Filomat 31:15 (2017), 4845–4856
https://doi.org/10.2298/FIL1715845F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Asymptotic for LS Estimators in the EV Regression Model for Dependent Errors

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Abstract. We study consistency and asymptotic normality of LS estimators in the EV (errors in variables) regression model under weak dependent errors that involve a wide range of linear and nonlinear time series. In our investigations we use a functional dependence measure of Wu [16]. Our results without mixing conditions complete the known asymptotic results for independent and dependent data obtained by Miao et al. [7]-[10].

1. Introduction

We consider the following simple linear errors in variables (EV) model:

$$\eta_i = \theta + \beta x_i + \varepsilon_i, \quad \xi_i = x_i + \delta_i, \quad 1 \leq i \leq n, \tag{1}$$

where θ, β are unknown parameters, x_i are nonrandom design points, $(\varepsilon_1, \delta_1), (\varepsilon_2, \delta_2), \ldots$ are random errors and η_i, ξ_i, $i = 1, 2, \ldots,$ are observable random variables. From (1), we have

$$\eta_i = \theta + \beta \xi_i + \nu_i, \quad \nu_i = \varepsilon_i - \beta \delta_i, \quad 1 \leq i \leq n.$$

Then, we get the least squares (LS) estimators of θ, β as

$$\hat{\beta}_n = \frac{\sum_{i=1}^n (\xi_i - \bar{\xi}_n)(\eta_i - \bar{\eta}_n)}{\sum_{i=1}^n (\xi_i - \bar{\xi}_n)^2}, \tag{2}$$

$$\hat{\theta}_n = \bar{\eta}_n - \hat{\beta}_n \bar{\xi}_n, \tag{3}$$

where $\bar{\xi}_n = n^{-1} \sum_{i=1}^n \xi_i$, $\bar{\eta}_n = n^{-1} \sum_{i=1}^n \eta_i$.

Model (1) was proposed by Deaton [1] to correct for the effects of sampling error and is more practical than the ordinary regression model. In the case that the errors are sequences of independent random variables, Liu and Chen [4] gave the consistency of the LS estimators for the linear EV regression model. Miao et al. [7] and Miao and Young [8] gave the central limit theorem and the law of iterated logarithm

2010 *Mathematics Subject Classification*. Primary 62F12; Secondary 60F05
Keywords. EV model, LS estimator, dependent errors
Received: 07 July 2016; Accepted: 30 November 2016
Communicated by Miroslav Ristić
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for LS estimators in model (1). Miao et al. [9] obtained the consistency and asymptotic normality in model (1) under some mild assumptions. Recently, several authors have been dealing with asymptotic properties of LS estimators in simple linear EV regression when the errors are dependent. For example, Fan et al. [2] established the strong consistency and asymptotic normality of LS estimators when the errors form a stationary α-mixing sequence of random variables. Miao et al. [9] established similar results when the errors are m-dependent, martingale differences, ϕ-mixing, ρ-mixing or α-mixing. Miao et al. [10] proved the strong and weak consistency of LS estimators when the errors are martingale differences.

Basic Assumptions

Now, we present our assumptions of the dependence model of errors and the conditions for design (x_i).

For a r.v. X, denote $\|X\|_Q = \mathbb{E}^{1/Q}|X|^Q$; moreover, $S_n \sim b_n$ means that $S_n/b_n \to 1$ and $S_n >> b_n$ means that $S_n/b_n \to \infty$ as $n \to \infty$. We formulate the following assumptions.

(A1) S_n is a real sequence such that $S_n \sim \sum_{i=1}^n (x_i - \bar{x}_n)^2$ and

$$\max_{1\le i\le n} \frac{|x_i - \bar{x}_n|}{\sqrt{S_n}} = O(n^{-\gamma}) \tag{4}$$

for some $\gamma > 1/2 - 1/p$,

$$S_n >> n^{2-2/p} \tag{5}$$

for some $p > 2$.

(A2) $\varepsilon_i = g(\tilde{\varepsilon}_i, \tilde{\varepsilon}_{i-1}, \ldots)$ for some measurable function g, where $(\tilde{\varepsilon}_i)_{i\in\mathbb{Z}}$ is a sequence of zero mean i.i.d. random variables such that for some $Q > 1$, $\mathbb{E}|\varepsilon_1|^{2Q} < \infty$ and

$$\sum_{i=1}^{\infty} \left\|\varepsilon_i - \varepsilon_i'\right\|_Q < \infty, \tag{6}$$

where $\varepsilon_i' = g\left(\tilde{\varepsilon}_i, \tilde{\varepsilon}_{i-1}, \ldots, \tilde{\varepsilon}_0', \tilde{\varepsilon}_{-1}, \ldots\right)$, and $\tilde{\varepsilon}_0'$ is an independent copy of $\tilde{\varepsilon}_0$.

(A3) $\delta_i = h(\check{\varepsilon}_i, \check{\varepsilon}_{i-1}, \ldots)$ for some measurable function h, where $(\check{\varepsilon}_i)_{i\in\mathbb{Z}}$ is a sequence of zero mean i.i.d. random variables such that for some $Q > 1$, $\mathbb{E}|\delta_1|^{2Q} < \infty$ and

$$\sum_{i=1}^{\infty} \left\|\delta_i - \delta_i'\right\|_Q < \infty, \tag{7}$$

where $\delta_i' = h\left(\check{\varepsilon}_i, \check{\varepsilon}_{i-1}, \ldots, \check{\varepsilon}_0', \check{\varepsilon}_{-1}, \ldots\right)$, and $\check{\varepsilon}_0'$ is an independent copy of $\check{\varepsilon}_0$.

(A4) S_n is a real sequence satisfying

$$S_n \sim \sum_{i=1}^{n} (x_i - \bar{x}_n)^2,$$

$$S_n >> n^2,$$

$$\max_{1\le i\le n} \frac{|x_i - \bar{x}_n|}{\sqrt{S_n}} = o(1).$$

(A5) Let $\varepsilon_i = g(\tilde{\varepsilon}_i, \tilde{\varepsilon}_{i-1}, \ldots)$ for some measurable function g, where $(\tilde{\varepsilon}_i)_{i\in\mathbb{Z}}$ is a sequence of zero mean i.i.d. random variables and $\delta_i = h(\check{\varepsilon}_i, \check{\varepsilon}_{i-1}, \ldots)$ for some measurable function h, where $(\check{\varepsilon}_i)_{i\in\mathbb{Z}}$ is a sequence of zero mean i.i.d. random variables such that

$$\sum_{i=1}^{\infty} i \left\|\varepsilon_i - \varepsilon_i'\right\|_2 < \infty, \tag{8}$$

$$\sum_{i=1}^{\infty} i \left\| \delta_i - \delta_i^{'} \right\|_2 < \infty, \tag{9}$$

$\varepsilon_i^{'} = g\left(\tilde{\varepsilon}_i, \tilde{\varepsilon}_{i-1}, \ldots, \tilde{\varepsilon}_0^{'}, \tilde{\varepsilon}_{-1}, \ldots\right)$, and $\tilde{\varepsilon}_0^{'}$ is an independent copy of $\tilde{\varepsilon}_0$ and $\delta_i^{'} = h\left(\check{\varepsilon}_i, \check{\varepsilon}_{i-1}, \ldots, \check{\varepsilon}_0^{'}, \check{\varepsilon}_{-1}, \ldots\right)$, and $\check{\varepsilon}_0^{'}$ is an independent copy of $\check{\varepsilon}_0$. Additionally we assume that $(\tilde{\varepsilon}_i)_{i\in\mathbb{Z}}$ and $(\check{\varepsilon}_i)_{i\in\mathbb{Z}}$ are independent.

Our new contributions to EV regression theory are the strong consistency and asymptotic normality for LS estimators in (1), when the errors satisfy weak dependence assumptions (A2)-(A3) that involve linear and nonlinear time series. This is a new concept of dependence measure which is an alternative to mixing models and can be found in Wu [10]. The proposed dependence model involves the computation of moments and it is easy verifiable. Generally, this model might to be less restrictive than strong mixing conditions and martingale differences assumptions that are hard to be verified. The main concept of this model is based on projective criterion which is presented in [11]. The processes with geometric moment contraction (bilinear process and nonlinear moving average-see Examples 2.1, 2.3 in Shao [12]) are closed connected in our dependence model. Some related ideas of this dependence model we can found in [6], [13], [15]. For further information and examples one can see in [14], [16].

In particular, using this approach, we consider linear EV regression models when the errors are a GARCH process (Example 1), iterated random functions (Example 2) and a linear process (Example 3). Our results on the strong consistency of $\hat{\beta}_n$ and $\hat{\theta}_n$ (Theorems 2.1-2.2) under weak dependence assumptions are very similar to the results of Miao et al. [10] when the errors (ε_i) and (δ_i) are martingale differences:

$$\frac{\sqrt{S_n}}{n^{\frac{1}{p}}}\left(\hat{\beta}_n - \beta\right) \to^{a.s.} 0 \tag{10}$$

for some $p > 2$ and

$$\lim_{n\to\infty} \frac{S_n}{n^{2-2/p}} = \infty,$$

and

$$n^{1-\alpha}\left(\hat{\theta}_n - \theta\right) \to^{a.s.} 0 \tag{11}$$

for some $\alpha \in (1/2, 1]$. For a general weak dependence model (A2)-(A3) for the errors (ε_i) and (δ_i), we obtain (10)-(11) under the following additional assumption:

$$\max_{1\le i\le n} \frac{|x_i - \bar{x}_n|}{\sqrt{S_n}} = O\left(n^{-\gamma}\right)$$

for some $\gamma > 0$. We obtain the asymptotic normality of $\hat{\beta}_n$ (Theorem 2.3) under the same assumptions on the normalizing sequence S_n (see (A4)) as in Miao et al. [9] (in Theorem 2.5) for independent errors. We generalise Miao's result to dependent errors (A2)-(A3). More concrete, we assume that the errors (ε_i) and (δ_i) separately satisfy weak dependence assumptions but they are independent of each other as a random sequences (see (A5)). In particular, our result covers a wide range of dependent errors given in Examples 1-3. In our proof of asymptotic normality of $\hat{\theta}_n$ (Theorem 2.4) our conditions of design (x_n): (A4) and

$$\frac{\sqrt{n}\,|\bar{x}_n|}{\sqrt{S_n}} \to 0$$

are the same as the conditions in Miao et al. [9] (in Theorem 2.6) in the case of independent errors. Below we present the discussion with more details of the assumptions (A1)-(A5).

Discussion of the conditions (A1)-(A5)

Now, we present a few examples when the errors (ε_i) and (δ_i) satisfy the weak dependence conditions (A2)-(A3) or (A5) (especially (6)-(7) or (8)-(9)). Next, we give some remarks of the design (x_i). Let $(\tilde{\varepsilon}_k)_{k\in\mathbb{Z}}$ be a zero mean i.i.d. sequence.

Example 1. Let $(\varepsilon_i)_{k\in\mathbb{Z}}$ be a GARCH(p,q) sequence given by the relations

$$\varepsilon_k = \tilde{\varepsilon}_k L_k,$$

and

$$L_k^2 = \mu + \alpha_1 L_{k-1}^2 + \ldots + \alpha_p L_{k-p}^2 + \beta_1 \varepsilon_{k-1}^2 + \ldots + \beta_q \varepsilon_{k-q}^2,$$

where $\mu, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \in \mathbb{R}$ are parameters. If

$$\sum_{i=1}^{\max(p,q)} \left\|\alpha_i + \beta_i \varepsilon_i^2\right\|_2 < 1,$$

then (see Jirak [3], p. 6)

$$\left\|\varepsilon_i - \varepsilon_i^{'}\right\|_Q = O\left(\rho^i\right) \tag{12}$$

for some $0 < \rho < 1$. Hence (6) holds. Similarly we can obtain (8).

Example 2. Let $(\varepsilon_i)_{k\in\mathbb{Z}}$ be defined by the recursion

$$\varepsilon_k = f\left(\varepsilon_{k-1}, \tilde{\varepsilon}_k\right)$$

for some measurable function f. If

$$\mathbb{E} \sup_{x\neq y} \frac{\left|f(x, \tilde{\varepsilon}_k) - f(y, \tilde{\varepsilon}_k)\right|}{\left|x - y\right|} < 1$$

and

$$\left\|f\left(x_0, \tilde{\varepsilon}_k\right)\right\|_Q < \infty$$

for some x_0 (see Jirak [3], p. 6), then we have (12) and (6), (8).

Example 3. Let $(\varepsilon_i)_{k\in\mathbb{Z}}$ be a linear process

$$\varepsilon_k = \sum_{r=0}^{\infty} b_r \tilde{\varepsilon}_{k-r},$$

where the coefficients $(b_r)_{r=0}^{\infty}$ are such that

$$\sum_{r=0}^{\infty} |b_r| < \infty.$$

Then it is easy to see that

$$\left\|\varepsilon_i - \varepsilon_i^{'}\right\|_Q = |b_i| \left\|\tilde{\varepsilon}_0 - \tilde{\varepsilon}_0^{'}\right\|_Q .$$

Hence, if $\mathbb{E}|\tilde{\varepsilon}_0|^Q < \infty$, then (8) holds. If we additionally assume that

$$\sum_{i=1}^{\infty} i\,|b_i| < \infty,$$

then (9) holds.

Remark 1. For nonrandom design $x_i = \frac{i}{n^{\tilde{\alpha}}}$ for $0 < \tilde{\alpha} < 1/2$ by simple calculation we have $\bar{x}_n = \frac{n+1}{2n^{\tilde{\alpha}}}$, $\sum_{i=1}^n x_i^2 = \frac{n(n+1)(2n+1)}{6n^{2\tilde{\alpha}}}$. Therefore $\sum_{i=1}^n (x_i - \bar{x}_n)^2 = \frac{n(n^2-1)}{12n^{2\tilde{\alpha}}}$ and $S_n \sim \frac{n(n^2-1)}{12n^{2\tilde{\alpha}}}$.

Observe that

$$\frac{S_n}{n^{2-2/p}} \sim \frac{n(n^2-1)}{12n^{2\tilde{\alpha}+2-2/p}} \to \infty$$

and

$$\frac{S_n}{n^2} \sim \frac{n(n^2-1)}{12n^{2\tilde{\alpha}+2}} \to \infty$$

for $0 < \tilde{\alpha} < 1/2$. Similarly

$$\max_{1\le i\le n} \frac{|x_i - \bar{x}_n|}{\sqrt{S_n}} = O\left(\max_{1\le i\le n} \frac{|i-n-1|}{\sqrt{n(n^2-1)}}\right) = O\left(n^{-1/2}\right)$$

and condition (4) in (A1) holds for $\gamma = 1/2$. Hence conditions (A1), (A4) are satisfied.

Remark 2. The strong consistency of $\hat{\beta}_n$ and $\hat{\theta}_n$ (Theorems 2.1, 2.2) we obtain when $S_n >> n^{2-2/p}$ for some $p > 2$. If $S_n >> n^2$, then we obtain asymptotic normality of $\hat{\beta}_n$ and $\hat{\theta}_n$ (Theorems 2.3, 2.4).

Remark 3. Condition (16) in Theorem 2.2 is satisfied for nonrandom design $x_i = \frac{i}{n^{\tilde{\alpha}}}$ for $\tilde{\alpha} > 0$.

Remark 4. Independence of random sequences $(\tilde{\varepsilon}_i)_{i\in\mathbb{Z}}$ and $(\check{\varepsilon}_i)_{i\in\mathbb{Z}}$in condition (A5) means that the regression errors (ε_i) and (δ_i) are independent.

The rest of the paper is organized as follows. In Section 2 we state and prove our results on strong consistency of $\hat{\beta}_n$ and $\hat{\theta}_n$ (Theorems 2.1, 2.2), and asymptotic normality (Theorems 2.3, 2.4). Some auxiliary lemmas and their proofs are given in the Appendix.

2. Main Results

In this Section we show our main results: strong consistency and asymptotic normality of estimators $\hat{\beta}_n$ and $\hat{\theta}_n$.

2.1. Strong consistency

Theorem 2.1. *Let (A1) and (A2)-(A3) for $Q > p$ for some $p > 2$ be satisfied. Then*

$$\frac{\sqrt{S_n}}{n^{\frac{1}{p}}}\left(\hat{\beta}_n - \beta\right) \to^{a.s.} 0. \tag{13}$$

Proof. By simple calculation from (2), we have

$$\hat{\beta}_n - \beta = I + II + III, \tag{14}$$

where

$$\begin{aligned} I &= \frac{\sum_{i=1}^{n}\left(\delta_i-\bar{\delta}_n\right)\varepsilon_i}{\sum_{i=1}^{n}\left(\xi_i-\bar{\xi}_n\right)^2}, \\ II &= \frac{\sum_{i=1}^{n}\left(x_i-\bar{x}_n\right)\left(\varepsilon_i-\beta\delta_i\right)}{\sum_{i=1}^{n}\left(\xi_i-\bar{\xi}_n\right)^2}, \\ III &= -\frac{\beta\sum_{i=1}^{n}\left(\delta_i-\bar{\delta}_n\right)}{\sum_{i=1}^{n}\left(\xi_i-\bar{\xi}_n\right)^2}. \end{aligned}$$

Let us observe that

$$\frac{\sqrt{S_n}}{n^{\frac{1}{p}}}I=\frac{S_n}{\sum_{i=1}^{n}\left(\xi_i-\bar{\xi}_n\right)^2}\frac{1}{\sqrt{S_n}n^{\frac{1}{p}}}\sum_{i=1}^{n}\left(\delta_i-\bar{\delta}_n\right)\varepsilon_i.$$

Since from Lemma 3.1, $\frac{S_n}{\sum_{i=1}^{n}\left(\xi_i-\bar{\xi}_n\right)^2}\rightarrow^{a.s.}1$, in order to prove $\frac{\sqrt{S_n}}{n^{\frac{1}{p}}}I\rightarrow^{a.s.}0$ it is enough to show

$$\frac{1}{\sqrt{S_n}n^{\frac{1}{p}}}\sum_{i=1}^{n}\left(\delta_i-\bar{\delta}_n\right)\varepsilon_i\rightarrow^{a.s.}0. \tag{15}$$

Observe that

$$\frac{1}{\sqrt{S_n}n^{\frac{1}{p}}}\left|\sum_{i=1}^{n}\left(\delta_i-\bar{\delta}_n\right)\varepsilon_i\right|\leq\frac{1}{2\sqrt{S_n}n^{\frac{1}{p}}}\sum_{i=1}^{n}\left(\left(\delta_i-\bar{\delta}_n\right)^2+\left(\varepsilon_i-\bar{\varepsilon}_n\right)^2\right)$$

and

$$\frac{1}{\sqrt{S_n}n^{\frac{1}{p}}}\sum_{i=1}^{n}\left(\delta_i-\bar{\delta}_n\right)^2\leq\frac{1}{\sqrt{S_n}n^{\frac{1}{p}}}\sum_{i=1}^{n}\delta_i^2,$$

$$\frac{1}{\sqrt{S_n}n^{\frac{1}{p}}}\sum_{i=1}^{n}\left(\varepsilon_i-\bar{\varepsilon}_n\right)^2\leq\frac{1}{\sqrt{S_n}n^{\frac{1}{p}}}\sum_{i=1}^{n}\varepsilon_i^2.$$

Therefore from (5) and the Ergodic Theorem for $\left(\varepsilon_i^2\right)$, $\left(\delta_i^2\right)$, we have $\frac{1}{\sqrt{S_n}n^{\frac{1}{p}}}\sum_{i=1}^{n}\delta_i^2\rightarrow^{a.s.}0$ and $\frac{1}{\sqrt{S_n}n^{\frac{1}{p}}}\sum_{i=1}^{n}\varepsilon_i^2\rightarrow^{a.s.}0$. Hence, we obtain (15), which yields

$$\frac{\sqrt{S_n}}{n^{\frac{1}{p}}}I\rightarrow^{a.s.}0.$$

Similarly, it follows that $\frac{\sqrt{S_n}}{n^{\frac{1}{p}}}III\rightarrow^{a.s.}0$. Let us write

$$\frac{\sqrt{S_n}}{n^{\frac{1}{p}}}II=-\beta\frac{S_n}{\sum_{i=1}^{n}\left(\xi_i-\bar{\xi}_n\right)^2}\frac{1}{\sqrt{S_n}n^{\frac{1}{p}}}\sum_{i=1}^{n}\left(x_i-\bar{x}_n\right)\left(\varepsilon_i-\beta\delta_i\right).$$

From Lemma 3.1, we have $\frac{S_n}{\sum_{i=1}^{n}\left(\xi_i-\bar{\xi}_n\right)^2}\rightarrow^{a.s.}1$ and setting $\omega_{i,n}=\frac{x_i-\bar{x}_n}{\sqrt{S_n}}$ in Lemma 3.2, we deduce that

$$\max_{1\leq i\leq n}\left|\omega_{i,n}\right|=O\left(n^{-\gamma}\right)$$

for $\gamma > 1/2 - 1/p$,

$$\sum_{i=1}^{n} \omega_{i,n}^2 = \mathcal{O}(1)$$

and consequently

$$\frac{1}{\sqrt{S_n} n^{\frac{1}{p}}} \sum_{i=1}^{n} (x_i - \bar{x}_n)(\varepsilon_i - \beta\delta_i) \to^{a.s.} 0.$$

Therefore we get $\frac{\sqrt{S_n}}{n^{\frac{1}{p}}} II \to^{a.s.} 0$ and taking account into (14) the proof of (13) is finished. □

Theorem 2.2. *Suppose (A1), (A2)-(A3) hold for $Q > 1/(\alpha - \frac{1}{2})$ and suppose that*

$$\frac{n^{1-\alpha+1/p}}{\sqrt{S_n}} |\bar{x}_n| = \mathcal{O}(1) \tag{16}$$

for some $\alpha \in (1/2 + 1/p, 1]$ and $p > 2$. Then

$$n^{1-\alpha}\left(\hat{\theta}_n - \theta\right) \to^{a.s.} 0.$$

Proof. Let $S_k = \sum_{i=1}^{k} \varepsilon_i$. First, observe that from Theorem 1 (iii) ([16]) for $Q > 2$, we have

$$\left\| \max_{k \le n} S_k \right\|_Q \le C_Q n^{1/2} \Theta_{0,Q},$$

where C_Q is a constant dependent on Q, $\Theta_{0,Q} = \sum_{i=0}^{\infty} \|\mathcal{P}_0 \varepsilon_i\|_Q$ and $\mathcal{P}_0 \varepsilon_i = \mathbb{E}\left(\varepsilon_i | \mathcal{F}_0^{\varepsilon}\right) - \mathbb{E}\left(\varepsilon_i | \mathcal{F}_{-1}^{\varepsilon}\right)$ with the σ-field $\mathcal{F}_i^{\varepsilon} = \sigma(\tilde{\varepsilon}_i, \tilde{\varepsilon}_{i-1}, \ldots)$. Since $\|\mathcal{P}_0 \varepsilon_i\|_Q \le \left\|\varepsilon_i - \varepsilon_i'\right\|_Q$, then from (A2) we obtain

$$\left\| \sum_{i=1}^{n} \varepsilon_i \right\|_Q \le C_Q' n^{1/2} \tag{17}$$

for some constant C_Q'. From Markov's inequality and (17) for $Q > 1/(\alpha - \frac{1}{2})$, we have, for any $r > 0$,

$$\mathbb{P}\left(\left| \sum_{i=1}^{n} \varepsilon_i \right| \ge r n^{\alpha} \right) \le \frac{\mathbb{E}\left| \sum_{i=1}^{n} \varepsilon_i \right|^Q}{r^Q n^{Q\alpha}} \le C n^{-Q(\alpha - \frac{1}{2})}.$$

Since $\sum_{n=1}^{\infty} n^{-Q(\alpha-\frac{1}{2})} < \infty$, we obtain $n^{1-\alpha}\bar{\varepsilon}_n \to^{a.s.} 0$. Similarly $n^{1-\alpha}\bar{\delta}_n \to^{a.s.} 0$. Hence, the proof is the same as in Miao et al. ([10] proof of Theorem 3.2). □

2.2. Asymptotic normality

Note that $w_{i,n} := \frac{x_i - \bar{x}_n}{\sqrt{S_n}}$ and $v_i := \varepsilon_i - \delta_i\beta$, $\varepsilon_i = g(\tilde{\varepsilon}_i, \tilde{\varepsilon}_{i-1}, \ldots)$, $\delta_i = h(\check{\varepsilon}_i, \check{\varepsilon}_{i-1}, \ldots)$ for some measurable functions g, h and a mean zero i.i.d. random sequences $(\tilde{\varepsilon}_i)_{i\in\mathbb{Z}}$, $(\check{\varepsilon}_i)_{i\in\mathbb{Z}}$. Additionally we assume that the sequences $(\tilde{\varepsilon}_i)_{i\in\mathbb{Z}}$, $(\check{\varepsilon}_i)_{i\in\mathbb{Z}}$ are independent. Let $D_k^{\varepsilon} = \sum_{i=k}^{\infty} w_{i,n} \mathcal{P}_k \varepsilon_i$, where $\mathcal{P}_k \varepsilon_i = \mathbb{E}\left(\varepsilon_i | \mathcal{F}_k^{\varepsilon}\right) - \mathbb{E}\left(\varepsilon_i | \mathcal{F}_{k-1}^{\varepsilon}\right)$ with the σ-field $\mathcal{F}_i^{\varepsilon} = \sigma(\tilde{\varepsilon}_i, \tilde{\varepsilon}_{i-1}, \ldots)$, $D_k^{\delta} = \sum_{i=k}^{\infty} w_{i,n} \mathcal{P}_k \delta_i$, where $\mathcal{P}_k \delta_i = \mathbb{E}\left(\delta_i | \mathcal{F}_k^{\delta}\right) - \mathbb{E}\left(\delta_i | \mathcal{F}_{k-1}^{\delta}\right)$ with the σ-field $\mathcal{F}_i^{\delta} = \sigma(\check{\varepsilon}_i, \check{\varepsilon}_{i-1}, \ldots)$.

Theorem 2.3. *Under (A4)-(A5), we have*

$$\sqrt{S_n}\left(\hat{\beta}_n - \beta\right) \to^d N\left(0, \sigma_w^2\right),$$

where $\sigma_w^2 = \lim_{n\to\infty} Var\left(\sum_{i=1}^n \left(D_i^\varepsilon + \beta D_i^\delta\right)\right) < \infty$.

Proof. Observe that

$$\sqrt{S_n}\left(\hat{\beta}_n - \beta\right) = \sqrt{S_n}\,(I + II + III),$$

where

$$\begin{aligned} I &= \frac{\sum_{i=1}^n (\delta_i - \bar{\delta}_n)\,\varepsilon_i}{\sum_{i=1}^n (\xi_i - \bar{\xi}_n)^2}, \\ II &= \frac{\sum_{i=1}^n (x_i - \bar{x}_n)\,\nu_i}{\sum_{i=1}^n (\xi_i - \bar{\xi}_n)^2}, \\ III &= -\frac{\beta \sum_{i=1}^n (\delta_i - \bar{\delta}_n)}{\sum_{i=1}^n (\xi_i - \bar{\xi}_n)^2}, \end{aligned}$$

with $\nu_i = \varepsilon_i - \beta\delta_i$. From (A4), it follows that $S_n / \sum_{i=1}^n (\xi_i - \bar{\xi}_n)^2 \to^{a.s.} 1$ (see Lemma 3.1), which yields

$$\sqrt{S_n} I \sim^{a.s.} \frac{\sum_{i=1}^n (\delta_i - \bar{\delta}_n)\,\varepsilon_i}{\sqrt{S_n}}.$$

Since

$$\frac{\left|\sum_{i=1}^n (\delta_i - \bar{\delta}_n)\,\varepsilon_i\right|}{\sqrt{S_n}} \le \frac{1}{\sqrt{S_n}}\left(\sum_{i=1}^n \delta_i^2 + \varepsilon_i^2\right)$$

and similarly

$$\frac{\left|\sum_{i=1}^n (\delta_i - \bar{\delta}_n)\right|}{\sqrt{S_n}} \le \frac{1}{\sqrt{S_n}}\left(\sum_{i=1}^n \delta_i^2\right).$$

By the Ergodic Theorem for (δ_i^2), (ε_i^2), we have $\frac{1}{n}\sum_{i=1}^n \left(\delta_i^2 + \varepsilon_i^2\right) \to^{a.s.} \mathbb{E}(\delta_1^2 + \varepsilon_1^2)$ and $\frac{1}{n}\sum_{i=1}^n \delta_i^2 \to^{a.s} \mathbb{E}(\delta_1^2)$. Hence,

$$\frac{\left|\sum_{i=1}^n (\delta_i - \bar{\delta}_n)\,\varepsilon_i\right|}{\sqrt{S_n}} = O\left(\frac{n}{\sqrt{S_n}}\right)$$

and

$$\frac{\left|\sum_{i=1}^n (\delta_i - \bar{\delta}_n)\right|}{\sqrt{S_n}} = O\left(\frac{n}{\sqrt{S_n}}\right).$$

Therefore by (A4), we have $\sqrt{S_n}\,(I + III) \to^{a.s.} 0$. Now, we will show $\sqrt{S_n} II \to^d N(0, \sigma_w^2)$. Under our assumptions, it is sufficient to prove

$$\sum_{i=1}^n w_{i,n}\nu_i \to^d N(0, \sigma_w^2). \tag{18}$$

Since (ε_i) and (δ_i) are independent, then if we show that

$$\sum_{i=1}^{n} w_{i,n}\varepsilon_i \to^d N(0,\sigma_1^2) \tag{19}$$

and

$$\sum_{i=1}^{n} w_{i,n}\delta_i \to^d N(0,\sigma_2^2), \tag{20}$$

where $\sigma_1^2 := \lim_{n\to\infty} Var\left(\sum_{i=1}^{n} D_i^{\varepsilon}\right) < \infty$, $\sigma_2^2 := \lim_{n\to\infty} Var\left(\sum_{i=1}^{n} D_i^{\delta}\right) < \infty$, then we will get (18).

Now, we will prove (19). Adopting the reasoning in Wu [16] we have the decomposition

$$\sum_{i=1}^{n} w_{i,n}\varepsilon_i = M_n^{\varepsilon} + R_n^{\varepsilon}, \tag{21}$$

where $M_n^{\varepsilon} = \sum_{i=1}^{n} D_i^{\varepsilon}$ and $R_n^{\varepsilon} = \sum_{i=1}^{n} w_{i,n}\varepsilon_i - M_n^{\varepsilon}$. Observe that $\left(D_i^{\varepsilon}\right)$ are martingale differences with respect to $\left(\mathcal{F}_i^{\varepsilon}\right)$, and from (A4) the weights $w_{i,n}$ satisfy conditions $\sum_{i=1}^{n} w_{i,n}^2 = O(1)$ and $\max_{1\le i\le n}\left|w_{i,n}\right| = o(1)$. Let $D_i = \sum_{j=i}^{\infty} \mathcal{P}_i\varepsilon_j$. Moreover,

$$\begin{aligned}
\mathbb{E}\max_{1\le i\le n}\left|D_i^{\varepsilon}\right| &\le \max_{1\le i\le n}\left|w_{i,n}\right| \mathbb{E}\max_{1\le i\le n}\left|D_i\right| \\
&\le \max_{1\le i\le n}\left|w_{i,n}\right| \sum_{i=1}^{n} \mathbb{E}\left|D_i\right| \\
&\le \max_{1\le i\le n}\left|w_{i,n}\right| \sum_{i=1}^{n} \left\|\sum_{j=i}^{\infty} \mathcal{P}_i v_j\right\|_2 \\
&\le \max_{1\le i\le n}\left|w_{i,n}\right| \sum_{i=1}^{n} \sum_{j=i}^{\infty} \left\|\mathcal{P}_i v_j\right\|_2 .
\end{aligned}$$

Observe that for $j \ge i$,

$$\left\|\mathcal{P}_i\varepsilon_j\right\|_2 \le \left\|\varepsilon_j - \varepsilon_j^{'}\right\|_2 . \tag{22}$$

Indeed from stationarity

$$\left\|\mathcal{P}_i\varepsilon_j\right\|_2 = \left\|\mathcal{P}_0\varepsilon_{j-i}\right\|_2$$

and

$$\mathbb{E}\left(\varepsilon_{j-i}|\mathcal{F}_{-1}^{\varepsilon}\right) = \mathbb{E}\left(\varepsilon_{j-i}^{'}|\mathcal{F}_0^{\varepsilon}\right).$$

Applying Jensen inequality, we have

$$\left\|\mathcal{P}_0\varepsilon_{j-i}\right\|_2 = \left\|\mathbb{E}\left(\varepsilon_{j-i}|\mathcal{F}_0^{\varepsilon}\right) - \mathbb{E}\left(\varepsilon_{j-i}^{'}|\mathcal{F}_0^{\varepsilon}\right)\right\|_2 \le \left\|\varepsilon_{j-i} - \varepsilon_{j-i}^{'}\right\|_2 .$$

From stationarity, we have $\left\|\varepsilon_{j-i} - \varepsilon_{j-i}^{'}\right\|_2 = \left\|\varepsilon_j - \varepsilon_j^{'}\right\|_2$. Hence we obtain (22) and

$$\begin{aligned}
\mathbb{E}\max_{1\le i\le n}\left|D_i^{\varepsilon}\right| &\le \max_{1\le i\le n}\left|w_{i,n}\right| \sum_{i=1}^{n} \sum_{j=i}^{\infty} \left\|\varepsilon_j - \varepsilon_j^{'}\right\|_2 \\
&\le \max_{1\le i\le n}\left|w_{i,n}\right| \sum_{i=1}^{\infty} i\left\|\varepsilon_i - \varepsilon_i^{'}\right\|_2 .
\end{aligned}$$

Therefore (8) and $\max_{1\le i\le n}\left|w_{i,n}\right| \to 0$ imply $\mathbb{E}\max_{1\le i\le n}\left|D_i^{\varepsilon}\right| \to 0$, and by the Central Limit Theorem for martingale differences, we have $M_n^{\varepsilon} \to^d N(0,\sigma_1^2)$. Reasoning as in Wu ([16] proof of Theorem 1 (i)) we obtain

$$\left\|R_n^{\varepsilon}\right\|_2 \le C\sqrt{\sum_{j=1}^{n} w_{j,n}^2\Theta_{j,2}^2}, \tag{23}$$

where $\Theta_{j,2} = \sum_{i=j}^{\infty}\|\mathcal{P}_0\varepsilon_i\|_2$. Since $\|\mathcal{P}_0\varepsilon_i\|_2 \le \left\|\varepsilon_i - \varepsilon_i^{'}\right\|_2$, from (23) we have

$$\begin{aligned}\left\|R_n^{\varepsilon}\right\|_2 &\le C\max_{1\le i\le n}\left|w_{i,n}\right|\sqrt{\sum_{j=1}^{n}\left(\sum_{i=j}^{\infty}\left\|\varepsilon_i-\varepsilon_i^{'}\right\|_2\right)^2}\\ &\le C\max_{1\le i\le n}\left|w_{i,n}\right|\sum_{i=1}^{\infty} i\left\|\varepsilon_i-\varepsilon_i^{'}\right\|_2. \end{aligned}\tag{24}$$

Finally, (8) and $\max_{1\le i\le n}\left|w_{i,n}\right| = o(1)$ imply $\left\|R_n^{\varepsilon}\right\|_2 \to 0$, which completes the proof of (19). Similarly, we have the decomposition

$$\sum_{i=1}^{n} w_{i,n}\delta_i = M_n^{\delta} + R_n^{\delta}, \tag{25}$$

where $M_n^{\delta} = \sum_{i=1}^{n} D_i^{\delta}$ and $R_n^{\delta} = \sum_{i=1}^{n} w_{i,n}\delta_i - M_n^{\delta}$ and we can obtain (20). □

Theorem 2.4. *Under (A4)-(A5) and*

$$\frac{\sqrt{n}\,|\bar{x}_n|}{\sqrt{S_n}} \to 0, \tag{26}$$

we have

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \to^d N\left(0,\sigma^2\right),$$

where $\sigma^2 = \left\|D_1^{\varepsilon} + \beta D_1^{\delta}\right\|_2^2$.

Proof. We have the decomposition

$$\begin{aligned}\sqrt{n}\left(\hat{\theta}_n - \theta\right) &= \sqrt{n}\bar{x}_n\left(\beta-\hat{\beta}_n\right) + \sqrt{n}\bar{\delta}_n\left(\beta-\hat{\beta}_n\right) + \sqrt{n}\left(\bar{\varepsilon}_n - \bar{\delta}_n\beta\right)\\ &= \sqrt{n}\left(\beta-\hat{\beta}_n\right)\left(\bar{x}_n+\bar{\delta}_n\right) + \sqrt{n}\left(\bar{\varepsilon}_n - \bar{\delta}_n\beta\right).\end{aligned}$$

Observe that

$$\sqrt{n}\left(\hat{\beta}_n-\beta\right)\left(\bar{x}_n+\bar{\delta}_n\right) = \frac{\sqrt{n}}{\sqrt{S_n}}\sqrt{S_n}\left(\hat{\beta}_n-\beta\right)\left(\bar{x}_n+\bar{\delta}_n\right)$$

and from the assumption (26) and the asymptotic normality of $\hat{\beta}_n$ (see Theorem 2.3), we have $\sqrt{S_n}\left(\hat{\beta}_n-\beta\right) = O_P(1)$ and $\frac{\sqrt{n}}{\sqrt{S_n}}\left(\bar{x}_n+\bar{\delta}_n\right) \to^{\mathbb{P}} 0$. Hence $\sqrt{n}\left(\hat{\beta}_n-\beta\right)\left(\bar{x}_n+\bar{\delta}_n\right) \to^{\mathbb{P}} 0$.
Since $\lim_{n\to\infty} Var\left(\frac{1}{n}\sum_{i=1}^{n}\left(D_i^{\varepsilon}+\beta D_i^{\delta}\right)\right) = \left\|D_1^{\varepsilon}+\beta D_1^{\delta}\right\|_2^2$, reasoning as in the proof of Theorem 2.3 setting $w_{i,n} = \frac{1}{\sqrt{n}}$ we obtain $\sqrt{n}\bar{v}_n \to^d N\left(0,\sigma^2\right)$ for $\sigma^2 = \left\|D_1^{\varepsilon}+\beta D_1^{\delta}\right\|_2^2$, which completes the proof. □

3. Appendix

In this section, we establish some auxiliary lemmas for the proofs of our main results.

Lemma 3.1. *Under (5), we have*

$$S_n / \sum_{i=1}^{n} (\xi_i - \bar{\xi}_n)^2 \to^{a.s.} 1 \tag{27}$$

as $n \to \infty$.

Proof. Observe that (for details see Miao et al. [10], (23))

$$\left| \frac{1}{S_n} \sum_{i=1}^{n} (\xi_i - \bar{\xi}_n)^2 - 1 \right| \le 2 \sqrt{\frac{1}{S_n} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2} + \frac{1}{S_n} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 \tag{28}$$

and

$$\frac{1}{\sqrt{S_n} n^{1/p}} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 \le \frac{1}{\sqrt{S_n} n^{1/p}} \sum_{i=1}^{n} \delta_i^2. \tag{29}$$

From (5) and the Ergodic Theorem for $\left(\delta_i^2\right)$, we have $\frac{1}{\sqrt{S_n} n^{1/p}} \sum_{i=1}^{n} \delta_i^2 \to^{a.s.} 0$, then from (5), (29) we have $\frac{n^{1/p}}{\sqrt{S_n}} \le \frac{n^{1-1/p}}{\sqrt{S_n}}$ and $\frac{n^{1/p}}{\sqrt{S_n}} \to 0$. Therefore, we have

$$\frac{1}{S_n} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 = \frac{n^{1/p}}{\sqrt{S_n}} \frac{1}{\sqrt{S_n} n^{1/p}} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 \to^{a.s.} 0.$$

Hence from (28), we obtain (27). □

Lemma 3.2. *Let assumptions (A2)-(A3) be satisfied for $Q > p$ for some $p > 2$, $(\omega_{i,n})_{i=1}^n$ be a sequence of real weights such that $\max_{1 \le i \le n} \left|\omega_{i,n}\right| = O\left(n^{-\gamma}\right)$ for some $\gamma > 1/2 - 1/p$ and $\sum_{i=1}^{n} \omega_{i,n}^2 = O(1)$. Moreover we assume that (6)-(9) hold. Then*

$$\frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} \omega_{i,n} \varepsilon_i \to^{a.s.} 0 \tag{30}$$

and

$$\frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} \omega_{i,n} \delta_i \to^{a.s.} 0. \tag{31}$$

Proof. From decompositions (21), (25), we have

$$\frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} \omega_{i,n} \varepsilon_i = \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} D_i^{\varepsilon} + \frac{1}{n^{\frac{1}{p}}} R_n^{\varepsilon} \tag{32}$$

and

$$\frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} \omega_{i,n} \delta_i = \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n} D_i^{\delta} + \frac{1}{n^{\frac{1}{p}}} R_n^{\delta}. \tag{33}$$

Since $\sum_{i=1}^{n} D_i^{\varepsilon}$ and $\sum_{i=1}^{n} D_i^{\delta}$ are the sums of the weighted maringale differences, then from Lemma 2.4 in [10], we get

$$\frac{1}{n^{\frac{1}{p}}}\sum_{i=1}^{n} D_i^{\varepsilon} \to^{a.s.} 0 \text{ and } \frac{1}{n^{\frac{1}{p}}}\sum_{i=1}^{n} D_i^{\delta} \to^{a.s.} 0.$$

Using Markov's inequality for any $r > 0$ we get

$$\mathbb{P}\left(\left|R_n^{\varepsilon}\right| \geq rn^{\frac{1}{p}}\right) \leq \frac{\mathbb{E}\left(R_n^{\varepsilon}\right)^2}{r^2 n^{\frac{2}{p}}}.$$

By (8) and (24) we have $\mathbb{E}\left(R_n^{\varepsilon}\right)^2 = O\left((\max_{1\leq i\leq n}|\omega_i|)^2\right) = O\left(n^{-2\gamma}\right)$. Therefore

$$\mathbb{P}\left(\left|R_n^{\varepsilon}\right| \geq rn^{\frac{1}{p}}\right) = O\left(\frac{1}{n^{\frac{2}{p}+2\gamma}}\right).$$

Since $\gamma > 1/2 - 1/p$, then $\frac{2}{p} + 2\gamma > 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{p}+2\gamma}} < \infty$, which implies $\frac{1}{n^{\frac{1}{p}}} R_n^{\varepsilon} \to^{a.s.} 0$ as $n \to \infty$. By similar arguments we have $\frac{1}{n^{\frac{1}{p}}} R_n^{\delta} \to^{a.s.} 0$ as $n \to \infty$. Hence, from (32)-(33) we obtain (30)-(31). □

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