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On Some Exact Distributions in Ranked Set Sampling

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Abstract. In this article we obtain the exact joint distribution of a ranked set sample. We show that this distribution belongs to the family of unified multivariate skew normal distributions. We also investigate a multivariate skew-t distribution 62J12 using ranked set samples as an application of our results. A numerical example is also provided to illustrate our results.

1. Preliminary

The theory of ranked set sampling (RSS) has been studied extensively in recent decades and is applicable when the observations are easier ranked than measured. The concept of ranked set sampling was first suggested by McIntyre (1952) who noted that it is much superior to the standard simple random sampling (SRS) for the estimation of the population mean.

Suppose m random samples with m units in each sample are selected from a normal population with mean μ and variance σ^2 and the following sets of m items are obtained.

$X_{11},\ X_{12}, ..., X_{1m}$
$X_{21},\ X_{22}, ..., X_{2m}$
$\vdots$
$X_{m1},\ X_{m2}, ...,\ X_{mm}.$

McIntyre's concept of RSS depends on measuring the first ordered unit from the first set, the second ordered unit from the second set and so on, until we reach the maximum unit from the last set. In other words, the items in the first set $X_{11},\ X_{12}, ..., X_{1m}$ are ranked by judgment and smallest is quantified. Then the items in the set $X_{21},\ X_{22}, ..., X_{2m}$ are ranked by judgment and the second smallest is quantified. The procedure is repeated until in the last set $X_{m1},\ X_{m2}, ...,\ X_{mm},$ the largest item is quantified. This completes a one sampling cycle and the set $X_{(1)1},\ X_{(2)1}, ..., X_{(m)1}$ is called a ranked set sample in the first cycle. If the cycle is repeated r times, we obtain a RSS of size rm units. Let $X_{(i)j}$ denote the $i-$th minimum of the $i-$th sample of size m in the $j-$th cycle, $i = 1, 2, ..., m,\ j = 1, 2, ..., r.$ An unbiased estimator of the population mean in the $j-$th cycle is

$\bar{X}_{RSS,\ j} = \frac{1}{m}\sum_{i=1}^{m} X_{(i)j}\ , \quad j = 1, 2, ..., r.$

Another unbiased estimator of the population mean is

$\bar{X}_{RSS} = \frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m} X_{(i)j}\ .$

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Al-Saleh and Al-Kadiri (2000) provided an explicit formula for obtaining the probability of all possible orderings of the elements of the RSS. The distribution of $X_{(r:k)}^{ORSS}$ in an *ordered ranked set sample* scheme was obtained by Balakrishnan and Li (2005). The results of these two papers were used by Li and Balakrishnan (2008) to obtain an exact null distribution for use in some non-parametric tests in RSS. There are large number of important papers in this direction and one may refer to Stokes (1977), Chen and Shen (2002) and Al-Saleh and Ananbeh (2005) for more details. Chen et al. (2003) provides a comprehensive review on various developments on RSS and its variates.

The skew normal distribution was first introduced by Azzalini (1985). Some extensions to the multivariate setting has been proposed by several authors, e.g., Azzalini and Dalla Valle (1996), Arellano-Valle and Azzalini (2006) and Genton (2004), etc. Following Arellano-Valle and Azzalini (2006) we say that the random vector $\mathbf{Y}$ has a unified multivariate skew normal distribution, if its density can be written as

$$f_{\mathbf{Y}}(\mathbf{y}) = \phi_d(\mathbf{y};\xi,\,\mathbf{\Omega})\,\frac{\Phi_m\left(\gamma + \mathbf{\Lambda}^T\mathbf{\Omega}^{-1}(\mathbf{y}-\xi)\,;\,\mathbf{\Gamma} - \mathbf{\Lambda}^T\mathbf{\Omega}^{-1}\mathbf{\Lambda}\right)}{\Phi_m\left(\gamma;\,\mathbf{\Gamma}\right)} \quad \mathbf{y}\in\mathbb{R}^d \tag{1}$$

where $\varphi_d(\,.,\,\xi,\,\mathbf{\Omega})$ is the density function of a d-dimensional normal with mean vector $\boldsymbol{\xi}$ and covariance matrix $\mathbf{\Omega}$ and $\Phi_m(.;\,\sum)$ is the multivariate normal cumulative function with the covariance matrix $\sum$. We write $\mathbf{Y} \sim SUN_{d,m}\left(\xi,\;\gamma,\;\mathbf{\Omega},\;\mathbf{\Gamma},\;\mathbf{\Lambda}\right)$.

In this paper we derive the distribution of the random vector $\mathbf{X}_{(m)} = (X_{(1)j},\;X_{(2)j},\ldots,\;X_{(m)j})^T$, $j = 1,2,\ldots,r$ and use this distribution to present a statistic test based on these ranked set samples. We use the connection of order statistics and skew normal distribution (see e.g., Loperfido, 2008 and Sheikhi et al., 2013) to show that this distribution belongs to the unified multivariate skew normal family.

Arellano-Valle and Azzalini (2006) introduced three types of a singular skew normal distribution. We say that $\mathbf{Y}$ has a singular unified skew normal distribution and write $\mathbf{Y} \sim SSUN_{d,m}(\xi,\boldsymbol{\delta},\mathbf{\Omega},\;\mathbf{\Gamma},\;\mathbf{\Lambda})$, if both the matrices $\mathbf{\Gamma}$ and $\mathbf{\Omega}$ are of full ranks ($rank(\mathbf{\Gamma})=m$ and $rank(\mathbf{\Omega})=d$) but $\Sigma^* = \begin{pmatrix}\mathbf{\Gamma} & \mathbf{\Lambda}^T\\ \mathbf{\Lambda} & \mathbf{\Omega}\end{pmatrix}$ is a singular matrix with $rank(\Sigma^*) < m+d$. For more details, see Arellano-Valle and Azzalini (2006) and Sheikhi and Jamalizadeh (2011).

Besides the skew normal distribution, the skew-t distribution has also received much attention. Similar to (1), the density of a unified multivariate skew-t random vector $\mathbf{W}$, denoted by $\mathbf{W} \sim SUT_{d,m}\left(\xi,\;\gamma,\;\mathbf{\Omega},\;\mathbf{\Gamma},\;\mathbf{\Lambda},v\right)$, is

$$g_{\mathbf{W}}(\mathbf{w}) = t_d(\mathbf{w};\xi,\,\mathbf{\Omega},\,v)\,\frac{T_m\left(\gamma + \mathbf{\Lambda}^T\mathbf{\Omega}^{-1}(\mathbf{w}-\xi)\,;\,\frac{(\mathbf{w}-\boldsymbol{\xi})^T\mathbf{\Omega}^{-1}(\mathbf{w}-\boldsymbol{\xi})}{v+d}(\mathbf{\Gamma} - \mathbf{\Lambda}^T\mathbf{\Omega}^{-1}\mathbf{\Lambda}),\,v+d\right)}{T_m\left(\gamma;\,\mathbf{\Gamma},\,v\right)} \quad \mathbf{w}\in\mathbb{R}^d \tag{2}$$

where $t_k(\,.,\,\xi,\,\mathbf{\Omega},v)$ is the pdf of a k-dimensional t-distribution with mean ξ, covariance matrix $\mathbf{\Omega}$ and degree of freedom v. Also, $T_m\,(.;\,\mathbf{\Omega},v)$ is the cdf of a centred k-dimensional t-distribution.

Now, let $\mathbf{X}$ be a random vector in $\mathbb{R}^n$ follows a multivariate normal distribution such that

$$\mathbf{X} \sim N_n\left(\mu\mathbf{1}_n,\;\mathbf{\Sigma} = \sigma^2\left\{(1-\rho)\,\mathbf{I}_n + \rho\mathbf{1}_n\mathbf{1}_n^T\right\}\right), \tag{3}$$

where ρ is the correlation coefficient between any two components of $\mathbf{X}$, $\mathbf{1}_n = (1,\ldots,1)^T$ has n components and $\mathbf{I}_n$ is the identity matrix of dimension n. In other words, the random vector $\mathbf{X}$ has an exchangeable multivariate normal distribution.

The aim of this section is to give a new proof of the independence between the mean and variance of random variables $X_1,X_2,\ldots,X_n$ when the corresponding joint distribution is multivariate exchangeable normal. The independence of the sample mean $\bar{X}$ and the sample variance S^2 when $X_1,X_2,\ldots,X_n$ is a random sample from a normal distribution has been proved by many authors (e.g., Basu, 1955). Arnold

(1973) has discussed the independence of squared order statistics, Azzalini and Capitanio (1999) have used the properties of skew normal distributions to prove the Cochran theorem, while Gupta and Huang (2002) have proved the independence of linear and quadratic forms of skew normal variables. Bathachria (1974) has established that given the value of order statistics, the concomitant variables are independent. In Asymptotic independence and limit theorem, Suresh (1993) has shown that the central concomitants and extreme concomitants are asymptotically independent.

In 2002, Loperfido first showed that ordered statistics and skew normal distributions are intimately connected. In this section we use this connection to present a new proof of the independence of $\bar{X}$ and S^2.

The following lemma is useful in what follows and is similar to the Proposition 6 in Azzalini and Capitanio (2001).

Lemma 1: *If* $\mathbf{Y} \sim SSUN_{d,m}(\boldsymbol{\xi}, \boldsymbol{\delta}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda}, \boldsymbol{\Omega})$ *where* $\boldsymbol{\Omega} = (\omega_{ij}), i, j = 1, 2, ..., d$ *and* $\boldsymbol{\Lambda} = (\lambda_1^T, \lambda_2^T, ..., \lambda_d^T)^T$. *Then the k-th component of* $\mathbf{Y}$ *is independent of the other components if the following two conditions hold:*

1) $\omega_{kj} = 0 \;\; \forall\, j \neq k$

2) $\lambda_k = \mathbf{0}$

Proof: Let

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_{-k} \\ y_k \end{pmatrix},\ \xi = \begin{pmatrix} \xi \mathbf{1}_{k-1} \\ \xi \end{pmatrix},\ \boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\Omega}_{-k-k} & \mathbf{0}^T \\ \mathbf{0} & \omega_{kk} \end{pmatrix}, \boldsymbol{\Lambda} = (\boldsymbol{\Lambda}_{-k}, \mathbf{0})^T$$

where $\mathbf{y}_{-k}$ is the vector $\mathbf{y}$ which its $k-$th component is removed and $\boldsymbol{\Omega}_{-k-k}$ is the covariance matrix of $\mathbf{y}_{-k}$. If the conditions (1) and (2) hold, we have

$$\varphi_d\left(\mathbf{y} - \xi; \boldsymbol{\Omega}\right) = \varphi\left(y_k - \xi; \sigma^2\right)\varphi_{d-1}\left(\mathbf{y}_{-k} - \boldsymbol{\xi}_{-k}; \boldsymbol{\Omega}_{-k}\right),$$

$$\begin{aligned} \Phi_m\left(\boldsymbol{\delta} + \boldsymbol{\Lambda}^T\boldsymbol{\Omega}^{-1}\left(\mathbf{y} - \boldsymbol{\xi}\right); \boldsymbol{\Gamma} - \boldsymbol{\Lambda}^T\boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}\right) &= \Phi_m\left(\boldsymbol{\delta} + \boldsymbol{\Lambda}_{-k}^T\boldsymbol{\Omega}_{-k}^{-1}\left(\mathbf{y} - \boldsymbol{\xi}\right); \boldsymbol{\Gamma} - \boldsymbol{\Lambda}_{-k}^T\boldsymbol{\Omega}_{-k}^{-1}\boldsymbol{\Lambda}_{-k}\right) \\ &= \Phi_m\left(\boldsymbol{\delta} + \boldsymbol{\Lambda}_{-k}^T\boldsymbol{\Omega}_{-k}^{-1}\left(\mathbf{y}_{-k} - \boldsymbol{\xi}_{-k}\right); \boldsymbol{\Gamma} - \boldsymbol{\Lambda}_{-k}^T\boldsymbol{\Omega}_{-k}^{-1}\boldsymbol{\Lambda}_{-k}\right). \end{aligned}$$

Hence,

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= \varphi\left(y_k - \xi; \sigma^2\right)\varphi_{d-1}\left(\mathbf{y}_{-k} - \boldsymbol{\xi}_{-k}; \boldsymbol{\Omega}_{-k}\right) \\ &\quad \times \frac{\Phi_m\left(\boldsymbol{\delta} + \boldsymbol{\Lambda}_{-k}^T\boldsymbol{\Omega}_{-k}^{-1}\left(\mathbf{y}_{-k} - \boldsymbol{\xi}_{-k}\right); \boldsymbol{\Gamma} - \boldsymbol{\Lambda}_{-k}^T\boldsymbol{\Omega}_{-k}^{-1}\boldsymbol{\Lambda}_{-k}\right)}{\Phi_m\left(\boldsymbol{\delta}; \boldsymbol{\Gamma}\right)}, \end{aligned}$$

which implies the assertion. ■

Suppose $S(\mathbf{X}) = \{\mathbf{X}^{(i)} = \mathbf{P}_i\mathbf{X}; i = 1, 2, ..., n!\}$ where $\mathbf{P}_i$ is an $n \times n$ permutation matrix. Let $\boldsymbol{\Delta}$ be the difference matrix of dimension $(2n-3) \times n$ such that its first $n-1$ rows are of the form $\mathbf{e}_{n,i}^T - \mathbf{e}_{n,1}^T$, $i = 2, ..., n$, and its last $n-2$ rows are of the form $\mathbf{e}_{n,i}^T - \mathbf{e}_{n,2}^T$, $i = 3, ..., n$ where $\mathbf{e}_{n,1}, \mathbf{e}_{n,2}, ..., \mathbf{e}_{n,n}$ are n-dimensional standard unit vectors. Then $\boldsymbol{\Delta}\mathbf{X} = (X_2 - X_1,\ X_3 - X_1, ...,\ X_n - X_1,\ X_3 - X_2,\ X_4 - X_2 ...,\ X_n - X_2)^T$.

We now have the following theorem.

Theorem 1: *Let the random vector* $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ *have a multivariate exchangeable normal distribution, then* $\bar{X}$ *and* S^2 *are independent.*

Proof: Since $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_{i:n}$ and $S^2 = \frac{1}{2n(n-1)}\sum_{i=1}^n\sum_{j=1}^n(X_{i:n} - X_{j:n})^2$, it is sufficient to show that $\sum_{i=1}^n X_{i:n}$ and $X_{i:n} - X_{j:n}$ are independent for all i and j. WLOG the proof reduces to showing that $\sum_{i=1}^n X_{i:n}$ is independent of $X_{1:n} - X_{2:n}$.

Let $\mathbf{a} = (1, 1, ..., 1)^T$ and $\mathbf{b} = (1, -1, 0, 0, ..., 0)^T$ be two n-dimensional vectors and $\mathbf{X}_{(n)} = (X_{1:n},\ X_{2:n}, ..., X_{n:n})^T$ denotes the vector of order statistics of $X_i's$, then $\mathbf{a}^T\mathbf{X}_{(n)} = \sum_{i=1}^n X_{i:n}$ and $\mathbf{b}^T\mathbf{X}_{(n)} = X_{1:n} - X_{2:n}$. The joint distribution of $\mathbf{a}^T\mathbf{X}_{(n)}$ and $\mathbf{b}^T\mathbf{X}_{(n)}$ is

$$\begin{aligned} F_{\mathbf{a}^T\mathbf{X}_{(n)},\mathbf{b}^T\mathbf{X}_{(n)}}(y_1, y_2) &= P\left(\mathbf{a}^T\mathbf{X}_{(n)} \le y_1, \mathbf{b}^T\mathbf{X}_{(n)} \le y_2\right) \\ &= \sum_{i=1}^{n!} P\left(\mathbf{a}^T\mathbf{X}^{(i)} \le y_1, \mathbf{b}^T\mathbf{X}^{(i)} \le y_2 | \boldsymbol{\Delta}\mathbf{X}^{(i)} \ge \mathbf{0}\right) P\left(\boldsymbol{\Delta}\mathbf{X}^{(i)} \ge \mathbf{0}\right) \end{aligned}$$

where $\mathbf{X}^{(i)}$ stands for the i-th permutation of $\mathbf{X}$. By the exchangeability assumption, we have $P\left(\Delta\mathbf{X}^{(i)} \geq \mathbf{0}\right) = \left(\frac{1}{n!}\right)$, $i = 1, 2, ..., n!$. Hence,

$$F_{\mathbf{a}^T\mathbf{X}_{(n)},\mathbf{b}^T\mathbf{X}_{(n)}}(y_1,\ y_2) = P\left(\mathbf{a}^T\mathbf{X} \leq y_1, \mathbf{b}^T\mathbf{X} \leq y_2 | \Delta\mathbf{X} \geq \mathbf{0}\right).$$

Moreover

$$\begin{pmatrix} \Delta\mathbf{X} \\ \mathbf{a}^T\mathbf{X} \\ \mathbf{b}^T\mathbf{X} \end{pmatrix} \sim N_{2n-1}\left(\begin{pmatrix} \mathbf{0} \\ n\mu \\ 0 \end{pmatrix}, \begin{pmatrix} \Delta\Sigma\Delta^T & \Delta\Sigma\mathbf{a} & \Delta\Sigma\mathbf{b} \\ & \mathbf{a}^T\Sigma\mathbf{a} & \mathbf{a}^T\Sigma\mathbf{b} \\ & & \mathbf{b}^T\Sigma\mathbf{b} \end{pmatrix}\right)$$

Since both $\Delta\mathbf{X}$ and $(\mathbf{a}^T\mathbf{X},\ \mathbf{b}^T\mathbf{X})^T$ are of full rank but $\begin{pmatrix} \Delta\mathbf{X} \\ \mathbf{a}^T\mathbf{X} \\ \mathbf{b}^T\mathbf{X} \end{pmatrix}$ is not, according to the Case (3) of Arellano-Valle and Azzalini (2006) we conclude that $(\mathbf{a}^T\mathbf{X},\ \mathbf{b}^T\mathbf{X})^T | \Delta\mathbf{X} > \mathbf{0} \sim SSUN_{2,2n-3}(\xi\ , \mathbf{0},\ \Omega,\ \Delta\Sigma\Delta^{\mathbf{T}},\ \Lambda)$ where

$$\xi = \begin{pmatrix} n\mu \\ 0 \end{pmatrix},\ \Omega = \begin{pmatrix} \mathbf{a}^T\Sigma\mathbf{a} & \mathbf{a}^T\Sigma\mathbf{b} \\ & \mathbf{b}^T\Sigma\mathbf{b} \end{pmatrix},\ \Lambda = \begin{pmatrix} \Delta\Sigma\mathbf{a} \\ \Delta\Sigma\mathbf{b} \end{pmatrix}.$$

We then easily obtain $\mathbf{a}^T\Sigma\mathbf{b}$ =0 and $\Delta\Sigma\mathbf{a} = \mathbf{0}_{2n-3}$, So Lemma 1 finishes the proof. ■

The following two corollaries are now a direct consequence of Theorem 1.

Corollary 1. *Let the random vector $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ have a multivariate exchangeable normal distribution, then $\bar{X}$ and range $W = X_{(n)} - X_{(1)}$ are independent.*

Corollary 2. *If $X_1, X_2, ..., X_n$ is a random sample form $N\left(\mu, \sigma^2\right)$, then $\bar{X}$ and S^2 are independent.*

The rest of this paper is organized as follows. In Section 2, we obtain the the exact distribution of the random vector $\mathbf{X}_{(m)} = (X_{(1)j},\ X_{(2)j}, ...,\ X_{(m)j})^T$ in a ranked set sample. In Section 3, by constructing an unified skew-t distribution, in a special case, we present an application of our results. Finally we illustrate our results by a numerical example.

2. Distribution of Ranked Set Samples

Let the random vectors $\mathbf{X}_j = (X_{j1},\ X_{j2}, ...,\ X_{jm})^T$ for $j = 1,\ 2, ..., m,$ be *iid* $N_m(\boldsymbol{\mu} = \mu\mathbf{1}_m,\ \sum = \sigma^2\mathbf{I}_m)$.

For $k = 1, 2, ..., m$, define the difference matrices Δ_k of dimension $m - 1 \times m$ such that the first $k - 1$ rows of Δ_k are $\mathbf{e}^T_{m,1} - \mathbf{e}^T_{m,i}$, $i = 2, 3, ..., k$, and the last $m - k$ are $\mathbf{e}^T_{m,i} - \mathbf{e}^T_{m,1}$, $i = k + 1, k + 2, ..., m - 1$, where $\mathbf{e}^T_{m,i}$ is the unit basis vector of dimension m. For example, for $m = 3$ we have

$$\Delta_1 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},\ \Delta_2 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix},\ \Delta_3 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Further let $\mathbf{X}^{(i)}_j$ be the permutation of the random vectors $\mathbf{X}_j$, such that its i−th element is located in the first place.

Theorem 2: *Under the ranked set sample setting,*

$$\mathbf{X}_{(m)} \sim SUN_{m,\ m(m-1)}(\mu\mathbf{1}_m,\ \mathbf{0}_{m(m-1)},\ \sigma^2\mathbf{I}_m,\ \Gamma,\ \Lambda).$$

where

$$\Gamma = \sigma^2\begin{pmatrix} \Delta_1\Delta_1^T & \Delta_1\Delta_2^T & ... & \Delta_1\Delta_m^T \\ & \Delta_2\Delta_2^T & ... & \Delta_2\Delta_m^T \\ & \vdots & \vdots & \vdots \\ & & .. . & \Delta_m\Delta_m^T \end{pmatrix},\ \Lambda = \sigma^2\begin{pmatrix} \boldsymbol{\delta}_1 \\ \boldsymbol{\delta}_2 \\ \vdots \\ \boldsymbol{\delta}_m \end{pmatrix}$$

and for $k = 1, 2, ..., m$, $\boldsymbol{\delta}_k$ is an $m - 1 \times m$ matrix which its k−th column is equal to the first column of Δ_k and all other components are zero.

Proof: WLOG we consider $j = 1$. By the theorem of total probability we have

$$\begin{aligned} F_{\mathbf{X}_{(m)}}(\mathbf{x}) &= P\left(X_{(1)1} \leq x_1,\ X_{(2)1} \leq x_2, ..., \ X_{(m)1} \leq x_m\right) \\ &= \sum_{k_1=1}^{m}\sum_{k_2=1}^{m}...\sum_{k_m=1}^{m} P\left(X_{1k_1} \leq x_1,\ X_{2k_2} \leq x_2, ... + X_{mk_m} \leq x_m | A_{k_1,\ k_2,\ ...,k_m}\right) P\left(A_{k_1,\ k_2,\ ...,k_m}\right) \end{aligned}$$

where $A_{k_1,\ k_2,\ ...,k_m} = \{\boldsymbol{\Delta}_1\mathbf{X}_1^{(k_1)} \geq \mathbf{0},\ \boldsymbol{\Delta}_2\mathbf{X}_2^{(k_2)} \geq \mathbf{0}, ... \ \boldsymbol{\Delta}_m\mathbf{X}_m^{(k_m)} \geq \mathbf{0}\}$. We note that for $i,\ j = 1,\ 2, ..., m,\ \mathbf{X}_j^{(i)}$ are *iid* $N_m\left(\boldsymbol{\mu},\ \sum\right)$ and by their independence we have

$$\begin{aligned} P\left(A_{k_1,\ k_2,\ ...,k_m}\right) &= P(\boldsymbol{\Delta}_1\mathbf{X}_1^{(k_1)} \geq \mathbf{0},\ \boldsymbol{\Delta}_2\mathbf{X}_2^{(k_2)} \geq \mathbf{0}, ... \ \boldsymbol{\Delta}_m\mathbf{X}_m^{(k_m)} \geq \mathbf{0}) \\ &= P(\boldsymbol{\Delta}_1\mathbf{X}_1^{(k_1)} \geq \mathbf{0})P(\boldsymbol{\Delta}_2\mathbf{X}_2^{(k_2)} \geq \mathbf{0})...P(\boldsymbol{\Delta}_m\mathbf{X}_m^{(k_m)} \geq \mathbf{0}) \\ &= \frac{(m-1)!}{m!} \times \frac{(m-1)!}{m!} \times ... \times \frac{(m-1)!}{m!} \\ &= (\frac{1}{m})^m \end{aligned}$$

Also, for $k = 1, 2, ..., m$ the joint distribution of $\left(\boldsymbol{\Delta}_1\mathbf{X}_1^{(k_1)},\ \boldsymbol{\Delta}_2\mathbf{X}_2^{(k_2)},\ ...,\ \boldsymbol{\Delta}_m\mathbf{X}_m^{(k_m)}\right)^T$ and $\mathbf{X}_{(m)}$ follows a m^2 dimensional multivariate normal distribution with

$$\boldsymbol{\mu} = \begin{pmatrix} \mathbf{0}_{m-1} \\ \mathbf{0}_{m-1} \\ \vdots \\ \mathbf{0}_{m-1} \\ \mu\mathbf{1}_m \end{pmatrix} \text{ and } \sum = \sigma^2 \begin{pmatrix} \boldsymbol{\Delta}_1\boldsymbol{\Delta}_1^T & \boldsymbol{\Delta}_1\boldsymbol{\Delta}_2^T & ... & \boldsymbol{\Delta}_1\boldsymbol{\Delta}_m^T & \boldsymbol{\delta}_1 \\ & \boldsymbol{\Delta}_2\boldsymbol{\Delta}_2^T & ... & \boldsymbol{\Delta}_2\boldsymbol{\Delta}_m^T & \boldsymbol{\delta}_2 \\ & \vdots & \vdots & \vdots & \vdots \\ & & ... & \boldsymbol{\Delta}_m\boldsymbol{\Delta}_m^T & \boldsymbol{\delta}_m \\ & & & & \mathbf{I}_m \end{pmatrix}.$$

Now, Similar to Sheikhi et al. (2013) we immediately conclude that the distribution of $\mathbf{X}_{(m)}$ is the same as

$(X_{1k_1},\ X_{2k_2}, ..., X_{mk_m})^T | A_{k_1,\ k_2,\ ...,k_m} \sim SUN_{m,\ m(m-1)}(\mu\mathbf{1}_m,\ \mathbf{0}_{m(m-1)},\ \sigma^2\mathbf{I}_m,\ \boldsymbol{\Gamma},\ \boldsymbol{\Lambda})$

where the parameters are as in the theorem. Since the terms in the summations do not depend on $k_1,\ k_2,\ ..., k_m$ we readily obtain

$$\begin{aligned} F_{\mathbf{X}_{(m)}}(\mathbf{x}) &= m^m \times F_{SUN}(\mathbf{x}\,; \mu\mathbf{1}_m,\ \mathbf{0}_{m(m-1)},\ \sigma^2\mathbf{I}_m,\ \boldsymbol{\Gamma},\ \boldsymbol{\Lambda})(\frac{1}{m})^m \\ &= F_{SUN}(x\,; \mu\mathbf{1}_m,\ \mathbf{0}_{m(m-1)},\ \sigma^2\mathbf{I}_m,\ \boldsymbol{\Gamma},\ \boldsymbol{\Lambda})\,. \end{aligned}$$

This proves the assertion. ■

3. An Application

In this section we present an application of our results which relies on the relation between skew normal and skew-t distribution. We first state the following lemma (See, e.g., Arellano Valle and Azzallini, 2006)

Lemma 2: If $\mathbf{Y} \sim SUN_{d,m}\left(\boldsymbol{\xi},\ \boldsymbol{\gamma},\ \boldsymbol{\Omega},\ \boldsymbol{\Gamma},\ \boldsymbol{\Lambda}\right)$ and $\mathbf{b}$ is an arbitray vector in $\mathbb{R}^d$ then

a) $\mathbf{b}^T\mathbf{Y} \sim SUN_{d,m}\left(\mathbf{b}^T\boldsymbol{\xi},\ \boldsymbol{\gamma},\ \mathbf{b}^T\boldsymbol{\Omega}\mathbf{b}\ ,\ \boldsymbol{\Gamma},\ \boldsymbol{\Lambda}\right)$

b) $M_{\left(\mathbf{Y}-\boldsymbol{\xi}\right)^T\mathbf{A}^{-1}\left(\mathbf{Y}-\boldsymbol{\xi}\right)}(t) = \left|\mathbf{I}_p - 2t\mathbf{A}\boldsymbol{\Omega}\right|^{-1/2} \frac{\Phi_m\left(\boldsymbol{\gamma};\ \mathbf{0},\ \boldsymbol{\Gamma}+2t\boldsymbol{\Lambda}^T\left(\mathbf{I}_p-2t\mathbf{A}\boldsymbol{\Omega}\right)^{-1}\mathbf{A}\boldsymbol{\Lambda}\right)}{\Phi_m\left(\boldsymbol{\gamma};\ \mathbf{0},\boldsymbol{\Gamma}\right)}.$

We now consider $m = 2$ and show that the distribution of $\frac{\bar{X}_{RSS}-\xi}{S_{RSS}/\sqrt{2}}$ is skew-t, where $\bar{X}_{RSS}$ is the mean of $X_{(1)1}$ and $X_{(2)1}$ and S_{RSS} is their standard deviation. Since $\bar{X}_{RSS}$ and S^2_{RSS} are independent via Lemma 1, we can readily deduce the following corollary

Corollary 3: If $m = 2$, the following results hold:
a) $\bar{X}_{RSS} \sim SUN_{1,2}\left(\mu,\ \mathbf{0}_2,\ \sigma^2, \mathbf{\Gamma},\ \mathbf{\Lambda}\right)$
b) $U = S^2_{RSS}/\sigma^2 \sim \chi^2_2$
c) $T = \frac{\bar{X}_{RSS}-\xi}{S_{RSS}/\sqrt{2}} \sim SUT_{1,2}\left(0,\ \mathbf{0}_2,\ 1,\ 2\sigma^2\mathbf{I_2},\ \mathbf{M}, 1\right)$.
Proof: Part (a) is a direct consequence of Theorem 2 and we only prove parts (b) and (c).
Let $\mathbf{X}_{(2)} = \left(X_{(1)},\ X_{(2)}\right)^T$ then by Theorem 2 we have $\mathbf{X}_{(2)} \sim SUN_{2,\,2}(\mu\mathbf{1}_2,\ \mathbf{0}_2,\ \sigma^2\mathbf{I}_2, 2\sigma^2\mathbf{I_2},\ \mathbf{\Lambda})$ where

$\mathbf{\Lambda} = \sigma^2 diag(-1, 1)$ and $U = \left(\mathbf{X}_{(2)}-\xi\right)^T\ \mathbf{\Omega}^{-1}\left(\mathbf{X}_{(2)}-\xi\right)$. Using lemma 2 we have $\boldsymbol{A} = \sigma^{-2}\mathbf{I}_2$ and

$$
\begin{aligned}
M_U(t) &= |\boldsymbol{I}_2 - 2t\boldsymbol{I}_2|^{-1/2}\frac{\Phi_m\left(\mathbf{0};\ \mathbf{0},\ \mathbf{\Gamma} + 2t\mathbf{\Lambda}^T\left(\boldsymbol{I}_p - 2t\boldsymbol{A}\mathbf{\Omega}\right)^{-1}\boldsymbol{A}\mathbf{\Lambda}\right)}{\Phi_m\left(\mathbf{0};\ \mathbf{0},\ \mathbf{\Gamma}\right)} \\
&= (1-2t)^{-1}\frac{\Phi_m\left(\mathbf{0};\ \mathbf{0},\ 2\sigma^2\boldsymbol{I_2} + 2t\sigma^2(1-2t)^{-2}\boldsymbol{I_2}\right)}{\Phi_m\left(\mathbf{0};\ \mathbf{0},\ 2\sigma^2\boldsymbol{I_2}\right)} \\
&= (1-2t)^{-1},
\end{aligned}
$$

which is the moment generating function of χ^2_2. Note that the last equality holds since both covariance matrices in the numerator and dominator are diagonal.

c) Using parts (a) and (b) we have
$Z = \frac{\bar{X}_{RSS}-\xi}{\sigma} \sim SUN_{1,2}\left(0,\ \mathbf{0}_2,\ 1\ , 2\sigma^2\mathbf{I_2},\ \mathbf{\Lambda}\right)$
and after some algebra we obtain

$$
\begin{aligned}
f_T(t) &= \frac{1}{\sqrt{\pi}(1+\frac{x^2}{2})^{3/2}}\int_0^\infty \frac{y^{3/2}e^{-t}\Phi_2\left(\frac{\sigma^2}{2}(\frac{2t}{1+\frac{t^2}{2}})^{1/2}(\mathbf{-1}, 1)^T; \boldsymbol{M}\right)}{\Phi_2(\mathbf{0};\ 2\sigma^2\boldsymbol{I})}dy \\
&= 4t(0,\ 1,\ 2)T_2\left(\frac{\sigma^2}{2}(\frac{2t}{1+\frac{t^2}{2}})^{1/2}(-1, 1)^T; \boldsymbol{M},\ 3\right),
\end{aligned}
$$

where $\mathbf{M} = \frac{1}{8}\begin{pmatrix} 7 & 1 \\ 1 & 7 \end{pmatrix}$, which is the distribution of $SUT_{1,2}\left(0,\ \mathbf{0}_2,\ 1,\ 2\sigma^2\mathbf{I_2},\ \mathbf{M}, 1\right)$. This ends the proof. ■

4. Numerical Study

In this section we present a numerical application of our results. Suppose a manufacturer of cable wire wants to assess if the diameter of the cable meets specifications. A cable wire must be 0.55 ± 0.05 cm in diameter to meet engineering specifications. Analyst evaluate the capability of the process to ensure it is meeting reliability requirements. Once every hour, for two hours, he samples two consecutive cable wires from the production line and records their diameters. He then repeats this cycle r times, for an arbitrary $j = 1, 2, \ldots, r$. An example of such data is available in Minitab Statistical Software. For this purpose, we consider the data of the first five cycles, are collected as

	First sample	Second sample
Cycle 1	0.529, 0.550	0.555, 0.541
Cycle 2	0.543, 0.557	0.559, 0.581
Cycle 3	0.493, 0.534	0.527, 0.511
Cycle 4	0.559, 0.519	0.562, 0.551
Cycle 5	0.545, 0.588	0.544, 0.561.

If we apply the ranked set sampling scheme to this data, we may estimate the distribution of mean of the ranked set sampling estimator in each cycle, i.e. $\bar{X}_{RSS,\ j},\ j = 1, 2, ..., 5$.

We assume that the diameter of these wires follows a normal distribution and the maximum likelihood estimators of mean and variance are obtained as $\hat{\mu} = 0.54748$ and $\hat{\sigma}^2 = 0.00046$. By part (a) of Corollary 3, we obtain the density of mean of the ranked set samples as $\bar{X}_{RSS} \sim SUN_{1,2}\left(\mu,\ \mathbf{0}_2,\ \sigma^2, \boldsymbol{\Gamma},\ \boldsymbol{\Lambda}\right)$, where $\boldsymbol{\Gamma} = 0.00046\begin{pmatrix} \boldsymbol{\Delta}_1\boldsymbol{\Delta}_1^T & \boldsymbol{\Delta}_1\boldsymbol{\Delta}_2^T \\ & \boldsymbol{\Delta}_2\boldsymbol{\Delta}_2^T \end{pmatrix}$ and $\boldsymbol{\Lambda} = 0.00046\begin{pmatrix} \boldsymbol{\delta}_1 \\ \boldsymbol{\delta}_2 \end{pmatrix}$. Therefore, the distribution of $T = \frac{\bar{X}_{RSS}-\xi}{S_{RSS}/\sqrt{2}}$ follows a unified skew-t, i.e., $SUT_{1,2}\left(0,\ \mathbf{0}_2,\ 1,\ 2\sigma^2\mathbf{I}_{2,},\ \mathbf{M}, 1\right)$.

As can be seen, this distribution only depends on the variance of population and enables the analyst to identity which product exceeds the control limits.

5. Conclusion

In this work we obtain the exact distribution of ranked set samples as well as the distribution of their mean. We model these distributions based on the unified multivariate normal distributions. The results can possibly be extended to double RSS, multistage RSS and (moving) extreme RSS schemes. For more information, see Al-Saleh and Al-Omari (2002), Al-Saleh and Samawi (2010) and Salehi and Jafari (2015). We hope to generalize our work to obtain the exact distribution of $\bar{X}_{RSS} = \frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m} X_{(i)j}$ in the both balanced and unbalanced cases. We also trying to obtain a test statistic from the ranked set samples when $m > 2$.

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