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A Lower Bound of Normalized Scalar Curvature for the Submanifolds of Locally Conformal Kaehler Space Form Using Casorati Curvatures

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Abstract. In the present paper, we prove the inequality between the normalized scalar curvature and the generalized normalized δ-Casorati curvatures for the submanifolds of locally conformal Kaehler space form and also consider the equality case of the inequality.

1. Introduction

The theory of Chen invariants, one of the most interesting research area of differential geometry started in 1993 by Chen [4]. In the initial paper Chen established inequalities between the scalar curvature, the sectional curvature(intrinsic invariants) and the squared norm of the mean curvature(the main extrinsic invariant) of a submanifold in a real space form. The same author obtained the inequalities for submanifolds between the k-Ricci curvature, the squared mean curvature and the shape operator in the real space form with arbitrary codimension [3]. Since then different geometers proved the similar inequalities for different submanifolds and ambient spaces [1, 2, 9, 12, 13].

The Casorati curvature(extrinsic invariant) of a submanifold of a Riemannian manifold introduced by Casorati defined as the normalized square length of the second fundamental form [8]. The concept of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold [20]. The geometrical meaning and the importance of the Casorati curvature discussed by some distinguished geometers [7, 14, 15, 17, 18]. Therefore it attracts the attention of geometers to obtain the optimal inequalities for the Casorati curvatures of the submanifolds of different ambient spaces [5, 10, 16, 23].

In this paper, we will study the inequalities for the generalized normalized δ-Casorati curvature for the submanifolds of locally conformal Kaehler space forms.

2. Preliminaries

Let $(\widetilde{M}, J, \widetilde{g})$ be a Hermitian manifold equipped with complex structure J and a Hermitian metric $\widetilde{g}$, is called a locally conformal Kaehler manifold if for each point $p \in \widetilde{M}$ has an open neighbourhood U with a differentiable map $\phi : U \to \mathbb{R}$ such that the local metric

$$g = e^{-2\phi}\widetilde{g}_{|U}$$

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is a Kaehler metric on U. The fundamental 2-form Ω of $\widetilde{M}$ is defined as

$$\Omega(X,Y) = g(JX,Y)$$

for any tangent vector field $X, Y \in T\widetilde{M}$. [19]

Proposition 2.1. *[22] A Hermitian manifold $\widetilde{M}$ is a locally conformal Kaehler manifold iff there exist a global 1-form ω, called the lee form, satisfying*

$$\widetilde{g}\big(\big(\widetilde{\nabla}_Z JX, Y\big)\big) = \omega(JX)\widetilde{g}(Y,Z) - \omega(X)\widetilde{g}(JY,Z) - \omega(JY)\widetilde{g}(X,Z) - \omega(Y)\widetilde{g}(JX,Z)$$

for all $X, Y, Z \in T\widetilde{M}$.

The 1-form ω is called the Lee form and its dual vector field is said to be the Lee vector field. On a locally conformal Kaehler manifold, a symmetric (0,2)-tensor $\widetilde{P}$ is defined as

$$\widetilde{P}(X,Y) = -\Big(\widetilde{\nabla}_X\omega\Big)Y - \omega(X)\omega(Y) + \frac{1}{2}\|\omega\|^2\widetilde{g}(X,Y),$$

where $\|\omega\|$ is the length of the Lee form ω with respect to $\widetilde{g}$. The tensor field $\widetilde{P}$ is said to be hybrid if

$$\widetilde{P}(JX,Y) + \widetilde{P}(X,JY) = 0,$$

for $X, Y \in T\widetilde{M}$.

Proposition 2.2. *[11] The Ricci tensor $\widetilde{S}$ in a locally conformal Kaehler manifold $\widetilde{M}$ of real dimension $2m$ satisfies*

$$\widetilde{S}(JX,Y) + \widetilde{S}(X,JY) = 2(m-1)\Big(\widetilde{P}(JX,Y) + \widetilde{P}(X,JY)\Big)$$

for all $X, Y \in T\widetilde{M}$. Thus, the tensor field $\widetilde{P}$ is hybrid iff Ricci tensor $\widetilde{S}$ is hybrid.

The locally conformal Kaehler manifold with constant holomorphic sectional curvature c is called locally conformal Kaehler space form and denoted by $\widetilde{M}(c)$. In the rest part of the paper we assume that $\widetilde{P}$ is hybrid in a locally conformal Kaehler space form.

The curvature tensor $\widetilde{R}$ of $\widetilde{M}(c)$ is given as [11, 21, 22]

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) \quad = \quad & \frac{c}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ & + \frac{c}{4}\{g(JY,Z)g(JX,W) - g(JX,Z)g(JY,W) - 2g(JX,Y)g(JZ,W)\} \\ & + \frac{3}{4}\{g(Y,Z)\widetilde{P}(X,W) - g(X,Z)\widetilde{P}(Y,W) + \widetilde{P}(Y,Z)g(X,W) - \widetilde{P}(X,Z)g(Y,W)\} \\ & - \frac{1}{4}\{g(JY,Z)\widetilde{P}(JX,W) - g(JX,Z)\widetilde{P}(JY,W) + \widetilde{P}(JY,Z)g(JX,W) \\ & - \widetilde{P}(JX,Z)g(JY,W) - 2\widetilde{P}(JX,Y)g(JZ,W) - 2g(JX,Y)\widetilde{P}(JZ,W)\} \qquad (1)\end{aligned}$$

for all $X, Y, Z, W \in T\widetilde{M}$.

Let M be an n-dimensional submanifold of a locally conformal Kaehler space form $\widetilde{M}$. Let ∇ and $\widetilde{\nabla}$ be the Levi-Civita connection on M and $\widetilde{M}$ respectively. The Gauss and Weingarten formulas are defined as

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\widetilde{\nabla}_X \xi = -S_\xi X + \nabla^\perp_X \xi,$$

for vector fields $X, Y \in TM$ and $\xi \in T^{\perp}M$. Where h, S and $\nabla^{\perp}$ is the second fundamental form, the shape operator and the normal connection respectively. The second fundamental form and the shape operator are related by the following equation

$$g(h(X,Y),\xi) = g(S_{\xi}X, Y),$$

for vector fields $X, Y \in TM$ and $\xi \in T^{\perp}M$.

The equation of Gauss is given by

$$R(X,Y,Z,W) = \widetilde{R}(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \tag{2}$$

for $X, Y, Z, W \in TM$, where $\widetilde{R}$ and R represent the curvature tensor of $\widetilde{M}$ and M respectively.

Let M be an n-dimensional submanifold of a locally conformal Kaehler space form $\widetilde{M}$ of dimension m. For any tangent vector field $X \in TM$, we can write $JX = PX + QX$, where P and Q are the tangential and normal components of JX respectively. If $P = 0$, the submanifold is said to be an anti-invariant submanifold and if $Q = 0$, the submanifold is said to be an invariant submanifold. The squared norm of P at $p \in M$ is defined as

$$||P||^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j), \tag{3}$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of the tangent space T_pM.

Let M be a Riemannian manifold and $K(\pi)$ denotes the sectional curvature of M of the plane section $\pi \subset T_pM$ at a point $p \in M$. If $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_m\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$ at any $p \in M$, then the scalar curvature τ at that point is given by

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j)$$

and the normalized scalar curvature ρ is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

The mean curvature vector denoted by H is defined as

$$H = \frac{1}{n}\sum_{i,j=1}^{n} h(e_i, e_i)$$

and also we put

$$h_{ij}^{\gamma} = g(h(e_i, e_j), e_{\gamma}),\ i, j \in 1, 2, .., n,\ \gamma \in \{n+1, ..., m\}.$$

The norm of the squared mean curvature of the submanifold is defined by

$$||H||^2 = \frac{1}{n^2}\sum_{\gamma=n+1}^{m}\left(\sum_{i=1}^{n} h_{ii}^{\gamma}\right)^2$$

and the squared norm of second fundamental form h is denoted by C defined as

$$C = \frac{1}{n}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{\gamma}\right)^2$$

known as Casorati curvature of the submanifold.

If we suppose that L is an r-dimensional subspace of TM, $r \geq 2$, and $\{e_1, e_2, \ldots, e_r\}$ is an orthonormal basis of L. then the scalar curvature of the r-plane section L is given as

$$\tau(L) = \sum_{1 \leq \gamma < \beta \leq r} K(e_\gamma \wedge e_\beta)$$

and the Casorati curvature C of the subspace L is as follows

$$C(L) = \frac{1}{r} \sum_{\gamma=n+1}^{m} \sum_{i,j=1}^{n} \left(h_{ij}^{\gamma}\right)^2$$

A point $p \in M$ is said to be an *invariantly quasi-umbilical point* if there exist $m - n$ mutually orthogonal unit normal vectors $\xi_{n+1}, \ldots, \xi_m$ such that the shape operators with respect to all directions ξ_γ have an eigenvalue of multiplicity $n - 1$ and that for each ξ_γ the distinguished eigen direction is the same. The submanifold is said to be an *invariantly quasi-umbilical submanifold* if each of its points is an invariantly quasi-umbilical point.

The normalized δ-Casorati curvature $\delta_c(n-1)$ and $\widehat{\delta_c}(n-1)$ are defined as

$$[\delta_c(n-1)]_p = \frac{1}{2} C_p + \frac{n+1}{2n} inf\{C(L) | L : \text{a hyperplane of } T_pM\} \tag{4}$$

and

$$[\widehat{\delta_c}(n-1)]_p = 2C_p + \frac{2n-1}{2n} sup\{C(L) | L : \text{a hyperplane of } T_pM\}. \tag{5}$$

Some authors use the coefficient $\frac{n+1}{2n(n-1)}$ instead of $\frac{2n-1}{2n}$ in the equation(5). It was pointed out that the coefficient $\frac{n+1}{2n(n-1)}$ is not suitable and therefore modified by the coefficient $\frac{2n-1}{2n}$. For a positive real number $t \neq n(n-1)$, put

$$a(t) = \frac{1}{nt}(n-1)(n+t)(n^2 - nt) \tag{6}$$

then the generalized normalized δ-Casorati curvatures $\delta_c(t; n-1)$ and $\widehat{\delta_c}(t; n-1)$ are given as

$$[\delta_c(t; n-1)]_p = tC_p + a(t) inf\{C(L) | L : \text{a hyperplane of } T_pM\}$$

if $0 < t < n^2 - n$, and

$$[\widehat{\delta_c}(t; n-1)]_p = rC_p + a(t) sup\{C(L) | L : \text{a hyperplane of } T_pM\}.$$

if $t > n^2 - n$.

3. Main Theorem

Theorem 3.1. *Let M be a submanifold of a locally conformal Kaehler space form $\widetilde{M}$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n-1)$ satisfies

$$\rho \leq \frac{\delta_c(t; n-1)}{n(n-1)} + \frac{c}{4} + \frac{3c}{4n(n-1)} \|P\|^2 + \frac{3}{2n} trace(\widetilde{P}) + \frac{3}{2n(n-1)} \sum_{i,j=1}^{n} g(Pe_i, e_j) \widetilde{P}(e_i, Je_j) \tag{7}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta_c}(t; n-1)$ satisfies

$$\rho \leq \frac{\widehat{\delta_c}(t; n-1)}{n(n-1)} + \frac{c}{4} + \frac{3c}{4n(n-1)} \|P\|^2 + \frac{3}{2n} trace(\widetilde{P}) + \frac{3}{2n(n-1)} \sum_{i,j=1}^{n} g(Pe_i, e_j) \widetilde{P}(e_i, Je_j) \tag{8}$$

for any real number $t > n(n-1)$. Moreover , the equality holds in (7) and (8) iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{M}$, such that with respect to suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \ldots, e_m\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+1, \ldots, m\}$, take the following form

$$S_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{t}a \end{pmatrix}, \qquad S_{n+2} = \cdots = S_m = 0. \tag{9}$$

Proof. Let $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_m\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$ respectively at any point $p \in M$. Putting $X = W = e_i$, $Y = Z = e_j$ in (1) and take $i \neq j$, we have

$$\begin{aligned} \widetilde{R}(e_i, e_j, e_j, e_i) \;=\; & \frac{c}{4}\{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\} \\ & + \frac{c}{4}\{g(Je_j, e_j)g(Je_i, e_i) - g(Je_i, e_j)g(Je_j, e_i) - 2g(Je_i, e_j)g(Je_j, e_i)\} \\ & + \frac{3}{4}\{g(e_j, e_j)\widetilde{P}(e_i, e_i) - g(e_i, e_j)\widetilde{P}(e_j, e_i) + \widetilde{P}(e_j, e_j)g(e_i, e_i) - \widetilde{P}(e_i, e_j)g(e_j, e_i)\} \\ & - \frac{1}{4}\{g(Je_j, e_j)\widetilde{P}(Je_i, e_i) - g(Je_i, e_j)\widetilde{P}(Je_j, e_i) + \widetilde{P}(Je_j, e_j)g(Je_i, e_i) \\ & - \widetilde{P}(Je_i, e_j)g(Je_j, e_i) - 2\widetilde{P}(Je_i, e_j)g(Je_j, e_i) - 2g(Je_i, e_j)\widetilde{P}(Je_j, e_i)\} \end{aligned} \tag{10}$$

From Gauss equation and (10), we have

$$\begin{aligned} & \frac{c}{4}\{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\} + \frac{3c}{4}\{g(Je_i, e_j)g(Je_i, e_j)\} + \\ & \frac{3}{4}\{g(e_j, e_j)\widetilde{P}(e_i, e_i) - g(e_i, e_j)\widetilde{P}(e_j, e_i) + \widetilde{P}(e_j, e_j)g(e_i, e_i) - \widetilde{P}(e_i, e_j)g(e_j, e_i)\} \\ & + \frac{3}{2}\{g(Pe_i, e_j)\widetilde{P}(e_i, Je_j)\} = R(e_i, e_j, e_j, e_i) + g(h(e_j, e_j), h(e_i, e_i)) - g(h(e_i, e_j), h(e_j, e_i)) \end{aligned} \tag{11}$$

By taking summation $1 \leq i, j \leq n$ and using (3) and (11), we get

$$2\tau = n^2\|H\|^2 - nC + \frac{n(n-1)c}{4} + \frac{3c}{4}\|P\|^2 + \frac{3(n-1)}{2}trace(\widetilde{P}) + \frac{3}{2}\sum_{i,j=1}^{n} g(Pe_i, e_j)\widetilde{P}(e_i, Je_j) \tag{12}$$

Define the following function, denoted by $\mathcal{Q}$, a quadratic polynomial in the components of the second fundamental form

$$\mathcal{Q} = tC + a(t)C(L) - 2\tau + \frac{n(n-1)c}{4} + \frac{3c}{4}\|P\|^2 + \frac{3(n-1)}{2}trace(\widetilde{P}) + \frac{3}{2}\sum_{i,j=1}^{n} g(Pe_i, e_j)\widetilde{P}(e_i, Je_j) \tag{13}$$

where L is the hyperplane of T_pM. Without loss of generality, we suppose that L is spanned by $\{e_1, \ldots, e_{n-1}\}$, it follows from (13) that

$$\mathcal{Q} = \frac{n+t}{n}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n}(h_{ij}^{\gamma})^2 + \frac{a(t)}{n-1}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n-1}(h_{ij}^{\gamma})^2 - \sum_{\gamma=n+1}^{m}\left(\sum_{i=1}^{n}h_{ii}^{\gamma}\right)^2$$

which can be easily written as

$$Q = \sum_{\gamma=n+1}^{m} \sum_{i=1}^{n-1} \Big[\Big(\frac{n+t}{n} + \frac{a(t)}{n-1}\Big)(h_{ii}^{\gamma})^2 + \frac{2(n+t)}{n}(h_{in}^{\gamma})^2\Big] + \sum_{n+1}^{m} \Big[2\Big(\frac{n+t}{n} + \frac{a(t)}{n-1}\Big) \sum_{(i<j)=1}^{n} (h_{ij}^{\gamma})^2 - 2\sum_{(i<j)=1}^{n} h_{ii}^{\gamma} h_{jj}^{\gamma} + \frac{t}{n}(h_{nn}^{\gamma})^2\Big] \tag{14}$$

From(14), we can see that the critical points

$$h^c = (h_{11}^{n+1}, h_{12}^{n+1}, \dots, h_{nn}^{n+1}, \dots, h_{11}^{m}, \dots, h_{nn}^{m})$$

of Q are the solutions of the following system of homogenous equations:

$$\begin{cases} \frac{\partial Q}{\partial h_{ii}^{\gamma}} = 2\Big(\frac{n+t}{n} + \frac{a(t)}{n-1}\Big)(h_{ii}^{\gamma}) - 2\sum_{k=1}^{n} h_{kk}^{\gamma} = 0 \\ \frac{\partial Q}{\partial h_{nn}^{\gamma}} = \frac{2t}{n} h_{nn}^{\gamma} - 2\sum_{k=1}^{n-1} h_{kk}^{\gamma} = 0 \\ \frac{\partial Q}{\partial h_{ij}^{\gamma}} = 4\Big(\frac{n+t}{n} + \frac{a(t)}{n-1}\Big)(h_{ij}^{\gamma}) = 0 \\ \frac{\partial Q}{\partial h_{in}^{\gamma}} = 4(\frac{n+t}{n}(h_{in}^{\gamma}) = 0, \end{cases} \tag{15}$$

where $i, j = \{1, 2, \dots, n-1\}, i \neq j$ and $\gamma \in \{n+1, n+2, \dots, m\}$.

Hence, every solution h^c has $h_{ij}^{\gamma} = 0$ for $i \neq j$ and the corresponding determinant to the first two equations of the above system is zero. Moreover, the Hessian matrix of Q is of the following form

$$\mathcal{H}(Q) = \begin{pmatrix} H_1 & O & O \\ O & H_2 & O \\ O & O & H_3 \end{pmatrix}$$

where

$$H_1 = \begin{pmatrix} 2\Big(\frac{n+t}{n} + \frac{a(t)}{n-1}\Big) - 2 & -2 & \dots & -2 & -2 \\ -2 & 2\Big(\frac{n+t}{n} + \frac{a(t)}{n-1}\Big) - 2 & \dots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \dots & 2\Big(\frac{n+t}{n} + \frac{a(t)}{n-1}\Big) - 2 & -2 \\ -2 & -2 & \dots & -2 & \frac{2t}{n} \end{pmatrix}$$

and H_2 and H_3 are the diagonal matrices and O is the null matrix of the respective dimensions. H_2 and H_3 are respectively given as

$$H_2 = diag\Big(4\Big(\frac{n+t}{n} + \frac{a(t)}{n-1}\Big), 4\Big(\frac{n+t}{n} + \frac{a(t)}{n-1}\Big), \dots, 4\Big(\frac{n+t}{n} + \frac{a(t)}{n-1}\Big)\Big)$$

and

$$H_3 = diag\Big(\frac{4(n+t)}{n}, \frac{4(n+t)}{n}, \dots, \frac{4(n+t)}{n}\Big).$$

Hence, we find that $\mathcal{H}(Q)$ has the following eigenvalues

$$\lambda_{11} = 0,\ \lambda_{22} = 2\Big(\frac{2t}{n} + \frac{a(t)}{n-1}\Big),\ \lambda_{33} = \dots = \lambda_{nn} = 2\Big(\frac{n+t}{n} + \frac{a(t)}{n-1}\Big),$$

$$\lambda_{ij} = 4\Big(\frac{n+t}{n} + \frac{a(t)}{n-1}\Big), \lambda_{in} = \frac{4(n+t)}{n}, \ \forall\ i, j \in \{1, 2, \dots, n-1\},\ i \neq j.$$

Thus, Q is parabolic and reaches at minimum $Q(h^c) = 0$ for the solution h^c of the system (15). Hence $Q \geq 0$ and hence

$$2\tau \leq tC + a(t)C(L) + \frac{n(n-1)c}{4} + \frac{3c}{4}\|P\|^2 + \frac{3(n-1)}{2}trace(\widetilde{P}) + \frac{3}{2}\sum_{i,j=1}^{n} g(Pe_i, e_j)\widetilde{P}(e_i, Je_j)$$

whereby, we obtain

$$\rho \leq \frac{t}{n(n-1)}C + \frac{a(t)}{n(n-1)}C(L) + \frac{c}{4} + \frac{3c}{4n(n-1)}\|P\|^2 + \frac{3}{2n}trace(\widetilde{P}) + \frac{3}{2n(n-1)}\sum_{i,j=1}^{n} g(Pe_i, e_j)\widetilde{P}(e_i, Je_j)$$

for every tangent hyperplane L of M. If we take the infimum over all tangent hyperplanes L, the result trivially follows. Moreover the equality sign holds iff

$$h^{\gamma}_{ij} = 0,\ \forall\ i, j \in \{1, \dots, n\},\ i \neq j \text{ and } \gamma \in \{n+1, \dots, m\} \tag{16}$$

and

$$h^{\gamma}_{nn} = \frac{n(n-1)}{t}h^{\gamma}_{11} = \cdots = \frac{n(n-1)}{t}h^{\gamma}_{n-1n-1}, \forall \gamma \in \{n+1, \dots, m\} \tag{17}$$

From (16) and (17), we obtain that the equality holds if and only if the submanifold is invariantly quasi-umbilical with normal connections in $\widetilde{M}$, such that the shape operator takes the form (9) with respect to the orthonormal tangent and orthonormal normal frames.

In the same way, we can prove (ii). □

Corollary 3.2. *Let M be a submanifold of a locally conformal Kaehler space form $\widetilde{M}$. Then*
(i) The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies

$$\rho \leq \delta_c(n-1) + \frac{c}{4} + \frac{3c}{4n(n-1)}\|P\|^2 + \frac{3}{2n}trace(\widetilde{P}) + \frac{3}{2n(n-1)}\sum_{i,j=1}^{n} g(Pe_i, e_j)\widetilde{P}(e_i, Je_j)$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{M}$, such that with respect to suitable tangent orthonormal frame $\{e_1, \dots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \dots, e_m\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+1, \dots, m\}$, take the following form

$$S_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 & 0 \\ 0 & a & 0 & \dots & 0 & 0 \\ 0 & 0 & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & 0 \\ 0 & 0 & 0 & \dots & 0 & 2a \end{pmatrix}, \ S_{n+2} = \cdots = S_m = 0.$$

(ii) The normalized δ-Casorati curvature $\widehat{\delta}_c(n-1)$ satisfies

$$\rho \leq \widehat{\delta}_c(n-1) + \frac{c}{4} + \frac{3c}{4n(n-1)}\|P\|^2 + \frac{3}{2n}trace(\widetilde{P}) + \frac{3}{2n(n-1)}\sum_{i,j=1}^{n} g(Pe_i, e_j)\widetilde{P}(e_i, Je_j)$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{M}$, such that with respect to suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ a and normal orthonormal frame $\{e_{n+1}, \ldots, e_m\}$, the shape operator $S_r \equiv S_{e_r}$, $r \in \{n+1, \ldots, m\}$, take the following form

$$S_{n+1} = \begin{pmatrix} 2a & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2a & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2a & 0 \\ 0 & 0 & 0 & \ldots & 0 & a \end{pmatrix}, S_{n+2} = \cdots = S_m = 0.$$

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