with the definitions given above, $\mathcal{B}_\alpha^{p,q} = H^{p,q,-\alpha}$, for $\alpha < 0$, and $\mathcal{B}_\alpha^{p,q} = H_{1+\alpha}^{p,q,1}$, for $\alpha > 0$.

We are now ready to state our main results.

**Theorem 1.1.** *The following assertions are equivalent:*
(a) *The operator $\mathcal{L}$ acts as a bounded operator from $H_\nu^{p,q,\alpha}$ into $H_\nu^{p,q,\alpha}$;*
(b) $\kappa_{p,\alpha,\nu} := \nu - \alpha - \frac{1}{p} + 1 > 0$.

**Theorem 1.2.** *Let $\kappa_{p,\alpha,\nu} \leq 0$. Then the following conditions are equivalent:*
(a) *The operator $\overline{\mathcal{L}}$ can be extended to a bounded operator from $H_\nu^{p,q,\alpha}$ to $\mathcal{H}(\mathbb{D})$;*
(b) *$\mathcal{L}$ acts as a bounded operator from $H_\nu^{p,q,\alpha}$ to $\mathcal{H}(\mathbb{D})$;*
(c) $H_\nu^{p,q,\alpha} \subseteq D^1 l^1$;
(d) $0 < p \leq 2$, $0 < q \leq 1$ *and* $\kappa_{p,\alpha,\nu} = 0$.

Theorem 1.1 and Theorem 1.2 have been proven in [7] when $1 \le p \le \infty$, $0 < q \le \infty$ and $\alpha > 0$. So it remains to prove them for $0 < p < 1$, $0 < q \le \infty$ and $\alpha > 0$.

Theorem 1.1 imply the following extension of Corollary 6.2 in [5].

**Corollary 1.3.** *Let $0 < p, q \le \infty$ and $\alpha \in \mathbb{R}$. Then the Libera operator $\mathcal{L}$ maps $\mathcal{B}^{p,q}_\alpha$ into $\mathcal{B}^{p,q}_\alpha$ if and only if $\alpha > \frac{1}{p} - 1$.*

## 2. $H^{p,q,\alpha}_\nu$ as a Subspace of $D^1 l^1$

It is easily seen that if $g(z) = \sum\limits_{n=0}^{\infty} \widehat{g}(n) z^n \in D^1 l^1$ then $\mathcal{L}g \in \mathcal{H}(\mathbb{D})$. In this section we determine the indices $p, q, \alpha, \nu$ for which $H^{p,q,\alpha}_\nu \subset D^1 l^1$. The problem was solved in [7], when $1 \le p \le \infty$. Our approach is different from that given in [7] and it is applicable when $0 < p < 1$ as well as in the case $1 \le p \le \infty$.

If $X$ and $Y$ are sequence spaces, we say that a sequence $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ is a multiplier from $X$ into $Y$, $\lambda \in (X, Y)$, if for any sequence $x = \{x_n\} \in X$, a sequence $\lambda * x = \{\lambda_n x_n\} \in Y$.

Now we determine the indices $p, q, \alpha, \nu$ for which $H^{p,q,\alpha}_\nu \subseteq D^1 l^1$. It is easy to see that $H^{p,q,\alpha}_\nu \subseteq D^1 l^1$ if and only if the sequence $\left\{\frac{1}{n+1}\right\}_{n=0}^{\infty}$ is a multiplier from the space $H^{p,q,\alpha}_\nu$ into $l^1$. Having in mind this and known results about multipliers from $H^{p,q,\alpha}_\nu$ into $l^1$ ([3], [4], [8]) we have the following

**Theorem 2.1.**
(1) *Let $0 < p \le 1$.*
(1a) *If $0 < q \le 1$, then $H^{p,q,\alpha}_\nu \subseteq D^1 l^1$ if and only if $\nu \ge \alpha + \frac{1}{p} - 1$;*
(1b) *If $1 < q \le \infty$, then $H^{p,q,\alpha}_\nu \subseteq D^1 l^1$ if and only if $\nu > \alpha + \frac{1}{p} - 1$;*
(2) *Let $2 \le p \le \infty$.*
(2a) *If $0 < q \le 1$, then $H^{p,q,\alpha}_\nu \subseteq D^1 l^1$ if and only if $\nu - \alpha \ge -\frac{1}{2}$;*
(2b) *If $1 < q \le \infty$, then $H^{p,q,\alpha}_\nu \subseteq D^1 l^1$ if and only if $\nu - \alpha > -\frac{1}{2}$;*
(3) *Let $1 < p < 2$.*
(3a) *If $0 < q \le 1$, then $H^{p,q,\alpha}_\nu \subseteq D^1 l^1$ if $\nu \ge \alpha + \frac{1}{p} - 1$;*
(3b) *If $1 < q \le \infty$, then $H^{p,q,\alpha}_\nu \subseteq D^1 l^1$ if $\nu > \alpha + \frac{1}{p} - 1$.*

## 3. Proof of Theorem 1.1

In the paper we use a sequence $\{V_n\}_{n=0}^{\infty}$ constructed in the following way (see, e.g., [2], [6]).
Let $\omega$ be a $C^\infty$ function on $\mathbb{R}$ such that
$\omega(t) = 1$ for $t \le 1$,
$\omega(t) = 0$ for $t \ge 2$,
$\omega$ is decreasing and positive on the interval $(1, 2)$.
Let $\varphi(t) = \omega\left(\frac{t}{2}\right) - \omega(t)$, and let $V_0(z) = 1 + z$, and, for $n \ge 1$,

$$V_n(z) = \sum_{k=2^{n-1}}^{2^{n+1}} \varphi\left(\frac{k}{2^{n-1}}\right) z^k = \sum_{k=0}^{\infty} \varphi\left(\frac{k}{2^{n-1}}\right) z^k.$$

These polynomials have the following properties

$$g(z) = \sum_{n=0}^{\infty} (V_n * g)(z), \text{ for } g \in \mathcal{H}(\mathbb{D}), \tag{3.1}$$

$$\|V_n * g\|_p \le C_p \|g\|_p, \text{ for } g \in H^p, p > 0, \tag{3.2}$$

$$\|V_n\|_p \asymp 2^{n\left(1 - \frac{1}{p}\right)}, \ 0 < p \le \infty. \tag{3.3}$$

In [6] the following characterization of $H^{p,q,\alpha}_\nu$ was proved.

**Lemma 3.1 ([6]).** *Let $0 < p \le \infty$, $0 < q \le \infty$, $\alpha > 0$ and let $\nu \in \mathbb{R}$. A function $g \in \mathcal{H}(\mathbb{D})$ is in $H_\nu^{p,q,\alpha}$ if and only if*

$$\sum_{n=0}^{\infty} 2^{n(\nu-\alpha)q} \|V_n * g\|_p^q < \infty,\ 0 < q < \infty,$$

*and if $q = \infty$, $g \in H_\nu^{p,\infty,\alpha}$ (resp. $h_\nu^{p,\infty,\alpha}$) if and only if $\|V_n * g\|_p = O\left(2^{n(\alpha-\nu)}\right)$ (resp. $\|V_n * g\|_p = o\left(2^{n(\alpha-\nu)}\right)$, $n \to \infty$).*

**Lemma 3.2.** *If $n$ is a positive integer,*

$$P(z) = \sum_{k=n}^{4n} \lambda_k z^k,$$

*where $\{\lambda_k\}$ is a complex sequence, and*

$$Q(z) = \sum_{k=n}^{4n} \log^\delta(k+1)\lambda_k z^k,\ \delta \in \mathbb{R},$$

*then there is an absolute constant $C$ depending only on $\delta$ such that*

$$C^{-1}\|Q\|_p \le \log^\delta(n+1)\|P\|_p \le C\|Q\|_p,\ 0 < p < \infty.$$

*Proof.* In the proof we use next well known results (see [4], [6]).

(1) *Let $f(z) = \sum_{k=m}^{n} a_k z^k$, $0 \le m \le n$. Then*

$$r^n\|f\|_p \le M_p(r,f) \le r^m\|f\|_p,\ 0 < r < 1.$$

(2) *If $f \in H^p$ and $g \in H^q$, where $0 < p \le 1$ and $p \le q$, then*

$$M_q(r, f * g) \le (1-r)^{1-\frac{1}{p}}\|f\|_p\|g\|_q,\ 0 < r < 1.$$

(3) *Let $\psi$ be a $C^\infty$ function with support in $[a,b]$ such that $b - a \ge 1$ and $p > 0$. Then for every integer $N$, $Np > 1$, there is a constant $C = C_{p,N}$ such that*

$$\|W_\psi\|_p \le C\|\psi^{(N)}\|_\infty (b-a)^{N+1-\frac{1}{p}},$$

*where $W_\psi(z) = \sum_{k=-\infty}^{\infty} \psi(k) z^k$.*

Suppose first that $0 < p \le 1$. Let $\varphi$ be a $C^\infty$ function with support in $\left(\frac{1}{2}, 5\right)$ such that $\varphi(x) = 1$ for $x \in [1,4]$ and $\lambda_k := 0$ for $k \notin \{n, n+1, \dots, 4n\}$. Consider the function $\psi(x) = \varphi\left(\frac{x}{n}\right)\log^\delta(x+1)$. Then we have $Q$ as

$$Q(z) = \sum_{k=0}^{\infty} \varphi\left(\frac{k}{n}\right)\log^\delta(k+1)\lambda_k z^k = \sum_{k=0}^{\infty} \psi(k)\lambda_k z^k.$$

Note that $\operatorname{supp}\psi \subset \left(\frac{n}{2}, 5n\right)$. Hence,

$$\begin{aligned} \|Q\|_p = \|W_\psi * P\|_p &\le Cn^{\frac{1}{p}-1}\|W_\psi\|_p\|P\|_p \\ &\le Cn^{\frac{1}{p}-1}\|\psi^{(N)}\|_\infty n^{N+1-\frac{1}{p}}\|P\|_p \\ &= Cn^N\|\psi^{(N)}\|_\infty\|P\|_p. \end{aligned}$$

In order to estimate $\|\psi^{(N)}\|_\infty$ we use Leibniz's rule and easily proved inequality

$$\left|\left(\log^\delta(x+1)\right)^{(j)}\right| \le C\frac{\log^\delta(n+1)}{n^j},\ \frac{n}{2} \le x \le 5n,\ j = 0,1,\dots,$$

to obtain

$$\begin{aligned}\left|\psi^{(N)}(x)\right| &\le C\sum_{j=0}^{N}\left|\varphi^{(N-j)}\left(\tfrac{x}{n}\right)\right| n^{j-N}\tfrac{\log^\delta(n+1)}{n^j} \\ &\le Cn^{-N}\log^\delta(n+1),\end{aligned}$$

where $C$ is independent of $n$. Therefore

$$\|Q\|_p \le C\log^\delta(n+1)\|P\|_p.$$

To prove the reverse inequality we write $P$ as

$$P(z) = \sum_{k=n}^{4n} \log^{-\delta}(k+1)\xi_k z^k,$$

where $\xi_k = \lambda_k \log^\delta(k+1)$. Applying the above case, we get

$$\|P\|_p \le C\log^{-\delta}(n+1)\|Q\|_p,$$

which completes the proof in the case $0 < p \le 1$.

If $p > 1$, then we use inequality

$$\|Q\|_p = \|W_\psi * P\|_p \le \|W_\psi\|_1\|P\|_p$$

and

$$\|W_\psi\|_1 \le C\log^\delta(n+1) \text{ (above case } p = 1),$$

to complete the proof of the lemma. □

We note that the more general result that the one stated in Lemma 3.2 could be found in [6]. For the sake of completness the proof of Lemma 3.2 is included.

We also need two lemmas, first is due to Hardy and Littlewood (see [1]) and the second one to Pavlović (see [7]).

**Lemma 3.3.**
(a) $H_\nu^{p,q,\alpha} \subseteq h_\nu^{p,\infty,\alpha} \subseteq H_\nu^{p,\infty,\alpha}$, *for* $0 < q < \infty$;
(b) *Let* $0 < p < u \le \infty$ *and* $\beta = \frac{1}{p} - \frac{1}{u} + \alpha$. *Then* $H_\nu^{p,q,\alpha} \subsetneq H_\nu^{u,q,\beta}$.

**Lemma 3.4 ([7]).** *If $a$ and $b$ are positive real numbers such that $a + b \le 1$ and if $g \in H^p$, $0 < p \le \infty$, then*

$$\left(\int_0^{2\pi}\left|g(a+be^{it})\right|^p dt\right)^{\frac{1}{p}} \le \left(\frac{2a+b}{b}\right)^{\frac{1}{p}}\left(\int_0^{2\pi}\left|g((a+b)e^{it})\right|^p dt\right)^{\frac{1}{p}}.$$

The following two real variables results are well known (see [7]).

**Sublemma 3.5.** *Let $0 < q < 1$, $\beta > 0$ and let $u : (0,1) \to [0,\infty)$ be a nondecreasing function. Then*

$$\left(\int_r^1 u(s)(1-s)^{\beta-1}ds\right)^q \le C\int_r^1 u^q(s)(1-s)^{q\beta-1}ds,$$

*where $C$ is independent of $u$.*

**Sublemma 3.6.** *Let $1 < q < \infty$, $1 + \delta > \varepsilon > 0$ and $u \geq 0$ a measurable function defined on $(r, 1)$. Then*

$$\left(\int_r^1 u(s)(1-s)^{\delta} ds\right)^q \leq C(1-r)^{\varepsilon q} \int_r^1 u^q(s)(1-s)^{(1+\delta-\varepsilon)q-1} ds.$$

*Proof.* [Proof of implication (b) ⇒ (a) in Theorem 1.1] The implication (b) ⇒ (a) has been proven in [7] for $1 \leq p \leq \infty$ and nonnegative integer $\nu$. We prove it for $0 < p < 1$ and $\nu \in \mathbb{R}$. Our proof shows that it holds when $1 \leq p \leq \infty$ and $\nu \in \mathbb{R}$.

Let $\nu > \alpha + \frac{1}{p} - 1$, $0 < p < 1$ and $g \in H_\nu^{p,q,\alpha}$. Let $n$ be a positive integer such that $n - 1 < \nu \leq n$. It is known that $\|D^\nu g\|_{p,q,\alpha} < \infty$ if and only if $\|g^{(n)}\|_{p,q,\beta} < \infty$, where $\beta = \alpha + n - \nu > 0$. The same equivalence holds for $\mathcal{L}g$. So it remains to prove that $\|(\mathcal{L}g)^{(n)}\|_{p,q,\beta} < \infty$.

Let $r_k = 1 - \frac{1}{2^k}$, $k \geq 0$. Since

$$(\mathcal{L}g)^{(n)}(z) = \int_0^1 (1-t)^n g^{(n)}(t + (1-t)z) dt,$$

we have that

$$\begin{aligned}
\int_0^{2\pi} \left|(\mathcal{L}g)^{(n)}(re^{i\theta})\right|^p d\theta &\leq \int_0^{2\pi} d\theta \left(\int_0^1 (1-t)^n |g^{(n)}(t + (1-t)re^{i\theta})| dt\right)^p \\
&= \int_0^{2\pi} d\theta \left(\sum_{k=0}^{\infty} \int_{r_k}^{r_{k+1}} (1-t)^n |g^{(n)}(t + (1-t)re^{i\theta})| dt\right)^p \\
&\leq C \sum_{k=0}^{\infty} 2^{-knp-kp} \int_0^{2\pi} |g^{(n)}(t_k + (1-t_k)re^{i\theta})|^p d\theta,
\end{aligned} \tag{3.4}$$

where $t_k \in [r_k, r_{k+1}]$. By using Lemma 3.4 we find that

$$\int_0^{2\pi} |g^{(n)}(t_k + (1-t_k)re^{i\theta})|^p d\theta \leq \frac{2t_k + (1-t_k)r}{(1-t_k)r} \int_0^{2\pi} |g^{(n)}((t_k + (1-t_k)r)e^{i\theta})|^p d\theta. \tag{3.5}$$

By using (3.4) and (3.5) we find

$$rM_p^p(r, (\mathcal{L}g)^{(n)}) \leq C \int_0^1 (1-t)^{np+p-2} M_p^p(t + (1-t)r, g^{(n)}) dt. \tag{3.6}$$

Substituting $t + (1-t)r = s$, we obtain

$$rM_p^p(r, (\mathcal{L}g)^{(n)}) \leq C(1-r)^{-np-p+1} \int_r^1 (1-s)^{np+p-2} M_p^p(s, g^{(n)}) ds. \tag{3.7}$$

Let now $0 < q \leq p < 1$. By using Sublemma 3.5 and (3.7) we obtain

$$\begin{aligned}
\|(\mathcal{L}g)^{(n)}\|_{p,q,\beta}^q &\asymp \int_0^1 r^{\frac{q}{p}} (1-r)^{q\beta-1} M_p^q(r, (\mathcal{L}g)^{(n)}) dr \\
&\leq C \int_0^1 (1-r)^{q\beta-1-nq-q+\frac{q}{p}} \left(\int_r^1 (1-s)^{np+p-2} M_p^p(s, g^{(n)}) ds\right)^{\frac{q}{p}} dr \\
&\leq C \int_0^1 (1-r)^{q\beta-1-nq-q+\frac{q}{p}} \int_r^1 (1-s)^{nq+q-\frac{q}{p}-1} M_p^q(s, g^{(n)}) ds dr \\
&= C \int_0^1 (1-s)^{nq+q-\frac{q}{p}-1} M_p^q(s, g^{(n)}) ds \int_0^s (1-r)^{q\beta-1-nq-q+\frac{q}{p}} dr \\
&\leq C \int_0^1 (1-s)^{q\beta-1} M_p^q(s, g^{(n)}) ds \\
&< \infty.
\end{aligned} \tag{3.8}$$

Here we used Fubini's theorem and the fact that $\nu > \alpha + \frac{1}{p} - 1$.

Case $p \le q < \infty$. Now we use (3.7) and Sublemma 3.6. We choose $\varepsilon > 0$ so that $\nu > \alpha + \frac{1}{p} - 1 + \frac{\varepsilon}{p}$ and we find that

$$\begin{aligned} & \int_0^1 r^{\frac{q}{p}}(1-r)^{q\beta-1}M_p^q(r,(\mathcal{L}g)^{(n)})dr \\ \le\; & C\int_0^1(1-r)^{q\beta-1-nq-q+\frac{q}{p}}\Big(\int_r^1(1-s)^{np+p-2}M_p^p(s,g^{(n)})ds\Big)^{\frac{q}{p}}dr \\ \le\; & C\int_0^1(1-r)^{q\beta-1-nq-q+\frac{q}{p}+\frac{\varepsilon q}{p}}dr\int_r^1(1-s)^{(1+np+p-2-\varepsilon)\frac{q}{p}-1}M_p^q(s,g^{(n)})ds \\ =\; & C\int_0^1(1-s)^{nq+q-\frac{q}{p}-\frac{\varepsilon q}{p}-1}M_p^q(s,g^{(n)})ds\int_0^s(1-r)^{q\beta-1-nq-q+\frac{q}{p}+\frac{\varepsilon q}{p}}dr \\ \le\; & C\int_0^1(1-s)^{q\beta-1}M_p^q(s,g^{(n)})ds \\ <\; & \infty. \end{aligned} \tag{3.9}$$

Let $q = \infty$. If $\sup_{0\le r<1}(1-r)^{\beta p}M_p^p(r,g^{(n)}) < \infty$, then by using (3.7) we find that

$$\begin{aligned} & \sup_{0\le r<1}(1-r)^{\beta p}M_p^p(r,(\mathcal{L}g)^{(n)}) \\ \le\; & C\sup_{0\le r<1}(1-r)^{\beta p-np-p+1}\int_r^1(1-s)^{np+p-2}M_p^p(s,g^{(n)})ds \\ \le\; & C\sup_{0\le r<1}(1-r)^{\beta p-np-p+1}\int_r^1(1-s)^{-\beta p+np+p-2}ds \\ \le\; & C. \end{aligned} \tag{3.10}$$

The implication (b) $\Rightarrow$ (a), for $0 < p < 1$, follows from (3.8), (3.9) and (3.10).

To prove that (a) $\Rightarrow$ (b) we need the following two propositions. See [7].

**Proposition 3.7.** *Let $0 < p \le \infty$.*
(a) *If $\kappa_{p,\alpha,\nu} = \nu - \alpha - \frac{1}{p} + 1 \le 0$ then $\overline{\mathcal{L}}$ cannot be extended from $h_\nu^{p,\infty,\alpha}$ to $\mathcal{H}(\mathbb{D})$, and consequently cannot be extended from $H_\nu^{p,\infty,\alpha}$ to $\mathcal{H}(\mathbb{D})$.*
(b) *If $\kappa_{p.\alpha,\nu} < 0$ and $0 < q < \infty$, then $\overline{\mathcal{L}}$ cannot be extended from $H_\nu^{p,q,\alpha}$ to $\mathcal{H}(\mathbb{D})$.*

*Proof.* (a) Let $f_\rho(z) = \frac{1}{1-\rho z}$ and let $n$ be a positive integer such that $n > \alpha + \frac{1}{p} - 1$.

If $0 < p < \infty$, then we have

$$M_p(r, f_\rho^{(n)}) \le C\left(\int_0^{2\pi}\frac{d\theta}{|1-r\rho e^{i\theta}|^{(n+1)p}}\right)^{\frac{1}{p}} \le \frac{C}{(1-r\rho)^{n+1-\frac{1}{p}}}.$$

Hence,

$$\sup_{0\le r<1}(1-r)^{\alpha+n-\nu}M_p(r,f_\rho^{(n)}) \le C\sup_{0\le r<1}\frac{(1-r)^{\alpha+n-\nu}}{(1-r\rho)^{n+1-\frac{1}{p}}} \le C,$$

since $\alpha - \nu + \frac{1}{p} - 1 \ge 0 \Leftrightarrow \kappa_{p,\alpha,\nu} = \nu - \alpha + \frac{1}{p} + 1 \le 0$.

If $p = \infty$, then

$$(1-r)^{\alpha+n-\nu}M_\infty(r,f_\rho^{(n)}) \le C\frac{(1-r)^{\alpha+n-\nu}}{(1-r\rho)^{n+1}} \le C.$$

The argument given above shows that the set $\{f_\rho : 0 < \rho < 1\}$ is bounded in $h_\nu^{p,\infty,\alpha}$. Thus if $\overline{\mathcal{L}}$ has an extension to a bounded operator from $h_\nu^{p,\infty,\alpha}$ to $\mathcal{H}(\mathbb{D})$, then the set $\{\overline{\mathcal{L}}f_\rho(0)\}$ is bounded because the functional $g \to g(0)$ is bounded on $\mathcal{H}(\mathbb{D})$. However,

$$\overline{\mathcal{L}}f_\rho(0) = \sum_{n=0}^{\infty}\frac{\rho^n}{n+1} \to \infty, \text{ as } \rho \to 1,$$

which is a contradiction.

(b) Let $\kappa_{p,\alpha,\nu} < 0$, $0 < q < \infty$. Choose $0 < \beta < \alpha$ such that $\kappa_{p,\beta,\nu} = 0$. Then we have $H_\nu^{p,\infty,\beta} \subseteq H_\nu^{p,q,\alpha}$. According to (a), that is proved, $\overline{\mathcal{L}}$ cannot be extended to a bounded operator from $H_\nu^{p,\infty,\beta}$ to $\mathcal{H}(\mathbb{D})$ and consequently neither from $H_\nu^{p,q,\alpha}$ to $\mathcal{H}(\mathbb{D})$. $\square$

**Proposition 3.8.** *If $0 < p \leq \infty$, $0 < q < \infty$ and $\kappa_{p,\alpha,\nu} = 0$, then the function*

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\log^{\varepsilon}(n+2)}, \text{ where } \max\left\{\frac{1}{q}, 1\right\} < \varepsilon \leq 1 + \frac{1}{q},$$

*belongs to $H_\nu^{p,q,\alpha}$, the function $\mathcal{L}f$ is well defined, but $\mathcal{L}f$ is not in $H_\nu^{p,q,\alpha}$.*

*Proof.* First, we show that $f \in H_\nu^{p,q,\alpha}$. By using Lemma 3.1, Lemma 3.2 and (3.3) we find that

$$\begin{aligned}
\|D^\nu f\|_{p,q,\alpha}^q &\asymp \sum_{n=0}^{\infty} 2^{n(\nu-\alpha)q} \|V_n * f\|_p^q \\
&\asymp \sum_{n=0}^{\infty} 2^{n(\nu-\alpha)q} \left(\log^{-\varepsilon}(2^{n+1})\right)^q \|V_n\|_p^q \\
&\asymp \sum_{n=0}^{\infty} 2^{n(\nu-\alpha)q} (n+1)^{-\varepsilon q} 2^{n\left(1-\frac{1}{p}\right)q} \\
&\asymp \sum_{n=0}^{\infty} (n+1)^{-\varepsilon q} \\
&< \infty,
\end{aligned}$$

because $\varepsilon q > 1$. Thus, $f \in H_\nu^{p,q,\alpha}$.

Now we show that $\mathcal{L}f \notin H_\nu^{p,q,\alpha}$. Since by Lemma 3.3 we have

$$H_\nu^{p,q,\alpha} \subseteq H_\nu^{\infty,q,\beta}, \text{ where } \beta = \alpha + \frac{1}{p},$$

it is enough to show that $\mathcal{L}f \notin H_\nu^{\infty,q,\beta}$.

Let $n$ be a positive integer such that $n > \nu$ and $\gamma = \beta + n - \nu$. Then $H_\nu^{\infty,q,\beta} = H_n^{\infty,q,\gamma}$. Now we show that $\mathcal{L}f \notin H_n^{\infty,q,\gamma}$, and consequently $\mathcal{L}f \notin H_\nu^{p,q,\alpha}$. Note that the function $\mathcal{L}f$ is well defined because $\varepsilon > 1$, which implies

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)\log^{\varepsilon}(k+2)} < \infty.$$

Note also that $\kappa_{\infty,\gamma,n} = 0$. Since the coefficients $c_k$ of $(\mathcal{L}f)^{(n)}$, are nonnegative, we see that $\mathcal{L}f$ is in $H_n^{\infty,q,\gamma}$ if and only if

$$\int_0^1 \left(\sum_{k=0}^{\infty} c_k r^k\right)^q (1-r)^{q\gamma-1} dr < \infty,$$

which is equivalent to

$$\sum_{k=0}^{\infty} 2^{-k\gamma q} \left(\sum_{j=2^k}^{2^{k+1}-1} c_j\right)^q < \infty.$$

Since $c_j \asymp (j+1)^n b_j$ $(j \to \infty)$, the latter is equivalent to

$$\sum_{k=0}^{\infty} 2^{-k(\gamma-n)q} \left(\sum_{j=2^k}^{2^{k+1}-1} b_j\right)^q < \infty,$$

where $b_j$ are coefficients of $\mathcal{L}f$. Since $b_j \downarrow 0$ $(j \to \infty)$, we get the equivalent condition

$$\sum_{k=0}^{\infty} 2^{-k(\gamma-n-1)q} b_{2^k}^q < \infty.$$

This condition is not satisfied because $n - \gamma + 1 = 0$ and

$$b_{2^k} = \sum_{j=2^k}^{\infty} \frac{1}{(j+1)\log^{\varepsilon}(j+2)} \asymp (k+1)^{1-\varepsilon}$$

and whence

$$\sum_{k=0}^{\infty} (k+1)^{(1-\varepsilon)q} = \infty,$$

because $(\varepsilon - 1)q \leq 1$. □

*Proof.* [Proof of implication (a) ⇒ (b) in Theorem 1.1]

It suffices to show that (a) doesn't hold in the following three cases:

$$q = \infty, \ \nu - \alpha - \frac{1}{p} + 1 \leq 0; \tag{3.11}$$

$$0 < q < \infty, \ \nu - \alpha - \frac{1}{p} + 1 < 0; \tag{3.12}$$

$$0 < q < \infty, \ \nu - \alpha - \frac{1}{p} + 1 = 0. \tag{3.13}$$

Proposition 3.7(a) shows that (a) in Theorem 1.1 doesn't hold if (3.11) is satisfied, while from Proposition 3.7(b) it follows that (a) doesn't hold if (3.12) is satisfied.

Proposition 3.8 shows that if (3.13) holds, then (a) doesn't hold. □

## 4. Proof of Theorem 1.2

*Proof.* [Proof of Theorem 1.2] It is clear that (c) ⇒ (b) ⇒ (a) and the implication (d) ⇒ (c) follows from Theorem 2.1. It remains to be proven that (a) ⇒ (d), that is, that (a) doesn't hold in the following cases:

$$0 < p \leq \infty, \ 0 < q \leq \infty, \ \kappa_{p,\alpha,\nu} < 0; \tag{4.1}$$

$$0 < p \leq \infty, \ 1 < q \leq \infty, \ \kappa_{p,\alpha,\nu} = 0; \tag{4.2}$$

$$2 < p \leq \infty, \ 0 < q \leq 1, \ \kappa_{p,\alpha,\nu} = 0. \tag{4.3}$$

Proposition 3.7 shows that (a) doesn't hold if (4.1) holds.

Case $0 < p \leq \infty$, $1 < q \leq \infty$, $\kappa_{p,\alpha,\nu} = 0$.

In view of the Proposition 3.7 we can assume that $1 < q < \infty$.

Let

$$f_{\varepsilon,\rho}(z) = \sum_{k=0}^{\infty} \frac{\rho^k z^k}{\log^{1+\varepsilon}(k+2)}, \ 0 < \varepsilon < \frac{1}{2}, \ 0 < \rho < 1.$$

As in the proof of Proposition 3.8 we find that

$$\|D^{\nu} f_{\varepsilon,\rho}\|_{p,q,\alpha}^q \leq C_q \sum_{k=0}^{\infty} (k+1)^{-(1+\varepsilon)q} \leq C_q \sum_{k=0}^{\infty} (k+1)^{-q} < \infty,$$

where $C_q$ is independent of $\varepsilon$ and $\rho$. The function $f_{\varepsilon,\rho}$ belongs to $\mathcal{H}(\overline{\mathbb{D}})$ and the set

$$\left\{ f_{\varepsilon,\rho} : 0 < \rho < 1, 0 < \varepsilon < \frac{1}{2} \right\}$$

is bounded in $H_\nu^{p,q,\alpha}$. On the other hand

$$\mathcal{L} f_{\varepsilon,\rho}(0) = \sum_{k=0}^{\infty} \frac{\rho^k}{(k+1)\log^{1+\varepsilon}(k+2)} \to \infty, \text{ as } \rho \to 1 \text{ and } \varepsilon \to 0.$$

The result follows.

If $2 < p \leq \infty$, $0 < q \leq 1$, $\kappa_{p,\alpha,\nu} = 0$, then $\overline{\mathcal{L}}$ cannot be extended to be a bounded operator from $H_\nu^{p,q,\alpha}$ to $\mathcal{H}(\mathbb{D})$. See Proposition 32 in [7]. □

**Corollary 4.1.** *Let* $1 < p < 2$.
(a) *If* $0 < q \leq 1$, *then* $H_\nu^{p,q,\alpha} \subseteq D^1 l^1$ *if and only if* $\kappa_{p,\alpha,\nu} \geq 0$.
(b) *If* $1 < q \leq \infty$, *then* $H_\nu^{p,q,\alpha} \subseteq D^1 l^1$ *if and only if* $\kappa_{p,\alpha,\nu} > 0$.

## 5. Summary

**Theorem 5.1.** *Let* $0 < p \leq \infty$, $0 < q \leq \infty$ *and* $\alpha > 0$. *Then*
(5.1) $\mathcal{L}$ *acts as a bounded operator from* $H_\nu^{p,q,\alpha}$ *to* $H_\nu^{p,q,\alpha}$ *if and only if* $\kappa_{p,\alpha,\nu} > 0$ (Theorem 1.1).
(5.2) $\mathcal{L}$ *acts from* $H_\nu^{p,q,\alpha}$ *to* $\mathcal{H}(\mathbb{D})$, *but not to* $H_\nu^{p,q,\alpha}$ *if* $0 < p \leq 2$, $0 < q \leq 1$ *and* $\kappa_{p,\alpha,\nu} = 0$ (Theorem 1.2).
(5.3) $\mathcal{L}$ *acts from* $H_\nu^{p,q,\alpha}$ *to* $H_\nu^{p,q,\alpha}$, *but* $H_\nu^{p,q,\alpha}$ *is not a subset of* $D^1 l^1$ *if* $\kappa_{p,\alpha,\nu} > 0$ *and*
$2 < p \leq \infty$ *and* $\nu - \alpha \leq -\frac{1}{2}$, *when* $1 < q \leq \infty$
*or*
$2 < p \leq \infty$ *and* $\nu - \alpha < -\frac{1}{2}$, *when* $0 < q \leq 1$
(Theorem 2.1, (2a) and (2b), and Theorem 1.1).
(5.4) *The operator* $\overline{\mathcal{L}}$ *cannot be extended to a bounded operator from* $H_\nu^{p,q,\alpha}$ *to* $\mathcal{H}(\mathbb{D})$ *if*
(5.4a) $\kappa_{p,\alpha,\nu} < 0$ (Proposition 3.7)
*or*
(5.4b) $\kappa_{p,\alpha,\nu} = 0$, $2 < p \leq \infty$ (Theorem 1.2)
*or*
(5.4c) $\kappa_{p,\alpha,\nu} = 0$, $1 < q \leq \infty$ (Theorem 1.2).

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