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# On Perturbed Monomials on 2-adic Spheres Around 1

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**Abstract.** We provide a complete description of ergodic perturbed monomials on 2-adic spheres around the unity.

## 1. Introduction

As it was mentioned in [4], non-expanding dynamics on the ring of $p$-adic integers $\mathbb{Z}_p$ have been explicitly studied in many papers like [3], [2], [8] and [7]. Recently, some results on dynamical systems were considered on spheres [4] and on general compact sets [9]. First results on ergodicity for monomial dynamical systems on $p$-adic spheres were obtained in [6]. Later, ergodicity criteria for locally analytic dynamical systems on $p$-adic spheres were studied in [1].

Let $\mathbb{Z}_2$ denote the ring of 2-adic integers endowed with its ultra-metric norm $|\cdot|$ and natural probability measure $\mu$. It is known that each element $x$ from $\mathbb{Z}_2$ has the form $x = \sum_{i=0}^{\infty} x_i 2^i$, where $x_i \in \{0, 1\}$.

Let $S$ consist of a collection of $2^n\mathbb{Z}_2$-cosets and for arbitrary $x \in S$ let the elements $x, f(x), \dots, f^{k-1}(x)$ be representatives of distinct classes of $2^n\mathbb{Z}_2$-cosets, where $k = 2^n\mu(S)$.

An isometric function $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ is said to be transitive modulo $2^n$ on $S$ if the set $\{x, f(x), \dots, f^{k-1}(x)\}$ is composed of only one cycle. In other words, $f^k(x) = x \ (mod\ 2^n)$, but $f^r(x) \neq x \ (mod\ 2^n)$, for all $r < k$.

We recall that in [2, Theorem 1.1.] and [3, Proposition 4.35.] we find that an isometric function $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ is ergodic on $S$ if and only if it is transitive modulo $2^n$ on the set $S$ for every positive integer $n$. Moreover, [4, Section 4.] is about perturbed monomials on spheres $S_{2^r}(1)$ centered at 1 with radius $2^{-r}$. These are functions of the form $f(x) = x^s + 2^{r+1}u(x)$, where the function $u$ is 1-Lipschitz. Our attempt is to study these functions with arbitrary functions $u$ defined on the ring $\mathbb{Z}_2$. We describe all ergodic perturbed monomials of this form on $S_{2^r}(1)$ for different values of integers $s$ and $r$. Then, [4, Theorem 4.1.] is obtained as a direct consequence of this description. Our results are based on some reformulation of [7, Lemma 3.12.] on a compact set of $\mathbb{Z}_2$ which consists of two disjoint balls of the same measure.

## 2. Main Results

**Lemma 2.1.** *Let $a$ and $b$ be different nonnegative integers. Let $a, b < 2^k$, where $k$ is some positive integer. Set $S = (a + 2^k\mathbb{Z}_2) \uplus (b + 2^k\mathbb{Z}_2)$. Let $f : S \to S$ be isometric. Then, $f$ is ergodic on $S$ if and only if the following conditions are satisfied*

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*(1)* $f(a) - b = 0\ (mod\ 2^k)$,

*(2)* $f(a) + f(b) = a + b + 2^k\ (mod\ 2^{k+1})$,

*(3)* $\sum_{\epsilon_k,\dots,\epsilon_{k+n}\in\{0,1\}} \left( f(a + \sum_{i=k}^{k+n} \epsilon_i 2^i) + f(b + \sum_{i=k}^{k+n} \epsilon_i 2^i) \right)$
$= \sum_{\epsilon_k,\dots,\epsilon_{k+n}\in\{0,1\}} \left( a + \sum_{i=k}^{k+n} \epsilon_i 2^i + b + \sum_{i=k}^{k+n} \epsilon_i 2^i \right) + 2^{k+n+1}\ (mod\ 2^{k+n+2}), \forall n \ge 0.$

*Proof.* Recall that according to [7, Lemma 3.12.], an isometric function $g$ is transitive modulo $2^{n+1}$, $n \ge 1$, if and only if it is transitive modulo $2^n$ and $S_n$ is odd, where

$$S_n = \sum_{0\le m\le 2^n-1} g_{m_n},\ g(m) = \sum_{i=0}^{\infty} g_{m_i} 2^i,\ g_{m_i} \in \{0, 1\}, \forall i \ge 0.$$

This can be expressed as

$$\sum_{m=0}^{2^n-1} g(m) = \sum_{m=0}^{2^n-1} m + 2^n\ (mod\ 2^{n+1}). \tag{2.1}$$

Let $\psi : \mathbb{Z}_2 \to S$ defined by $\psi(x) = \begin{cases} 2^{k-1}x + a, & x \in 2\mathbb{Z}_2; \\ 2^{k-1}(x-1) + b, & x \in 1 + 2\mathbb{Z}_2. \end{cases}$

It is clear that $g := \psi^{-1} \circ f \circ \psi$ is ergodic on $\mathbb{Z}_2$ if and only if $f$ is ergodic on $S$. For $n \ge 2$ (2.1) can be written as

$$(2.1) \Leftrightarrow \sum_{m=0}^{2^n-1} \psi^{-1} \circ f \circ \psi(m) = 2^{n-1}(2^n - 1) + 2^n\ (mod\ 2^{n+1})$$

$$\Leftrightarrow \sum_{0\le m\le 2^n-1,\ m\ even} \psi^{-1} \circ f(2^{k-1}m + a) + \sum_{0\le m\le 2^n-1,\ m\ odd} \psi^{-1} \circ f(2^{k-1}(m-1) + b) = 2^{n-1}\ (mod\ 2^{n+1})$$

$$\Leftrightarrow \sum_{0\le m\le 2^n-1,\ m\ even} \left( \frac{f(2^{k-1}m + a) - b}{2^{k-1}} + 1 + \frac{f(2^{k-1}m + b) - a}{2^{k-1}} \right) = 2^{n-1}\ (mod\ 2^{n+1})$$

$$\Leftrightarrow \sum_{0\le m\le 2^n-1,\ m\ even} \left( f(2^{k-1}m + a) + f(2^{k-1}m + b) \right) = 2^{n-1}(a + b)\ (mod\ 2^{n+k})$$

$$\Leftrightarrow \sum_{\epsilon_k,\dots,\epsilon_{k+n-2}\in\{0,1\}} \left( f(a + \sum_{i=k}^{k+n-2} \epsilon_i 2^i) + f(b + \sum_{i=k}^{k+n-2} \epsilon_i 2^i) \right) = \sum_{\epsilon_k,\dots,\epsilon_{k+n-2}\in\{0,1\}} \left( a + \sum_{i=k}^{k+n-2} \epsilon_i 2^i + b + \sum_{i=k}^{k+n-2} \epsilon_i 2^i \right) + 2^{n+k-1}\ (mod\ 2^{n+k}).$$

On the other hand it is clear that $f$ is transitive modulo $2^k$ on $S$ if and only if Condition (1) is satisfied and Condition (2) is equivalent to (2.1) for $n = 1$. □

**Theorem 2.2.** *Let $s$ and $r$ be positive integers. Assume that $s = 1\ (mod\ 4)$. Let the functions $f$ and $u$ be defined on $\mathbb{Z}_2$ such that $f(x) = x^s + 2^{r+1}u(x)$. Denote by $S_{2^r}(1)$ the sphere of radius $2^{-r}$ centered at 1. Then, $f$ is ergodic on $S_{2^r}(1)$ if and only if $u$ satisfies the following conditions:*

*(1)* $|u(x) - u(y)| < 2^{r+1}|x - y|, \forall x, y \in S_{2^r}(1)$,

*(2)* $\sum_{\epsilon_{r+1},\dots,\epsilon_l\in\{0,1\}} u(1 + 2^r + \sum_{i=r+1}^{l} \epsilon_i 2^i) = 2^{l-r}\ (mod\ 2^{l-r+1}), \forall l \ge r.$

*Proof.* It is clear that $g(x) = x^s$ is isometric on $S_{2^r}(1)$. Then, $f$ is also isometric on this set if and only if Condition (1) is satisfied. On the other hand, applying Lemma 2.1 if $f$ is isometric on $S_{2^r}(1)$ then it is also ergodic on this set if and only the following formula holds:

$$\sum_{\epsilon_{r+1},\dots,\epsilon_l\in\{0,1\}} f(1+2^r+\sum_{i=r+1}^{l}\epsilon_i 2^i) = \sum_{\epsilon_{r+1},\dots,\epsilon_l\in\{0,1\}} (1+2^r+\sum_{i=r+1}^{l}\epsilon_i 2^i) + 2^{l+1}\ (mod\ 2^{l+2}), \forall l\geq r. \tag{2.2}$$

The main idea of the proof is based on the fact that the function $g$ is ergodic on some specific subsets depending on the values of $r$ and $l$. Lemma 2.1 is then applied on $g$ which yields that $f$ is ergodic if and only if $u$ satisfies statement (2) of this theorem.

Set $s = 1 + 2^k\ (mod\ 2^{k+1})$, for some $k \geq 2$. We first consider the case when $r = 1$.

For every positive even integer $m$ and all $x \in S_2(1)$, we have $x^m + 1 = 2\ (mod\ 4)$.

For every positive odd integer $m$ and all $|x| = 1$, we have

$$|x^m+1| = |x+1|\cdot|x^{m-1}-x^{m-2}+\dots+x^2-x+1| = |x+1|, \tag{2.3}$$

and

$$|x^m-1| = |x-1|\cdot|x^{m-1}+x^{m-2}+\dots+x^2+x+1| = |x-1|. \tag{2.4}$$

It follows that for $x \in S_2(1)$,

$$\begin{aligned}|x^s-x| &= |x^{s-1}-1| = |x^{\frac{s-1}{2}}-1|\cdot|x^{\frac{s-1}{2}}+1| = \frac{1}{2}|x^{\frac{s-1}{2}}+1| = \dots = \\ &= \frac{1}{2^{k-1}}|x^{\frac{s-1}{2^k}}-1|\cdot|x^{\frac{s-1}{2^k}}+1| = \frac{1}{2^{k-1}}|x-1|\cdot|x+1| = \frac{1}{2^k}|x+1|.\end{aligned} \tag{2.5}$$

Moreover,

$$|x^{s^2}-x| = |x^{s^2-1}-1| = |x^{(s-1)\frac{s+1}{2}}-1|\cdot|x^{(s-1)\frac{s+1}{2}}+1| = |x^{s-1}-1|\cdot|x^{s-1}+1| = \frac{1}{2}|x^{s-1}-1|. \tag{2.6}$$

By means of [9, Lemma 3.1.] and [5, Proposition 9] (see also [9, Lemma 3.3.] as a modified version of [5, Proposition 9]), we get that $g$ is ergodic on each set of the form $x + \frac{2^k}{|x+1|}\mathbb{Z}_2$, where $x \in S_2(1)$.

Now we verify that (2.2) is equivalent to Condition (2) for all $l \geq 1$. First, for $l \leq k$ and $x \in S_2(1)$ we have from (2.5)

$$|x^s-x| = \frac{1}{2^k}|x+1| \leq \frac{1}{2^l}|x+1| \leq \frac{1}{2^{l+2}}.$$

It follows immediately that Condition (2) is equivalent to (2.2). Now, let $l \geq k+1$. We have

$$x^s = x\ (mod\ 2^{l+2}), \forall x \in \{|x+1| \leq 2^{-l-2+k}\}. \tag{2.7}$$

Moreover, from Lemma 2.1, since as mentioned above $g$ is ergodic on each set of the form $x + \frac{2^k}{|x+1|}\mathbb{Z}_2$,

$$\begin{aligned}&\forall t\leq l-k, \forall x\in\{|x+1|=2^{-t}\}, \forall\epsilon_{t+1},\dots,\epsilon_{t+k-1}\in\{0,1\}:\\ &\sum_{\epsilon_{t+k},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i 2^i)^s = \sum_{\epsilon_{t+k},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i 2^i)+2^{l+1}\ (mod\ 2^{l+2}),\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&t = l-k+1, \forall x\in\{|x+1|=2^{-t}\}:\\ &(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i 2^i)^s = (3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i 2^i)+2^{l+1}\ (mod\ 2^{l+2}).\end{aligned} \tag{2.9}$$

It follows from (2.5) and (2.9) that

$$\forall t \le l-k+1, \forall x \in \{|x+1| = 2^{-t}\}:$$

$$\sum_{\epsilon_{t+1},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i 2^i)^s = \sum_{\epsilon_{t+1},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i 2^i) + \sum_{\epsilon_{t+1},\dots,\epsilon_{t+k-1}\in\{0,1\}}2^{l+1}\ (mod\ 2^{l+2})$$
$$= \sum_{\epsilon_{t+1},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i 2^i)\ (mod\ 2^{l+2}). \tag{2.10}$$

We obtain from (2.7) and (2.10)

$$\sum_{\epsilon_2,\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{l}\epsilon_i 2^i)^s = \sum_{t=2}^{l+1-k}\sum_{\epsilon_{t+1},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i 2^i)^s + \sum_{\epsilon_{l+2-k},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{l+1-k}2^i+\sum_{i=l+2-k}^{l}\epsilon_i 2^i)^s$$
$$= \sum_{t=2}^{l+1-k}\sum_{\epsilon_{t+1},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i 2^i) + \sum_{\epsilon_{l+2-k},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{l+1-k}2^i+\sum_{i=l+2-k}^{l}\epsilon_i 2^i)\ (mod\ 2^{l+2}) \tag{2.11}$$
$$= \sum_{\epsilon_2,\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{l}\epsilon_i 2^i)\ (mod\ 2^{l+2}),$$

which completes the proof for the case when $r = 1$.

Now, let $r \ge 2$. We have from (2.4) for $x \in S_{2^r}(1)$

$$|x^s - x| = |x^{s-1}-1| = |x^{\frac{s-1}{2}}+1|\cdot|x^{\frac{s-1}{2}}-1| = \frac{1}{2}|x^{\frac{s-1}{2}}-1| = \dots = \frac{1}{2^k}|x^{\frac{s-1}{2^k}}-1| = \frac{1}{2^k}|x-1|. \tag{2.12}$$

Also,

$$|x^{s^2}-x| = |x^{s^2-1}-1| = |x^{(s-1)\frac{s+1}{2}}-1|\cdot|x^{(s-1)\frac{s+1}{2}}+1| = |x^{s-1}-1|\cdot|x^{s-1}+1| = \frac{1}{2}|x^{s-1}-1|. \tag{2.13}$$

Hence, $g$ is ergodic on each set of the form $x + 2^{r+k}\mathbb{Z}_2$, where $x \in S_{2^r}(1)$. In order to see that Condition (2) is equivalent to (2.2), first consider the case when $l \le r+k-2$. From (2.12)

$$|x^s - x| = \frac{1}{2^k}|x-1| = \frac{1}{2^{k+r}} \le \frac{1}{2^{l+2}},$$

which gives immediately that Condition (2) is equivalent to (2.2).

Besides, when $l \ge r+k$, for all $\epsilon_{r+1},\dots,\epsilon_{r+k-1} \in \{0,1\}$ we have from ergodicity of function $g$ on the set $1+2^r+\sum_{i=r+1}^{r+k-1}\epsilon_i 2^i + 2^{r+k}\mathbb{Z}_2$:

$$\sum_{\epsilon_{r+k},\dots,\epsilon_l\in\{0,1\}}(1+2^r+\sum_{i=r+1}^{l}\epsilon_i 2^i)^s = \sum_{\epsilon_{r+k},\dots,\epsilon_l\in\{0,1\}}(1+2^r+\sum_{i=r+1}^{l}\epsilon_i 2^i) + 2^{l+1}\ (mod\ 2^{l+2}).$$

Also, for $l = r+k-1$ and all $\epsilon_{r+1},\dots,\epsilon_{r+k-1} \in \{0,1\}$

$$(1+2^r+\sum_{i=r+1}^{l}\epsilon_i 2^i)^s = (1+2^r+\sum_{i=r+1}^{l}\epsilon_i 2^i) + 2^{l+1}\ (mod\ 2^{l+2}).$$

This yields for $l \ge r+k-1$,

$$\sum_{\epsilon_{r+1},\dots,\epsilon_l\in\{0,1\}}(1+2^r+\sum_{i=r+1}^{l}\epsilon_i 2^i)^s = \sum_{\epsilon_{r+1},\dots,\epsilon_l\in\{0,1\}}(1+2^r+\sum_{i=r+1}^{l}\epsilon_i 2^i) + \sum_{\epsilon_{r+1},\dots,\epsilon_{r+k-1}\in\{0,1\}}2^{l+1}\ (mod\ 2^{l+2})$$

$$= \sum_{\epsilon_{r+1},\dots,\epsilon_l\in\{0,1\}} (1 + 2^r + \sum_{i=r+1}^{l} \epsilon_i 2^i) \ (mod\ 2^{l+2}).$$

□

**Theorem 2.3.** *Let $s$ and $r$ be positive integers. Assume that $s = 3\ (mod\ 4)$ and $r \geq 2$. Let the functions $f$ and $u$ be defined on $\mathbb{Z}_2$ such that $f(x) = x^s + 2^{r+1}u(x)$. Then, $f$ is ergodic on $S_{2^r}(1)$ if and only if $u$ satisfies the following conditions:*

*(1)* $|u(x) - u(y)| < 2^{r+1}|x - y|, \forall x, y \in S_{2^r}(1)$,

*(2)* $u(x) = 0\ (mod\ 2), \forall x \in S_{2^r}(1)$,

*(3)* $\sum_{\epsilon_{r+1},\dots,\epsilon_l\in\{0,1\}} u(1 + 2^r + \sum_{i=r+1}^{l} \epsilon_i 2^i) = 2^{l-r}\ (mod\ 2^{l-r+1}), \forall l \geq r + 1.$

*Proof.* Arguing as in the previous theorem, it suffices to prove that an isometric function $f$ is ergodic on $S_{2^r}(1)$ if and only if Conditions (2) and (3) are simultaneously satisfied. The sphere $S_{2^r}(1)$ can be expressed as $(x + 2^{r+2}\mathbb{Z}_2) \cup (x + 2^{r+1} + 2^{r+2}\mathbb{Z}_2)$, for all $x \in S_{2^r}(1)$. Notice that from Condition (1) of Lemma 2.1 $f$ is transitive modulo $2^{r+2}$ if and only if $f(x) = x + 2^{r+1}\ (mod\ 2^{r+2})$, for all $x \in S_{2^r}(1)$. Namely,

$$x^s - x + 2^{r+1}u(x) = 2^{r+1}\ (mod\ 2^{r+2}), \forall x \in S_{2^r}(1).$$

From (2.3) and (2.4)

$$|x^s - x| = |x^{s-1} - 1| = |x^{\frac{s-1}{2}} + 1| \cdot |x^{\frac{s-1}{2}} - 1| = |x + 1| \cdot |x - 1| = 2^{-r-1}. \tag{2.14}$$

Hence, $f$ is transitive modulo $2^{r+2}$ if and only if Condition (2) is true. It remains to prove that $f$ is transitive modulo $2^{l+2}$ for all $l \geq r + 1$ if and only if Condition (3) is valid.

If $s = 2^k - 1\ (mod\ 2^{k+1})$, let $l \in \{r + 1, \dots, k + r - 1\}$. Notice that from (2.3) and (2.4) we also have

$$\begin{aligned} |x^{s^2} - x| &= |x^{s^2-1} - 1| = |x^{\frac{s+1}{2}(s-1)} + 1| \cdot |x^{\frac{s+1}{2}(s-1)} - 1| = \frac{1}{2}|x^{\frac{s+1}{2}(s-1)} - 1| = \dots = \\ &= \frac{1}{2^k}|x^{\frac{s+1}{2^k}(s-1)} - 1| = \frac{1}{2^k}|x^{\frac{s+1}{2^k}\frac{s-1}{2}} + 1| \cdot |x^{\frac{s+1}{2^k}\frac{s-1}{2}} - 1| = \frac{1}{2^k}|x + 1| \cdot |x - 1| = 2^{-k-r-1}. \end{aligned} \tag{2.15}$$

From (2.14) and (2.15) we conclude that the function $g$ is ergodic on each set of the form $(x + 2^{k+r+1}\mathbb{Z}_2) \cup (x^s + 2^{k+r+1}\mathbb{Z}_2)$, where $x \in S_{2^r}(1)$. Hence, $\forall x \in S_{2^r}(1) : g^2(x + 2^{k+r+1}\mathbb{Z}_2) = x + 2^{k+r+1}\mathbb{Z}_2$. It follows that $\forall l \in \{r+1, \dots, k+r-1\}$, $\forall x \in S_{2^r}(1) : g^2(x+2^{l+2}\mathbb{Z}_2) = x+2^{l+2}\mathbb{Z}_2$. Therefore, $\forall l \in \{r+1, \dots, k+r-1\}$, $\forall x \in S_{2^r}(1)$ function $g$ is transitive modulo $2^{l+1}$ on the set $(x + 2^{l+1}\mathbb{Z}_2) \cup (x^s + 2^{l+1}\mathbb{Z}_2)$, but it is not transitive modulo $2^{l+2}$. According to Lemma 2.1, for $a = x$, $b = x^s\ (mod\ 2^{l+1})$ and $k = l + 1$, Condition (1) is verified, but Condition (2) which gives transitivity of $g$ modulo $2^{l+2}$ is not verified.

We get $g(a) + g(b) = a + b\ (mod\ 2^{l+2})$. Namely, if $x$ has the form $x = a = 1 + 2^r + \sum_{i=r+2}^{l} \epsilon_i 2^i$ and $x^s = b\ (mod\ 2^{l+1})$, where $b = 1 + 2^r + 2^{r+1} + \sum_{i=r+2}^{l} \epsilon'_i 2^i$, we must have

$$(1 + 2^r + \sum_{i=r+2}^{l} \epsilon_i 2^i)^s + (1 + 2^r + 2^{r+1} + \sum_{i=r+2}^{l} \epsilon'_i 2^i)^s = (1 + 2^r + \sum_{i=r+2}^{l} \epsilon_i 2^i) + (1 + 2^r + 2^{r+1} + \sum_{i=r+2}^{l} \epsilon'_i 2^i)\ (mod\ 2^{l+2}).$$

This yields

$$\sum_{\epsilon_{r+1},\dots,\epsilon_l\in\{0,1\}} (1 + 2^r + \sum_{i=r+1}^{l} \epsilon_i 2^i)^s = \sum_{\epsilon_{r+1},\dots,\epsilon_l\in\{0,1\}} (1 + 2^r + \sum_{i=r+1}^{l} \epsilon_i 2^i)\ (mod\ 2^{l+2}),$$

which implies that Condition (3) is equivalent to the transitivity of $f$ modulo $2^{l+2}$.

Besides, for $l \geq k+r$, since function $g$ is ergodic on each set of the form $(x + 2^{k+r+1}\mathbb{Z}_2) \cup (x^s + 2^{k+r+1}\mathbb{Z}_2)$, we get for all fixed $\epsilon_{r+2}, \ldots, \epsilon_{k+r} \in \{0,1\}$, if $\epsilon'_{r+2}, \ldots, \epsilon'_{k+r} \in \{0,1\}$ are such that

$$(1 + 2^r + \sum_{i=r+2}^{k+r} \epsilon_i 2^i)^s \in 1 + 2^r + 2^{r+1} + \sum_{i=r+2}^{k+r} \epsilon'_i 2^i + 2^{k+r+1}\mathbb{Z}_2,$$

according to Lemma 2.1 (3) we must have for $l \geq k+r+1$:

$$\sum_{\epsilon_{k+r+1},\ldots,\epsilon_l \in \{0,1\}} \left( (1 + 2^r + \sum_{i=r+2}^{l} \epsilon_i 2^i)^s + (1 + 2^r + 2^{r+1} + \sum_{i=r+2}^{k+r} \epsilon'_i 2^i + \sum_{i=r+k+1}^{l} \epsilon_i 2^i)^s \right)$$

$$= \sum_{\epsilon_{k+r+1},\ldots,\epsilon_l \in \{0,1\}} \left( (1 + 2^r + \sum_{i=r+2}^{l} \epsilon_i 2^i) + (1 + 2^r + 2^{r+1} + \sum_{i=r+2}^{k+r} \epsilon'_i 2^i + \sum_{i=r+k+1}^{l} \epsilon_i 2^i) \right) + 2^{l+1} \ (mod\ 2^{l+2}),$$

and for $l = r+k$, by Lemma 2.1 (2), we have:

$$(1 + 2^r + \sum_{i=r+2}^{r+k} \epsilon_i 2^i)^s + (1 + 2^r + 2^{r+1} + \sum_{i=r+2}^{k+r} \epsilon'_i 2^i)^s$$

$$= (1 + 2^r + \sum_{i=r+2}^{r+k} \epsilon_i 2^i) + (1 + 2^r + 2^{r+1} + \sum_{i=r+2}^{k+r} \epsilon'_i 2^i) + 2^{l+1} \ (mod\ 2^{l+2}).$$

Therefore, for every $l \geq k+r$, we also get that $f$ is transitive modulo $2^{l+2}$ if and only if Condition (3) is true because

$$\sum_{\epsilon_{r+1},\ldots,\epsilon_l \in \{0,1\}} (1 + 2^r + \sum_{i=r+1}^{l} \epsilon_i 2^i)^s = \sum_{\epsilon_{r+1},\ldots,\epsilon_l \in \{0,1\}} (1 + 2^r + \sum_{i=r+1}^{l} \epsilon_i 2^i) + \sum_{\epsilon_{r+2},\ldots,\epsilon_{r+k} \in \{0,1\}} 2^{l+1} \ (mod\ 2^{l+2})$$

$$= \sum_{\epsilon_{r+1},\ldots,\epsilon_l \in \{0,1\}} (1 + 2^r + \sum_{i=r+1}^{l} \epsilon_i 2^i) \ (mod\ 2^{l+2}).$$

□

**Theorem 2.4.** *Let $s = 3\ (mod\ 4)$. Let the functions $f$ and $u$ be defined on $\mathbb{Z}_2$ such that $f(x) = x^s + 4u(x)$. Then, $f$ is ergodic on $S_2(1)$ if and only if $u$ satisfies the following conditions:*

*(1)* $|u(x) - u(y)| < 4|x-y|, \forall x, y \in S_2(1)$,
*(2)* $u(x) = 1\ (mod\ 2), \forall x \in S_2(1)$,
*(3)* $\sum_{\epsilon_2,\ldots,\epsilon_l \in \{0,1\}} u(3 + \sum_{i=2}^{l} \epsilon_i 2^i) = 0\ (mod\ 2^{l+2}), \forall l \geq 2.$

*Proof.* As seen above, $f$ is isometric if and only if Condition (1) is true. Assume that Condition (1) is satisfied. As seen in (2.14) we have that

$$|x^s - x| = |x+1| \cdot |x-1| = \frac{1}{2}|x+1| \leq \frac{1}{8}, \forall x \in S_2(1). \tag{2.16}$$

Therefore, $f$ is transitive modulo 8 if and only if $4u(x) = 4\ \ (mod\ 8)$, which is equivalent to Condition (2).
Recall that for $s = 2^k - 1\ (mod\ 2^{k+1})$, from (2.15) we have

$$|x^{s^2} - x| = \frac{1}{2^k}|x+1| \cdot |x-1| = \frac{1}{2^{k+1}}|x+1|. \tag{2.17}$$

This means that $g$ is ergodic on each set of the form $(x + \frac{2^{k+1}}{|x+1|}\mathbb{Z}_2) \cup (x^s + \frac{2^{k+1}}{|x+1|}\mathbb{Z}_2)$, where $x \in S_2(1)$.

In order to prove that $f$ is transitive modulo $2^{l+2}$, for all $l \geq 2$ if and only if Condition (3) is satisfied it suffices to verify that

$$\sum_{\epsilon_2,\ldots,\epsilon_l \in \{0,1\}} (3 + \sum_{i=2}^{l} \epsilon_i 2^i)^s = \sum_{\epsilon_2,\ldots,\epsilon_l \in \{0,1\}} (3 + \sum_{i=2}^{l} \epsilon_i 2^i) + 2^{l+1} \ (mod\ 2^{l+2}). \tag{2.18}$$

Indeed, take first $l \leq k+1$. For all $t \geq 2$ and $x \in \{|x+1| = 2^{-t}\}$, the function $g$ is not transitive modulo $2^{l+2}$ on $(x + 2^{l+1}\mathbb{Z}_2) \cup (x^s + 2^{l+1}\mathbb{Z}_2)$.

Then, from Lemma 2.1, if $t \leq l-1$, for all fixed $\epsilon_{t+2}, \ldots, \epsilon_l \in \{0,1\}$, if $\epsilon'_{t+2}, \ldots, \epsilon'_l \in \{0,1\}$ are such that

$$(3 + \sum_{i=2}^{t-1} 2^i + \sum_{i=t+2}^{l} \epsilon_i 2^i)^s \in 3 + \sum_{i=2}^{t-1} 2^i + 2^{t+1} + \sum_{i=t+2}^{l} \epsilon'_i 2^i + 2^{l+1}\mathbb{Z}_2,$$

we must have

$$\begin{aligned} &(3 + \sum_{i=2}^{t-1} 2^i + \sum_{i=t+2}^{l} \epsilon_i 2^i)^s + (3 + \sum_{i=2}^{t-1} 2^i + 2^{t+1} + \sum_{i=t+2}^{l} \epsilon'_i 2^i)^s = \\ &3 + \sum_{i=2}^{t-1} 2^i + \sum_{i=t+2}^{l} \epsilon_i 2^i + 3 + \sum_{i=2}^{t-1} 2^i + 2^{t+1} + \sum_{i=t+2}^{l} \epsilon'_i 2^i \ (mod\ 2^{l+2}). \end{aligned} \tag{2.19}$$

Besides, for $t = l$, we have

$$(3 + \sum_{i=2}^{l-1} 2^i)^s = 3 + \sum_{i=2}^{l-1} 2^i + 2^{l+1} \ (mod\ 2^{l+2}). \tag{2.20}$$

While, when $t \geq l+1$, we have from (2.16),

$$x^s = x \ (mod\ 2^{l+2}), \forall x \in \{|x+1| = 2^{-t}\}. \tag{2.21}$$

Using (2.19), (2.20) and (2.21), we get

$$\begin{aligned} \sum_{\epsilon_2,\ldots,\epsilon_l \in \{0,1\}} (3 + \sum_{i=2}^{l} \epsilon_i 2^i)^s &= \sum_{t=2}^{l-1} \sum_{\epsilon_{t+1},\ldots,\epsilon_l \in \{0,1\}} (3 + \sum_{i=2}^{t-1} 2^i + \sum_{i=t+1}^{l} \epsilon_i 2^i)^s + (3 + \sum_{i=2}^{l-1} 2^i)^s + (3 + \sum_{i=2}^{l} 2^i)^s \\ &= \sum_{t=2}^{l-1} \sum_{\epsilon_{t+1},\ldots,\epsilon_l \in \{0,1\}} (3 + \sum_{i=2}^{t-1} 2^i + \sum_{i=t+1}^{l} \epsilon_i 2^i) + (3 + \sum_{i=2}^{l-1} 2^i) + 2^{l+1} + (3 + \sum_{i=2}^{l} 2^i) \ (mod\ 2^{l+2}) \\ &= \sum_{\epsilon_2,\ldots,\epsilon_l \in \{0,1\}} (3 + \sum_{i=2}^{l} \epsilon_i 2^i) + 2^{l+1} \ (mod\ 2^{l+2}). \end{aligned}$$

This proves (2.18) for $l \in \{2, \ldots, k+1\}$.

In a similar way, if $l \geq k+2$ and $t \in \{2, \ldots, l-k\}$, then since $g$ is ergodic on $(x + 2^{t+k+1}\mathbb{Z}_2) \cup (x^s + 2^{t+k+1}\mathbb{Z}_2)$, for $x \in \{|x+1| = 2^{-t}\}$, we get for all fixed $\epsilon_{t+2}, \ldots, \epsilon_{t+k} \in \{0,1\}$, if $\epsilon'_{t+2}, \ldots, \epsilon'_{t+k} \in \{0,1\}$ are such that

$$(3 + \sum_{i=2}^{t-1} 2^i + \sum_{i=t+2}^{t+k} \epsilon_i 2^i)^s \in 3 + \sum_{i=2}^{t-1} 2^i + 2^{t+1} + \sum_{i=t+2}^{t+k} \epsilon'_i 2^i + 2^{t+k+1}\mathbb{Z}_2,$$

we have from Lemma 2.1

$$\sum_{\epsilon_{t+k+1},\dots,\epsilon_l\in\{0,1\}}\left[(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+2}^{l}\epsilon_i2^i)^s+(3+\sum_{i=2}^{t-1}2^i+2^{t+1}+\sum_{i=t+2}^{t+k}\epsilon_i'2^i+\sum_{i=t+k+1}^{l}\epsilon_i2^i)^s\right]$$

$$=\sum_{\epsilon_{t+k+1},\dots,\epsilon_l\in\{0,1\}}\left[(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+2}^{l}\epsilon_i2^i)+(3+\sum_{i=2}^{t-1}2^i+2^{t+1}+\sum_{i=t+2}^{t+k}\epsilon_i'2^i+\sum_{i=t+k+1}^{l}\epsilon_i2^i)\right]+2^{l+1}\ (mod\ 2^{l+2}),$$

where if $t = l - k$, the sum over $\epsilon_{t+k+1},\dots,\epsilon_l \in \{0, 1\}$ contains only one term and $\sum\limits_{i=t+k+1}^{l}\epsilon_i2^i = 0$.

Therefore,

$$\begin{aligned}\sum_{\epsilon_{t+1},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i2^i)^s &= \sum_{\epsilon_{t+1},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i2^i)+\sum_{\epsilon_{t+2},\dots,\epsilon_{t+k}\in\{0,1\}}2^{l+1}\ (mod\ 2^{l+2})\\ &= \sum_{\epsilon_{t+1},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i2^i)\ (mod\ 2^{l+2}).\end{aligned} \tag{2.22}$$

Meanwhile, for $t \in \{l - k + 1, \dots, l - 1\}$, as seen above function $g$ is not transitive modulo $2^{l+2}$ on the set $(x + 2^{l+1}\mathbb{Z}_2) \cup (x^s + 2^{l+1}\mathbb{Z}_2)$, for $x \in \{|x + 1| = 2^{-t}\}$. Hence, for all fixed $\epsilon_{t+2},\dots,\epsilon_l \in \{0, 1\}$, if $\epsilon'_{t+2},\dots,\epsilon'_l \in \{0, 1\}$ are such that

$$(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+2}^{l}\epsilon_i2^i)^s\in 3+\sum_{i=2}^{t-1}2^i+2^{t+1}+\sum_{i=t+2}^{l}\epsilon_i'2^i+2^{l+1}\mathbb{Z}_2,$$

where $\sum\limits_{i=t+2}^{l}\epsilon_i2^i = \sum\limits_{i=t+2}^{l}\epsilon_i'2^i = 0$, for $t = l - 1$, we have

$$\begin{aligned}&(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+2}^{l}\epsilon_i2^i)^s+(3+\sum_{i=2}^{t-1}2^i+2^{t+1}+\sum_{i=t+2}^{l}\epsilon_i'2^i)^s\\ &=(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+2}^{l}\epsilon_i2^i)+(3+\sum_{i=2}^{t-1}2^i+2^{t+1}+\sum_{i=t+2}^{l}\epsilon_i'2^i)\ (mod\ 2^{l+2}).\end{aligned} \tag{2.23}$$

For $t = l$, we get

$$(3+\sum_{i=2}^{l-1}2^i)^s = 3+\sum_{i=2}^{l-1}2^i+2^{l+1}\ (mod\ 2^{l+2}). \tag{2.24}$$

Finally, when $t \geq l + 1$, we have from (2.16)

$$(3+\sum_{i=2}^{t-1}2^i)^s = 3+\sum_{i=2}^{t-1}2^i\ (mod\ 2^{l+2}). \tag{2.25}$$

We conclude from (2.22), (2.23), (2.24) and (2.25)

$$\sum_{\epsilon_2,\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{l}\epsilon_i2^i)^s=\sum_{t=2}^{l-1}\sum_{\epsilon_{t+1},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i2^i)^s+(3+\sum_{i=2}^{l-1}2^i)^s+(3+\sum_{i=2}^{l}2^i)^s=$$

$$=\sum_{t=2}^{l-1}\sum_{\epsilon_{t+1},\dots,\epsilon_l\in\{0,1\}}(3+\sum_{i=2}^{t-1}2^i+\sum_{i=t+1}^{l}\epsilon_i2^i)+(3+\sum_{i=2}^{l-1}2^i)+2^{l+1}+(3+\sum_{i=2}^{l}2^i)\ (mod\ 2^{l+2})$$

$$= \sum_{\epsilon_2,\ldots,\epsilon_l \in \{0,1\}} (3 + \sum_{i=2}^{l} \epsilon_i 2^i) + 2^{l+1} \ (mod\ 2^{l+2}).$$

□

**Corollary 2.5.** *[4, Theorem 4.1.] Let $u$ be a 1-Lipschitz function defined on $\mathbb{Z}_2$. Let $s$ and $r$ be positive integers, then the function $f(x) = x^s + 2^{r+1}u(x)$ is ergodic on $S_{2^r}(1)$ if and only if $s = 1\ (mod\ 4)$ and $u(1) = 1\ (mod\ 2)$.*

*Proof.* Assume first that $f$ is ergodic and $u(1) = 0\ (mod\ 2)$. It is clear that in this case $u$ does not satisfy the conditions of Theorems 2.2 and 2.4. It follows that $s = 3\ (mod\ 4)$ and $r \geq 2$. Meanwhile, we prove that $u$ does not verify Condition (3) of Theorem 2.3. Indeed, since $u$ is 1-Lipschitz

$$u(1 + 2^r) + u(1 + 2^r + 2^{r+1}) = 2u(1 + 2^r)\ (mod\ 4) = 0\ (mod\ 4),$$

which contradicts Condition (3) of Theorem 2.3 for $l = r + 1$.

In this part we assume that $f$ is ergodic and $u(1) = 1\ (mod\ 2)$ and $s = 3\ (mod\ 4)$. By means of Theorem 2.3, we get that $r = 1$. By Theorem 2.4 $u$ satisfies Conditions (1), (2) and (3). Meanwhile,

$$u(3) + u(7) = 2u(3)\ (mod\ 4) = 2\ (mod\ 4),$$

which contradicts Condition (3) of Theorem 2.4.

On the other hand, if $s = 1\ (mod\ 4)$ and $u(1) = 1\ (mod\ 2)$, then we claim that $u$ satisfies all conditions of Theorem 2.2. Indeed, for $l = r$ Condition (2) of Theorem 2.2 is equivalent to $u(1 + 2^r) = 1\ (mod\ 2)$, which is true by assumption.

Suppose that Condition (2) of Theorem 2.2 is satisfied for all $l \in \{r, \ldots, l_0\}$, for some $l_0 \geq r$.

$$\sum_{\epsilon_{r+1},\ldots,\epsilon_{l_0+1} \in \{0,1\}} u(1 + 2^r + \sum_{i=r+1}^{l_0+1} \epsilon_i 2^i) = \sum_{\epsilon_{r+1},\ldots,\epsilon_{l_0} \in \{0,1\}} \left[ u(1 + 2^r + \sum_{i=r+1}^{l_0} \epsilon_i 2^i) + u(1 + 2^r + \sum_{i=r+1}^{l_0} \epsilon_i 2^i + 2^{l_0+1}) \right]$$

$$= 2 \sum_{\epsilon_{r+1},\ldots,\epsilon_{l_0} \in \{0,1\}} u(1 + 2^r + \sum_{i=r+1}^{l_0} \epsilon_i 2^i)\ (mod\ 2^{l_0+1}) = 2^{l_0+1-r}\ (mod\ 2^{l_0+1-r+1}),$$

because

$$u(1 + 2^r + \sum_{i=r+1}^{l_0} \epsilon_i 2^i + 2^{l_0+1}) - u(1 + 2^r + \sum_{i=r+1}^{l_0} \epsilon_i 2^i) = 0\ (mod\ 2^{l_0+1}).$$

This proves Condition (2) for all $l \geq r$. □

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