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On the Number of Perfect Matchings for Some Certain Types of Bipartite Graphs

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Abstract. In this paper, we consider the relationships between the number of perfect matchings (1-factors) for some certain types of bipartite graphs and Fibonacci and Lucas numbers.

1. Introduction

The permanent of an $n \times n$ matrix $A = \left(a_{ij}\right)$ is defined by

$$\operatorname{per}(A) = \sum_{\sigma \epsilon S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n. The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive [1].

Let A be an $n \times n$ matrix, then Brualdi and Cvetkovic show that

$$\operatorname{per}(PAQ) = \operatorname{per}(A) \tag{1}$$

for all permutation matrices P and Q of order n [2]. They also show that if

$$A = \begin{pmatrix} B & 0 \\ X & C \end{pmatrix},$$

where B and C are square matrices, then

$$\operatorname{per}(A) = \operatorname{per}(B)\operatorname{per}(C) \quad [2]. \tag{2}$$

Permanents have many applications in physics, chemistry and electrical engineering. One can find the basic properties and more applications of permanents [1-5]. Some of the most important applications of permanents are via graph theory. A more difficult problem with many applications is the enumeration

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of perfect matchings of a graph. It is clearly known that bipartite graphs have an important place in graph theory. The enumeration or actual construction of perfect matching of a bipartite graph has many applications, for example, in maximal flow problems and in assignment and scheduling problems arising in operational research [1]. The number of perfect matchings of bipartite graphs also plays a significant role in organic chemistry [3].

A bipartite graph G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex in V_1 and a vertex in V_2. A perfect matching (or 1-factor) of a graph with $2n$ vertices is a spanning subgraph of G in which every vertex has degree 1. Let $A(G)$ be adjacency matrix of the bipartite graph G, and let $\mu(G)$ denote the number of perfect matchings of G. Then, one can find the following fact in [1]: $\mu(G) = \sqrt{\operatorname{per}(A(G))}$.

Let G be a bipartite graph whose vertex set V is partitioned into two subsets V_1 and V_2 such that $|V_1| = |V_2| = n$. We construct the bipartite adjacent matrix $B(G) = \left(b_{ij}\right)$ of G as following: $b_{ij} = 1$ if and only if G contains an edge from $v_i \in V_1$ to $v_j \in V_2$, and otherwise $b_{ij} = 0$. Then, the number of perfect matchings of bipartite graph G is equal to the permanent of its bipartite adjacency matrix [1].

Fibonacci and Lucas numbers belong to a large family of positive integers. They have many interesting properties and applications to almost every field of science and art. They continue to provide invaluable opportunities for exploration, and contribute handsomely to the beauty of mathematics, especially number theory [6-7].

The well-known Fibonacci sequence $\{F(n)\}_{n=0}^{\infty}$ and Lucas sequence $\{L(n)\}_{n=0}^{\infty}$ are defined by the recurrence relation

$$\begin{aligned} F(n) &= F(n-1) + F(n-2), \qquad F(0) = 0 \text{ and } F(1) = 1, \\ L(n) &= L(n-1) + L(n-2), \qquad L(0) = 2 \text{ and } L(1) = 1. \end{aligned}$$

for $n \geq 2$. These sequences are respectively named as $A000045$ and $A000032$ in [8].

The following well-known identity gives the relationship between Lucas numbers and Fibonacci numbers. For $n \geq 1$,

$$L(n) = F(n-1) + F(n+1) = F(n) + 2F(n-1). \tag{3}$$

The relationships between perfect matchings (1-factors) of bipartite graphs and Fibonacci, Lucas numbers and their generalizations have been extensively discussed by many researchers. For example, In [9], Lee et al. consider a bipartite graph $G\left(A_n = \left(a_{i,j}\right)\right)$ with bipartite adjacency matrix is the $n \times n$ tridiagonal matrix of the form

$$A_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 1 & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}, \tag{4}$$

with entries are

$$a_{i,j} = \begin{cases} 1, & \text{if } \left|j-i\right| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then they obtain the number of perfect matchings of $G(A_n)$ as

$$\operatorname{per}(A_n) = F(n+1), \tag{5}$$

where $F(n)$ is the nth Fibonacci number. They also consider a bipartite graph $G\left(\mathcal{F}_n^k\right)$ with bipartite adjacency matrix $\mathcal{F}_n^k = \left(f_{i,j}\right)$ such that $f_{i,j} = 1$ if $-1 \le j - i \le k-1$ and $f_{i,j} = 0$ otherwise, for $k \le n+1$. Then the number of perfect matchings of $G\left(\mathcal{F}_n^k\right)$ is $g^k(n+k-1)$ where $g^k(n)$ is the nth k-Fibonacci number.

In [10], Lee considers a bipartite graph $G\left(C_n = \left(c_{i,j}\right)\right)$ with bipartite adjacency matrix is the $n \times n$ matrix of the form

$$C_n = \begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}, \tag{6}$$

with entries are

$$c_{i,j} = \begin{cases} 1, & \text{if } i = 1 \text{ and } j = 1,\ j = 3, \\ 1, & \text{if } \left|j - i\right|_{2 \le i \le n} \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $n \ge 3$, they obtain the number of perfect matchings of $G(C_n)$ as

$$\operatorname{per}(C_n) = L(n-1), \tag{7}$$

where $L(n)$ is the nth Lucas number. He also considers a bipartite graph $G\left(\mathcal{L}_n^k\right)$ with bipartite adjacency matrix $\mathcal{L}_n^k = \mathcal{F}_n^k + E_{1,k+1} - \sum_{j=2}^{k} E_{1,j}$ for $n \ge 3$, where $E_{i,j}$ denotes the $n \times n$ matrix with 1 at the (i,j)-entry and zeros elsewhere. Then the number of perfect matchings of $G\left(\mathcal{L}_n^k\right)$ is $l^k(n-1)$, where $l^k(n)$ is the nth k-Lucas number.

In [11], Shiu et al. firstly define the (k,α)-sequences as: For $k \ge 2$, $n \ge 1$ and $\alpha = (a_1, a_2, \ldots a_m) \in R^m$, where R is a ring. The k-sequence $\left\{s_\alpha^k(n)\right\}$ is

$$\begin{aligned} s_\alpha^k(n) &= a_1 f^k(n+k-2) + a_2 f^k(n+k-3) + \ldots + a_m f^k(n+k-m-1) \\ &= \sum_{i=1}^{m} a_i f^k(n-1+k-i). \end{aligned}$$

The number $s_\alpha^k(n)$ is called nth (k,α)-number. Then they give the following result:

For a fixed $m \ge 1$, suppose $n, k \ge 2$ and $n \ge m$. Let $G\left(\mathcal{B}_n^k(\alpha)\right)$ a bipartite graph with bipartite adjacency matrix has the form

$$\mathcal{B}_n^k(\alpha) = \begin{pmatrix} a_1 & a_2 & \ldots & a_m & 0 & \ldots & 0 \\ 1 & & & & & & \\ 0 & & & \mathcal{F}_{n-1}^k & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{pmatrix}.$$

Then the number of perfect matching of $G\left(\mathcal{B}_n^k(\alpha)\right)$ is nth (k,α)-number $s_\alpha^k(n)$.

In [12], Kılıç et al. consider a bipartite graph $G(\mathcal{R}_n)$ with bipartite adjacency matrix $\mathcal{R}_n = \left(r_{i,j}\right)$ such that $r_{i,j} = 1$ if $-1 \le j - i \le 1$ or $i = 1$ and $r_{i,j} = 0$ otherwise. Then the number of perfect matchings of $G(\mathcal{R}_n)$

is $\sum_{i=0}^{n} F(i) = F(n+2) - 1,$ where $F(n)$ is the nth Fibonacci number. They also consider a bipartite graph $G(W_n)$ with bipartite adjacency matrix $W_n = \mathcal{R}_n + \mathcal{S}_n,$ where $\mathcal{S}_n$ denotes the $n \times n$ matrix with -1 at the $(1,2)$-entry, 1 at the $(2,4)$-entry and zeros elsewhere. Then for $n \geq 4,$ the number of perfect matchings of $G(W_n)$ is $\sum_{i=0}^{n-2} L(i) = L(n) - 1,$ where $L(n)$ is the nth Lucas number.

One can find more applications related with the number of perfect matchings of bipartite graphs and the well-known integer sequences [13-20].

In this paper, we consider the relationships between the number of perfect matchings for some certain types of bipartite graphs and Fibonacci and Lucas numbers.

Let us give the following lemma that will be needed later.

Lemma 1.1. *[14] Let $\{T_n,\ n = 1,2,\ldots\}$ be sequence of tridiagonal matrices of type $n \times n$ in the following form*

$$T_n = \begin{pmatrix} t_{1,1} & t_{1,2} & 0 & \cdots & \cdots & 0 \\ t_{2,1} & t_{2,2} & t_{2,3} & \ddots & & \vdots \\ 0 & t_{3,2} & t_{3,3} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & t_{n-1,n} \\ 0 & \cdots & \cdots & 0 & t_{n,n-1} & t_{n,n} \end{pmatrix}.$$

Then the succesive permanents of T_n are given by the recursive formula:

$$\begin{aligned} \operatorname{per}(T_1) &= t_{1,1}, \\ \operatorname{per}(T_2) &= t_{1,1}t_{2,2} + t_{1,2}t_{2,1}, \\ \operatorname{per}(T_n) &= t_{n,n}\operatorname{per}(T_{n-1}) + t_{n-1,n}t_{n,n-1}\operatorname{per}(T_{n-2}). \end{aligned}$$

2. Main Results

In this section, we firstly consider some certain types of bipartite graphs. Then we show that the numbers of perfect matchings for these graphs are equal to the well-known integer sequences.

Lee studies on the bipartite adjacent matrix A_n in (4), where its main diagonal is in the form of $(1,1,\ldots,1)$. Then we wonder what the result would be if the main diagonal is in the form of $(1,0,1,0,\ldots,1,0,\ldots)$. We interestingly obtain the following results.

Theorem 2.1. *Let $G\left(H_n = \left(h_{i,j}\right)\right)$ $(n = 2t,\ t \in \mathbb{N})$ be a bipartite graph with bipartite adjacency matrix is the $n \times n$ tridiagonal matrix of the form*

$$H_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & 1 & \ddots & & & \vdots \\ 0 & 1 & 1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & 1 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 1 & \ddots & 1 & 0 \\ \vdots & & & \ddots & \ddots & & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & \end{pmatrix}, \tag{8}$$

where

$$h_{i,j} = \begin{cases} 1, & \text{if } |j-i| = 1, \\ 1, & \text{if } i = j = 2m-1\ (m \in \mathbb{N}), \\ 0, & \text{otherwise.} \end{cases}$$

Then the number of perfect matchings of $G(H_n)$ *is* 1.

Proof. We prove the theorem by strong induction on t. The claim holds for $t = 1$ $(n = 2)$ as

$$\operatorname{per}(H_2) = \operatorname{per}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = 1.$$

Assume that the claim holds for every k, $2 \le k \le t$. That is, $\operatorname{per}(H_{2t}) = 1$. Then we must show that the claim is true for $t+1$. From Lemma 1.1 we get

$$\begin{aligned} \operatorname{per}(H_{2t+2}) &= h_{2t+2,2t+2}\operatorname{per}(H_{2t+1}) + h_{2t+2,2t+1}h_{2t+1,2t+2}\operatorname{per}(H_{2t}) \\ &= 0 \cdot \operatorname{per}(H_{2t+1}) + 1 \cdot 1 \\ &= 1. \end{aligned}$$

So, the proof is completed. □

Theorem 2.2. *Let* $G(H_n)$ $(n = 2t+1,\ t \in \mathbb{N})$ *be a bipartite graph whose with bipartite adjacency matrix* H_n *given by (8). Then the number of perfect matchings of* $G(H_n)$ *is* $\frac{n+1}{2}$.

Proof. We prove the theorem by strong induction on t. The claim holds for $t = 1$ $(n = 3)$ as

$$\operatorname{per}(H_3) = \operatorname{per}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 2.$$

Assume that the claim holds for every k, $2 \le k \le t$. That is, $\operatorname{per}(H_{2t+1}) = t+1$. Then we must show that the claim is true for $t+1$. From Lemma 1.1 we get

$$\operatorname{per}(H_{2t+3}) = h_{2t+3,2t+3}\operatorname{per}(H_{2t+2}) + h_{2t+3,2t+2}h_{2t+2,2t+3}\operatorname{per}(H_{2t+1}) \tag{9}$$

Since $\operatorname{per}(H_{2t+2}) = 1$ from Theorem 2.1 and $\operatorname{per}(H_{2t+1}) = t+1$, we get (9) as

$$\begin{aligned} \operatorname{per}(H_{2t+3}) &= 1 \cdot 1 + 1 \cdot (t+1) \\ &= t+2, \end{aligned}$$

which is desired. □

Considering the main diagonal of the matrix A_n is $(1,0,1,0,1,\dots,1)$, we obtain a different result. Let us give it.

Theorem 2.3. *Let* $G\left(K_n = \left(k_{i,j}\right)\right)$ *be a bipartite graph with bipartite adjacency matrix is the* $n \times n$ *tridiagonal matrix of the form*

$$K_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & 1 & \ddots & & & \vdots \\ 0 & 1 & 1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & 1 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 1 & 1 & 1 & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix},$$

where $k_{2,2} = k_{4,4} = 0$, *all other terms on the main diagonal are* 1, *all terms on the subdiagonal and superdiagonal are* 1 *and otherwise* $k_{i,j} = 0$. *Then for* $n \ge 3$, *the number of perfect matchings of* $G(K_n)$ *is the* $(n-3)$*rd Lucas number* $L(n-3)$.

Proof. By applying the Laplace expansion for permanent according to 4th row of K_n, we get

$$\operatorname{per}(K_n) = \operatorname{per}\begin{pmatrix} X_3 & \mathbf{0}_{3\times(n-1)} \\ * & A_{n-4} \end{pmatrix} + \operatorname{per}\begin{pmatrix} Y_4 & * \\ \mathbf{0}_{(n-5)\times 4} & A_{n-5} \end{pmatrix} \tag{10}$$

where X_3, Y_4 are respectively the matrices of order of 3 and 4 as

$$X_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \; Y_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$\mathbf{0}_{m\times n}$ is the $m \times n$ null matrix and A_n is the matrix given by (4). By using (2), we can write (10) as

$$\operatorname{per}(K_n) = \operatorname{per}(X_3)\operatorname{per}(A_{n-4}) + \operatorname{per}(Y_4)\operatorname{per}(A_{n-5}). \tag{11}$$

Taking into account $\operatorname{per}(X_3) = 1$, $\operatorname{per}(Y_4) = 2$ and (5), we get (11) as

$$\operatorname{per}(K_n) = F(n-3) + 2F(n-4).$$

The result follows by using (3). □

Next, we study bipartite graphs whose adjacency matrices have anti-diagonal forms.

Theorem 2.4. *Let $G\left(B_n = \left(b_{i,j}\right)\right)$ be a bipartite graph with bipartite adjacency matrix is $n \times n$ anti-tridiagonal matrix*

$$B_n = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 & 1 \\ \vdots & & \iddots & 1 & 1 & 1 \\ \vdots & \iddots & \iddots & \iddots & 1 & 0 \\ 0 & \iddots & \iddots & \iddots & \iddots & \vdots \\ 1 & 1 & \iddots & \iddots & & \vdots \\ 1 & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{12}$$

where

$$b_{i,j} = \begin{cases} 1, & \text{if } \left|i + j \bmod (n+1)\right| \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $n \ge 2$, the number of perfect matchings of $G(B_n)$ is the $(n+1)$st Fibonacci number $F(n+1)$.

Proof. Let J_n be the $n \times n$ backward identity matrix (see e.g. [21])

$$J_n = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 0 & 1 \\ \vdots & & \iddots & 0 & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & 0 & 0 \\ 0 & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & \iddots & & \vdots \\ 1 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{13}$$

With the help of the matrix J_n, we can write

$$B_n = J_n A_n \tag{14}$$

where A_n is the matrix given by (4). Taking into account per$(J_n) = 1$ and (1), we get (14) as

$$\text{per}(B_n) = \text{per}(A_n).$$

The result now follows by taking into account (5). □

Theorem 2.5. *Let $G\left(D_n = \left(d_{i,j}\right)\right)$ be a bipartite graph with bipartite adjacency matrix is*

$$D_n = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & 1 \\ \vdots & & 0 & 1 & 1 & 1 \\ \vdots & \iddots & \iddots & \iddots & 1 & 0 \\ 0 & \iddots & \iddots & \iddots & \iddots & \vdots \\ 1 & 1 & 1 & \iddots & & \vdots \\ 1 & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

where

$$d_{i,j} = \begin{cases} 1, & \text{if } i = 1 \text{ and } j = n-2,\ j = n, \\ 1, & \text{if } \left| i + j \bmod (n+1) \right|_{2 \le i \le n} \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $n \geq 3$, the number of perfect matchings of $G(D_n)$ is the $(n-1)$st Lucas number $L(n-1)$.

Proof. With the help of the matrix J_n in (13), we have

$$D_n = C_n J_n,$$

where C_n is the matrix given by (6). Taking into account per$(J_n) = 1$ and (1), we get the last equation as

$$\text{per}(D_n) = \text{per}(C_n).$$

The result follows by taking into account (7). □

Theorem 2.6. *Let $G\left(U_n = \left(u_{i,j}\right)\right)$ be a bipartite graph with bipartite adjacency matrix is*

$$U_n = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \\ \vdots & & \iddots & 1 & 1 & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & 1 & \iddots & \\ 0 & 1 & \iddots & \iddots & \iddots & & \vdots \\ 1 & 1 & 1 & \iddots & & & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \tag{15}$$

where

$$u_{i,j} = \begin{cases} 1, & \text{if } i = 1 \text{ and } 1 \le j \le n, \\ 1, & \text{if } \left| i + j \bmod (n+1) \right|_{2 \le i \le n} \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $n \geq 2$, the number of perfect matchings of $G(U_n)$ is $F(n+2) - 1$.

Proof. By applying the Laplace expansion for permanent according to the last column of U_n, we get

$$\operatorname{per}(U_n) = \operatorname{per}(B_{n-1}) + \operatorname{per}(U_{n-1}),$$

where is B_n is the matrix given by (12). By using Theorem 2.4 we get the last equation as

$$\operatorname{per}(U_n) - \operatorname{per}(U_{n-1}) = F(n). \tag{16}$$

The result follows by solving the difference equation (16). □

Theorem 2.7. *Let $G\left(V_n = \left(v_{i,j}\right)\right)$ be a bipartite graph with bipartite adjacency matrix is*

$$V_n = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 & 1 \\ \vdots & & \iddots & 0 & 1 & 1 & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & 1 & 1 & 0 & 0 \\ 0 & \iddots & \iddots & \iddots & 1 & \iddots & \iddots & \vdots \\ 0 & 1 & 1 & \iddots & \iddots & \iddots & & \vdots \\ 1 & 1 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

where

$$v_{i,j} = \begin{cases} 1, & \text{if } i = 1 \text{ and } 1 \le j \le n-2,\ j = n, \\ 1, & \text{if } i = 2 \text{ and } \ j = n-3, \\ 1, & \text{if } \left|i + j \bmod (n+1)\right|_{2\le i\le n} \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $n \ge 3$, the number of perfect matchings of $G(V_n)$ is $L(n) - 1$.

Proof. By applying the Laplace expansion for permanent according to penultimate column of V_n, we get

$$\operatorname{per}(V_n) = \operatorname{per}(B_{n-2}) + \operatorname{per}(U_{n-1}),$$

where B_n and U_{n-1} are respectively the matrices given by (12) and (15). Taking into account Theorem 2.4 and Theorem 2.6, we get

$$\operatorname{per}(V_n) = F(n-1) + F(n+1) - 1$$

The result follows by using (3). □

Rimas considers a pentadiagonal matrix $P_n = \left(p_{i,j}\right)$ (This matrix is $D_n(\alpha)$ for $\alpha = 1$ in [22] and [23]) as

$$P_n = \begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & 1 & \ddots & & \vdots \\ 1 & 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 \\ \vdots & & \ddots & 1 & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 1 \end{pmatrix}. \tag{17}$$

with the entries are

$$p_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 1, & \text{if } \left|j - i\right| = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, he gives a connection between the determinant of P_n and the determinant of A_n given by (4) by using Laplace expansion as

$$\det(P_n) = \begin{cases} \det\left(A_{\frac{n-1}{2}}\right)\det\left(A_{\frac{n+1}{2}}\right), & \text{if } n \text{ is odd,} \\ \left(\det\left(A_{\frac{n}{2}}\right)\right)^2, & \text{if } n \text{ is even} \end{cases}$$

[22, 23].

By the similar way, we have the following result for the permanent of P_n with the help of Laplace expansion for permanent.

Corollary 2.8. *Let P_n be the $n \times n$ pentadiagonal matrix given by (17). Then,*

$$\operatorname{per}(P_n) = \begin{cases} \operatorname{per}\left(A_{\frac{n-1}{2}}\right)\operatorname{per}\left(A_{\frac{n+1}{2}}\right), & \text{if } n \text{ is odd,} \\ \left(\operatorname{per}\left(A_{\frac{n}{2}}\right)\right)^2, & \text{if } n \text{ is even,} \end{cases}$$

where A_n is given by (4).

Now we can present the following theorem related to bipartite graph with bipartite adjacency matrix P_n given by (17).

Theorem 2.9. *Let $G(P_n)$ be a bipartite graph with bipartite adjacency matrix is P_n given by (17). Then the number of perfect matchings of $G(P_n)$ is*

$$\operatorname{per}(P_n) = \begin{cases} F\left(\frac{n+1}{2}\right)F\left(\frac{n+3}{2}\right), & \text{if } n \text{ is odd,} \\ \left(F\left(\frac{n}{2}+1\right)\right)^2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. The proof is obtained by taking into account (5) and Corollary 2.8. □

3. Conclusion

The results show that there is a strong connection between graph theory and number theory such that the bipartite graphs studied in the manuscript whose the numbers of perfect matchings correspond to some of the well-known integer sequences.

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