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# A Fixed Point Approach to the Stability of Sextic Lie $*-$Derivations

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**Abstract.** We obtain a general solution of the sextic functional equation $f(ax+by)+f(ax-by)+f(bx+ay)+f(bx-ay) = (ab)^2(a^2+b^2)[f(x+y)+f(x-y)]+2(a^2-b^2)(a^4-b^4)[f(x)+f(y)]$ and investigate the stability of sextic Lie $*$-derivations associated with the given functional equation via fixed point method. Also, we present a counterexample for a single case.

## 1. Introduction

The stability problem of functional equations originated from a question of Ulam ([19]) concerning the stability of group homomorphisms. Hyers ([7]) gave a first affirmative partial answer to the question of Ulam for Banach spaces. Afterwards, the result of Hyers was generalized by Aoki ([1]) for additive mapping. Also, Rassias ([16]) generalized Hyers' Theorem for a unbounded Cauchy difference controlled by $\varepsilon(\|x\|^p+\|y\|^p)$ $(0 \le p < 1)$. Gavruta ([6]) replaced the factor $\|x\|^p+\|y\|^p$ by a general control function $\phi(x,\ y)$. Later, the result of Rassias has provided a lot of influence in the development of what we call Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. In 1996, Isac and Rassias ([9]) were first to provide applications of new fixed point theorems for the proof of stability theory of functional equations. Jang and Park ([10]) investigated the stability of $*$-derivations and of quadratic $*$-derivations with Cauchy functional equation and the Jensen functional equation on Banach $*$-algebra. The stability of $*$-derivations on Banach $*$-algebra by using fixed point alternative was proved by Park and Bodaghi and also Yang et al.; see ([14]) and ([22]), respectively. Also, the stability of cubic Lie derivations was introduced by Fošner and Fošner; see ([5]). For further information about these topics, we also refer the reader to ([11]), ([8]), ([2]), ([3]), ([13]) and ([15]).

Xu and et al. ([20]) introduced the sextic functional equation

$$f(x+3y)+f(x-3y)-6[f(x+2y)+f(x-2y)]+15[f(x+y)+f(x-y)] \tag{1}$$
$$= 20f(x)+720f(y)\,.$$

In particular, Sahoo ([18]) and Xu and Rassias ([21]) determined the general solution of a given functional equation without assuming any regularity conditions on the unknown function. In fact, they proved that the solution of the given functional equation is equivalent to a symmetric and additive function in each variable.

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In this paper, we deal with the following functional equation:

$$f(ax+by)+f(ax-by)+f(bx+ay)+f(bx-ay) \\ = (ab)^2(a^2+b^2)[f(x+y)+f(x-y)]+2(a^2-b^2)(a^4-b^4)[f(x)+f(y)] \quad (2)$$

for all $x, y \in X$ and integers $a$, $b\,(a, b \neq 0, \pm 1$ and $a \neq \pm b)$. We will obtain the general solution of the functional equation (2) by using the symmetric and additive functions and investigate the Hyers-Ulam stability of the sextic Lie $*$-derivations associated with the given functional equation. Also, we will present a counterexample for a single case.

## 2. General Solution of a Sextic Functional Equation

In this section let $X$ and $Y$ be real vector spaces and we investigate the general solution of the functional equation (2). Before we proceed, we would like to introduce some basic definitions concerning $n$-additive symmetric mappings and key concepts which are found in ([18]) and ([21]). A function $A : X \to Y$ is said to be *additive* if $A(x+y) = A(x) + A(y)$ for all $x, y \in X$. Let $n$ be a positive integer. A function $A_n : X^n \to Y$ is called *n-additive* if it is additive in each of its variables. A function $A_n$ is said to be *symmetric* if $A_n(x_1, \cdots, x_n) = A_n(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$ for every permutation $\{\sigma(1), \cdots, \sigma(n)\}$ of $\{1, 2, \cdots, n\}$. If $A_n(x_1, x_2, \cdots, x_n)$ is an $n$-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x, x, \cdots, x)$ and $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and all $r \in \mathbb{Q}$. such a function $A^n(x)$ will be called a *monomial function* of degree $n$ (assuming $A^n \not\equiv 0$). Furthermore the resulting function after substitution $x_1 = x_2 = \cdots = x_s = x$ and $x_{s+1} = x_{s+2} = \cdots = x_n = y$ in $A_n(x_1, x_2, \cdots, x_n)$ will be denoted by $A^{s,n-s}(x, y)$.

**Theorem 2.1.** *A function $f : X \to Y$ is a solution of the functional equation (2) if and only if $f$ is of the form $f(x) = A^6(x)$ for all $x \in X$, where $A^6(x)$ is the diagonal of the 6-additive symmetric mapping $A_6 : X^6 \to Y$.*

*Proof.* Suppose $f$ satisfies the functional equation (2). Letting $x = y = 0$ in the equation (2), we have

$$(4a^6+4b^6-2a^4b^2-2a^2b^4-4)f(0) = 0 \quad (3)$$

for all $x \in X$ and integers $a$, $b\,(a, b \neq 0, \pm 1$ and $a \neq \pm b)$. Hence we get $f(0) = 0$. On taking $y = 0$ and $x = 0$ in the equation (2), we get

$$f(ax)+f(bx) = a^6 f(x) + b^6 f(x) \quad (4)$$

$$f(by)+f(-by)+f(ay)+f(-ay) \\ = (ab)^2(a^2+b^2)[f(y)+f(-y)]+2(a^2-b^2)(a^4-b^4)f(y) \quad (5)$$

for all $x, y \in X$, respectively. Replacing $x$ and $y$ by $-x$ and $x$ in the equations (4) and (5), respectively we have

$$f(-ax)+f(-bx) = a^6 f(-x) + b^6 f(-x) \quad (6)$$

and

$$f(bx)+f(-bx)+f(ax)+f(-ax) \\ = (ab)^2(a^2+b^2)[f(x)+f(-x)]+2(a^2-b^2)(a^4-b^4)f(x) \quad (7)$$

for all $x \in X$, respectively. If both equations (4) and (6) apply to the equation (7), we get

$$(a^6+b^6-a^4b^2-a^2b^4)f(-x)-(a^6+b^6-a^4b^2-a^2b^4)f(x) = 0,$$

that is, $f(-x) = f(x)$ for all $x \in X$. Now, we can rewrite the functional equation (2) in the following form

$$\begin{aligned}
&f(x) - \frac{1}{2(a^2-b^2)(a^4-b^4)}f(ax+by) - \frac{1}{2(a^2-b^2)(a^4-b^4)}f(ax-by) \\
&- \frac{1}{2(a^2-b^2)(a^4-b^4)}f(bx+ay) - \frac{1}{2(a^2-b^2)(a^4-b^4)}f(bx-ay) \\
&+ \frac{(ab)^2}{2(a^2-b^2)^2}f(x+y) + \frac{(ab)^2}{2(a^2-b^2)^2}f(x-y) + f(y) = 0
\end{aligned}$$

for all $x, y \in X$ and integers $a$, $b$ ($a, b \neq 0, \pm 1$ and $a \neq \pm b$). By Theorem 3.5 and 3.6 in ([21]), $f$ is a generalized polynomial function of degree at most 6, that is, $f$ is of the form

$$f(x) = A^6(x) + A^5(x) + A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x)$$

for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of $Y$ and $A^i(x)$ is the diagonal of the $i$-additive symmetric mapping $A_i : X^i \to Y$ for $i = 1, 2, \cdots, 6$. Since $f(0) = 0$ and $f(-x) = f(x)$ for all $x \in X$, we get $A^0(x) = A^0 = 0$ and $A^1(x) = A^3(x) = A^5(x) = 0$. Hence we have

$$f(x) = A^6(x) + A^4(x) + A^2(x),$$

for all $x \in X$. The equations (4), (5) and $A^n(rx) = r^n A^n(x)$ for all $r \in \mathbb{Q}$ imply that

$$A^4(x) = \frac{(a^2+b^2) - (a^6+b^6)}{(a^6+b^6) - (a^4+b^4)} A^2(x)$$

for all $x \in X$ and integers $a$, $b$ ($a, b \neq 0, \pm 1$ and $a \neq \pm b$). Hence $A^4(x) = A^2(x) = 0$, that is, $f(x) = A^6(x)$ for all $x \in X$, as desired. Conversely, assume that $f(x) = A^6(x)$ for all $x \in X$, where $A^6(x)$ is the diagonal of a 6-additive symmetric mapping $A_6 : X^6 \to Y$. Note that

$$\begin{aligned}
A^6(qx+ry) &= q^6A^6(x) + 6q^5rA^{5,1}(x,y) + 15q^4r^2A^{4,2}(x,y) + 20q^3r^3A^{3,3}(x,y) \\
&+ 15q^2r^4A^{2,4}(x,y) + 6qr^5A^{1,5}(x,y) + r^6A^6(y)
\end{aligned}$$

$$c^sA^{s,t}(x,y) = A^{s,t}(cx,y), \quad c^tA^{s,t}(x,y) = A^{s,t}(x,cy)$$

where $1 \leq s, t \leq 5$ and $c \in \mathbb{Q}$. Thus we may conclude that $f$ satisfies the equation (2). □

From now on, we call the mapping $f$ a *generalized sextic mapping* if $f$ satisfies the equation (2).

## 3. Hyers-Ulam-Rassias Stability of Sextic Lie ∗-Derivations

In this section, we will investigate the Hyers-Ulam-Rassias stability of functional equation $f$ in (2) when $b = 1$. Before proceeding this section, we will introduce some definitions and notations. We assume that $A$ is a complex normed ∗-algebra and $M$ is a Banach $A$-bimodule. We will use the same symbol $\|\cdot\|$ as norms on a normed algebra $A$ and a normed $A$-bimodule $M$. A mapping $f : A \to M$ is a *sextic homogeneous mapping* if $f(\mu a) = \mu^6 f(a)$, for all $a \in A$ and $\mu \in \mathbb{C}$. A sextic homogeneous mapping $f : A \to M$ is called a *sextic derivation* if

$$f(xy) = f(x)y^6 + x^6 f(y)$$

holds for all $x, y \in A$. For all $x, y \in A$, the symbol $[x, y]$ will denote the commutator $xy - yx$. We say that a sextic homogeneous mapping $f : A \to M$ is a sextic Lie derivation if

$$f([x, y]) = [f(x), y^6] + [x^6, f(y)]$$

for all $x, y \in A$. In addition, if $f$ satisfies in condition $f(x^*) = f(x)^*$ for all $x \in A$, then it is called the *sextic Lie ∗-derivation*.

**Example 3.1.** *Let $A = \mathbb{C}$ be a complex field endowed with the map $z \mapsto z^* = \bar{z}$ (where $\bar{z}$ is the complex conjugate of $z$). We define $f : A \to A$ by $f(a) = a^6$ for all $a \in A$. Then $f$ is sextic and*

$$f([a,\, b]) = [f(a),\, b^6] + [a^6,\, f(b)] = 0$$

*for all $a \in A$. Also,*

$$f(a^*) = f(\bar{a}) = \bar{a}^6 = \overline{f(a)} = f(a)^*$$

*for all $a \in A$. Thus $f$ is a sextic Lie $*$-derivation.*

In the following, $\mathbb{T}^1$ will stand for the set of all complex units, that is,

$$\mathbb{T}^1 = \{\mu \in \mathbb{C} \,|\, |\mu| = 1\}.$$

For the given mapping $f : A \to M$, we consider

$$\Delta_\mu f(x,y) := f(k\mu x + \mu y) + f(k\mu x - \mu y) + f(\mu x + k\mu y) + f(\mu x - k\mu y) \tag{8}$$
$$-\mu^6 k^2(k^2+1)[f(x+y) + f(x-y)] - 2\mu^6(k^2-1)(k^4-1)[f(x) + f(y)]$$

and

$$\Delta f(x,y) := f([x,\, y]) - [f(x),\, y^6] - [x^6,\, f(y)]$$

for all $x,\, y \in A$, $\mu \in \mathbb{C}$ and $k \in \mathbb{Z}\,(k \neq 0\,, \pm 1)$.

**Theorem 3.2.** *Suppose that $f : A \to M$ is a mapping with $f(0) = 0$ for which there exists a function $\phi : A^5 \to [0,\, \infty)$ such that*

$$\widetilde{\phi}(a,b,x,y,z) := \sum_{j=0}^{\infty} \frac{1}{|k|^{6j}} \phi(k^j a, k^j b, k^j x, k^j y, k^j z) < \infty \tag{9}$$

$$\|\Delta_\mu f(a,b)\| \leq \phi(a,b,0,0,0) \tag{10}$$

$$\|\Delta f(x,y) + f(z^*) - f(z)^*\| \leq \phi(0,0,x,y,z) \tag{11}$$

*for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}} = \{e^{i\theta} \,|\, 0 \leq \theta \leq \frac{2\pi}{n_0}\}$ and all $a,b,x,y,z \in A$ in which $n_0 \in \mathbb{N}$. Also, if for each fixed $a \in A$ the mapping $r \mapsto f(ra)$ from $\mathbb{R}$ to $M$ is continuous, then there exists a unique sextic Lie $*$-derivation $S : A \to M$ satisfying*

$$\|f(a) - S(a)\| \leq \frac{1}{2|k|^6} \widetilde{\phi}(a,0,0,0,0)\,, \tag{12}$$

*for all $a \in A$.*

*Proof.* Letting $b = 0$ and $\mu = 1$ in the inequality (10), we have

$$\|f(a) - \frac{1}{k^6} f(ka)\| \leq \frac{1}{2|k|^6} \phi(a,0,0,0,0) \tag{13}$$

for all $a \in A$. By using the induction, it is easy to show that

$$\|\frac{1}{k^{6n}} f(k^n a) - \frac{1}{k^{6m}} f(k^m a)\| \leq \frac{1}{2|k|^6} \sum_{j=m}^{n-1} \frac{\phi(k^j a,0,0,0,0)}{|k|^{6j}} \tag{14}$$

for $n > m \geq 0$ and $a \in A$ . The inequalities (9) and (14) imply that the sequence $\{\frac{1}{k^{6n}} f(k^n a)\}_{n=0}^{\infty}$ is a Cauchy sequence. Since $M$ is complete, the Cauchy sequence is convergent. Hence we can define a mapping $S : A \to M$ as

$$S(a) = \lim_{n\to\infty} \frac{1}{k^{6n}} f(k^n a) \tag{15}$$

for $a \in A$ . On taking $m = 0$ in the inequality (14), we have

$$\|\frac{1}{k^{6n}} f(k^n a) - f(a)\| \leq \frac{1}{2|k|^6} \sum_{j=0}^{n-1} \frac{\phi(k^j a, 0, 0, 0, 0)}{|k|^{6j}} \tag{16}$$

for $n > 0$ and $a \in A$ . On taking $n \to \infty$ in the inequality (16), the inequalities (9) implies that the inequality (12) holds.

Now, we will show that the mapping $L$ is a unique sextic Lie $*$-derivation satisfying the inequality (12). We note that

$$\|\Delta_\mu S(a,b)\| = \lim_{n\to\infty} \frac{1}{|k|^{6n}} \|\Delta_\mu f(k^n a, k^n b)\| \tag{17}$$

$$\leq \lim_{n\to\infty} \frac{\phi(k^n a, k^n b, 0, 0, 0)}{|k|^{6n}} = 0\,,$$

for all $a, b \in A$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ . On taking $\mu = 1$ in the inequality (17), it follows that the mapping $S$ is a sextic mapping. Also, the inequality (17) implies that $\Delta_\mu S(a, 0) = 0$ . Hence

$$S(\mu a) = \mu^6 S(a)$$

for all $a \in A$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ . Let $\mu \in \mathbb{T}^1 = \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$ . Then $\mu = e^{i\theta}$ , where $0 \leq \theta \leq 2\pi$ . Let $\mu_1 = \mu^{\frac{1}{n_0}} = e^{\frac{i\theta}{n_0}}$ . Hence we have $\mu_1 \in \mathbb{T}^1_{\frac{1}{n_0}}$ . Then

$$S(\mu a) = S(\mu_1^{n_0} a) = \mu_1^{6n_0} S(a) = \mu^6 S(a)$$

for all $\mu \in \mathbb{T}^1$ and $a \in A$ . Suppose that $\rho$ is any continuous linear functional on $A$ and $a$ is a fixed element in $A$ . Then we can define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(r) = \rho(S(ra))$$

for all $r \in \mathbb{R}$ . It is easy to check that $g$ is sextic. Let

$$g_n(r) = \rho\Big(\frac{f(k^n ra)}{k^{6n}}\Big)$$

for all $n \in \mathbb{N}$ and $r \in \mathbb{R}$ .

Note that $g$ is measurable because $g$ is the the pointwise limit of the sequence of measurable functions $g_n$ . Hence $g$ is continuous see ([4]) and

$$g(r) = r^6 g(1)$$

for all $r \in \mathbb{R}$ . Thus

$$\rho(S(ra)) = g(r) = r^6 g(1) = r^6 \rho(S(a)) = \rho(r^6 S(a))$$

for all $r \in \mathbb{R}$ . Since $\rho$ was an arbitrary continuous linear functional on $A$ we may conclude that

$$S(ra) = r^6 S(a)$$

for all $r \in \mathbb{R}$. Let $\mu \in \mathbb{C}\,(\mu \neq 0)$. Then $\frac{\mu}{|\mu|} \in \mathbb{T}^1$. Hence

$$S(\mu a) = S\Big(\frac{\mu}{|\mu|}|\mu|a\Big) = \Big(\frac{\mu}{|\mu|}\Big)^6 S(|\mu|a) = \Big(\frac{\mu}{|\mu|}\Big)^6 |\mu|^6 S(a) = \mu^6 S(a)$$

for all $a \in A$ and $\mu \in \mathbb{C}\,(\mu \neq 0)$. Since $a$ was an arbitrary element in $A$, we may conclude that $S$ is sextic homogeneous.

Next, replacing $x$ and $y$ by $k^n x$ and $k^n y$, respectively, and letting $z = 0$ in the inequality (11), we have

$$\begin{aligned} \|\Delta S(x, y)\| &= \lim_{n\to\infty} \|\frac{\Delta f(k^n x, k^n y)}{k^{6n}}\| \\ &\leq \lim_{n\to\infty} \frac{1}{|k|^{6n}} \phi(0, 0, k^n x, k^n y, 0) = 0 \end{aligned}$$

for all $x, y \in A$. Hence we have $\Delta S(x, y) = 0$ for all $x, y \in A$. That is, $S$ is a sextic Lie derivation. Letting $x = y = 0$ and replacing $z$ by $k^n z$ in the inequality (11), we get

$$\Big\|\frac{f(k^n z^*)}{k^{6n}} - \frac{f(k^n z)^*}{k^{6n}}\Big\| \leq \frac{\phi(0, 0, 0, 0, k^n z)}{|k|^{6n}} \tag{18}$$

for all $z \in A$. As $n \to \infty$ in the inequality (18), we have

$$S(z^*) = S(z)^*$$

for all $z \in A$. This means that $S$ is a sextic Lie $*$-derivation. Now, assume $S' : A \to A$ is another sextic $*$-derivation satisfying the inequality (12). Then

$$\begin{aligned} \|S(a) - S'(a)\| &= \frac{1}{|k|^{6n}} \|S(k^n a) - S'(k^n a)\| \\ &\leq \frac{1}{|k|^{6n}} \Big(\|S(k^n a) - f(k^n a)\| + \|f(k^n a) - S'(k^n a)\|\Big) \\ &\leq \frac{1}{2|k|^{6n+1}} \sum_{j=0}^{\infty} \frac{1}{|k|^{6j}} \phi(k^{j+n} a, 0, 0, 0, 0) \\ &\leq \frac{1}{2|k|^{6n+1}} \widetilde{\phi}(k^n a, 0, 0, 0, 0), \end{aligned}$$

which tends to zero as $n \to \infty$, for all $a \in A$. Thus $S(a) = S'(a)$ for all $a \in A$. This proves the uniqueness of $S$. □

**Corollary 3.3.** *Let $\theta$, $r$ be positive real numbers with $r < 6$ and let $f : A \to M$ be a mapping with $f(0) = 0$ such that*

$$\|\Delta_\mu f(a, b)\| \leq \theta(\|a\|^r + \|b\|^r) \tag{19}$$

$$\|\Delta f(x, y) + f(z^*) - f(z)^*\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{20}$$

*for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a, b, x, y, z \in A$. Then there exists a unique sextic Lie $*$-derivation $S : A \to M$ satisfying*

$$\|f(a) - S(a)\| \leq \frac{\theta\|a\|^r}{2(|k|^6 - |k|^r)}$$

*for all $a \in A$.*

*Proof.* On taking $\phi(a,b,x,y,z) = \theta(\|a\|^r + \|b\|^r + \|x\|^r + \|y\|^r + \|z\|^r)$ for all $a,b,x,y,z \in A$, it is easy to show that the inequalities (19) and (20) hold. Similar to the proof of Theorem 3.2, we have

$$\begin{aligned}\|f(a) - S(a)\| &\leq \frac{1}{2|k|^6}\widetilde{\phi}(a,0,0,0,0,) \\ &= \frac{\theta\|a\|^r}{2|k|^6}\sum_{j=0}^{\infty}\Big(\frac{|k|^r}{|k|^6}\Big)^j \\ &= \frac{\theta\|a\|^r}{2|k|^6}\frac{1}{1-\frac{|k|^r}{|k|^6}} = \frac{\theta\|a\|^r}{2(|k|^6 - |k|^r)}\end{aligned}$$

for all $a \in A$ and $r < 6$. □

In the following corollaries, we show the hyperstability for the sextic Lie $*$-derivations.

**Corollary 3.4.** *Let $r$ be positive real numbers with $r < 6$ and let $f : A \to M$ be a mapping with $f(0) = 0$ such that*

$$\|\Delta_\mu f(a,b)\| \leq \|a\|^r\|b\|^r$$

$$\|\Delta f(x,y) + f(z^*) - f(z)^*\| \leq \|x\|^r\|y\|^r\|z\|^r$$

*for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a,b,x,y,z \in A$. Then $f$ is a sextic Lie $*$-derivation on $A$.*

*Proof.* On taking $\phi(a,b,x,y,z) = (\|a\|^r + \|x\|^r)(\|b\|^r + \|y\|^r\|z\|^r)$, we have

$$\|\Delta_\mu f(a,b)\| \leq \phi(a,b,0,0,0) = \|a\|^r\|b\|^r$$

$$\|\Delta f(x,y) + f(z^*) - f(z)^*\| \leq \phi(0,0,x,y,z) = \|x\|^r\|y\|^r\|z\|^r$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a,b,x,y,z \in A$. Similar to the proof of Theorem 3.2, we have

$$\begin{aligned}\|f(a) - S(a)\| &\leq \frac{1}{2|k|^6}\widetilde{\phi}(a,0,0,0,0) \\ &= \frac{1}{2|k|^6}\sum_{j=0}^{\infty}\frac{1}{|k|^{6j}}\phi(a,0,0,0,0) = 0\end{aligned}$$

for all $a \in A$ and $r < 6$. Hence the inequality (12) implies that $f = S$, that is, $f$ is a sextic Lie $*$-derivation on $A$. □

**Corollary 3.5.** *Let $r$ be positive real numbers with $r < 6$ and let $f : A \to M$ be a mapping with $f(0) = 0$ such that*

$$\|\Delta_\mu f(a,b)\| \leq \|a\|^r\|b\|^r \tag{21}$$

$$\|\Delta f(x,y) + f(z^*) - f(z)^*\| \leq \|x\|^r(\|y\|^r + \|z\|^r) \tag{22}$$

*for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a,b,x,y,z \in A$. Then $f$ is a sextic Lie $*$-derivation on $A$.*

*Proof.* On taking $\phi(a,b,x,y,z) = (\|a\|^r + \|x\|^r)(\|b\|^r + \|y\|^r + \|z\|^r)$, it is easy to show that the inequalities (21) and (22) hold. Similar to the proof of Theorem 3.2, we my conclude that the inequality 12) is true, that is,

$$\|f(a) - S(a)\| \leq \frac{1}{2|k|^6}\sum_{j=0}^{\infty}\frac{1}{|k|^{6j}}\phi(a,0,0,0,0) = 0$$

for all $a \in A$ and $r < 6$. Hence the inequality (12) implies that $f = S$, that is, $f$ is a sextic Lie $*$-derivation on $A$. □

## 4. Stability of Sextic Lie *-Derivations via a Fixed Point Method

In this section, we will investigate the stability of the given functional equation (8) using the alternative fixed point method. Before proceeding the proof, we will state the theorem, the alternative of fixed point; see ([12]) and ([17]).

**Definition 4.1.** *Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies*
*(1) $d(x, y) = 0$ if and only if $x = y$;*
*(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;*
*(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.*

**Theorem 4.2 ( The alternative of fixed point ([12]), ([17]) ).** *Suppose that we are given a complete generalized metric space $(\Omega, d)$ and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant $l$. Then for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \text{ for all } n \ge 0,$$

*or there exists a natural number $n_0$ such that*

1. *$d(T^n x, T^{n+1} x) < \infty$ for all $n \ge n_0$;*
2. *The sequence $(T^n x)$ is convergent to a fixed point $y^*$ of $T$;*
3. *$y^*$ is the unique fixed point of $T$ in the set*

$$\triangle = \{y \in \Omega | d(T^{n_0} x, y) < \infty\};$$

4. *$d(y, y^*) \le \frac{1}{1-l} d(y, Ty)$ for all $y \in \triangle$.*

**Theorem 4.3.** *Let $f : A \to M$ be a continuous mapping with $f(0) = 0$ and let $\phi : A^5 \to [0, \infty)$ be a continuous mapping such that*

$$\|\Delta_\mu f(a, b)\| \le \phi(a, b, 0, 0, 0) \tag{23}$$

$$\|\Delta f(x, y) + f(z^*) - f(z)^*\| \le \phi(0, 0, x, y, z) \tag{24}$$

*for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a, b, x, y, z \in A$. If there exists a constant $l \in (0, 1)$ such that*

$$\phi(ka, kb, kx, ky, kz) \le |k|^6 l \phi(a, b, x, y, z) \tag{25}$$

*for all $a, b, x, y, z \in A$, then there exists a sextic Lie $*$-derivation $S : A \to M$ satisfying*

$$\|f(a) - S(a)\| \le \frac{1}{2|k|^6(1-l)} \phi(a, 0, 0, 0, 0) \tag{26}$$

*for all $a \in A$.*

*Proof.* Consider the set

$$\Omega = \{g \,|\, g : A \to A, g(0) = 0\}$$

and introduce the generalized metric on $\Omega$,

$$d(g, h) = \inf\{c \in (0, \infty) \,|\, \| g(a) - h(a) \| \le c\phi(a, 0, 0, 0, 0), \text{for all } a \in A\}.$$

It is easy to show that $(\Omega, d)$ is complete. Now we define a function $T : \Omega \to \Omega$ by

$$T(g)(a) = \frac{1}{k^6} g(ka) \tag{27}$$

for all $a \in A$. Note that for all $g, h \in \Omega$, let $c \in (0, \infty)$ be an arbitrary constant with $d(g,h) \leq c$. Then

$$\|g(a) - h(a)\| \leq c\phi(a,0,0,0,0) \tag{28}$$

for all $a \in A$. Letting $a = ka$ in the inequality (28) and using both inequalities (25) and (27), we have

$$\begin{aligned} \|T(g)(a) - T(h)(a)\| &= \frac{1}{|k|^6}\|g(ka) - h(ka)\| \\ &\leq \frac{1}{|k|^6} c\,\phi(ka,0,0,0,0) \leq c\,l\,\phi(a,0,0,0,0), \end{aligned}$$

that is,

$$d(Tg,\, Th) \leq c\,l.$$

Hence we have that

$$d(Tg,\, Th) \leq l\,d(g,\, h),$$

for all $g, h \in \Omega$, that is, $T$ is a strictly self-mapping of $\Omega$ with the Lipschitz constant $l$. Letting $\mu = 1$, $b = 0$ in the inequality (23), we get

$$\|\frac{1}{k^6} f(ka) - f(a)\| \leq \frac{1}{2|k|^6}\phi(a,0,0,0,0)$$

for all $a \in A$. This means that

$$d(Tf, f) \leq \frac{1}{2|k|^6}.$$

Since $\lim_{n\to\infty} d(T^n f, S) = 0$, there exists a fixed point $S$ of $T$ in $\Omega$ such that

$$S(a) = \lim_{n\to\infty} \frac{f(k^n a)}{k^{6n}}, \tag{29}$$

for all $a \in A$. Hence

$$d(f,S) \leq \frac{1}{1-l} d(Tf,f) \leq \frac{1}{2|k|^6}\frac{1}{1-l}.$$

This implies that the inequality (26) holds for all $a \in A$. Since $l \in (0,1)$, the inequality (25) shows that

$$\lim_{n\to\infty} \frac{\phi(k^n a, k^n b, k^n x, k^n y, k^n z)}{|k|^{6n}} = 0. \tag{30}$$

Replacing $a$ and $b$ by $k^n a$ and $k^n b$, respectively, in the inequality (23), we have

$$\frac{1}{|k|^{6n}}\|\Delta_\mu f(k^n a, k^n b)\| \leq \frac{\phi(k^n a, k^n b, 0,0,0)}{|k|^{6n}}.$$

On taking the limit as $n$ tend to infinity, we have $\Delta_\mu f(a,b) = 0$ for all $a, b \in A$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. The remains are similar to the proof of Theorem 3.2. □

**Corollary 4.4.** *Let $\theta$, $r$ be positive real numbers with $r < 6$ and let $f : A \to M$ be a mapping with $f(0) = 0$ such that*

$$\|\Delta_\mu f(a,b)\| \leq \theta(\|a\|^r + \|b\|^r) \tag{31}$$

$$\|\Delta f(x,y) + f(z^*) - f(z)^*\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{32}$$

*for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a,b,x,y,z \in A$. Then there exists a unique sextic Lie $*$-derivation $S : A \to M$ satisfying*

$$\|f(a) - S(a)\| \leq \frac{\theta\|a\|^r}{2|k|^6(1-l)}$$

*for all $a \in A$.*

*Proof.* On taking $\phi(a,b,x,y,z) = \theta(\|a\|^r + \|b\|^r + \|x\|^r + \|y\|^r + \|z\|^r)$ for all $a,b,x,y,z \in A$, it is easy to show that the inequalities (31) and (32) hold. Similar to the proof of Theorem 4.3, we have

$$\|f(a) - S(a)\| \le \frac{1}{2|k|^6(1-l)}\phi(a,0,0,0,0,) = \frac{\theta\|a\|^r}{2|k|^6(1-l)}$$

for all $a \in A$ and $r < 6$. □

In the following corollaries, we show the hyperstability for the sextic Lie ∗-derivations.

**Corollary 4.5.** *Let r be positive real numbers with $r < 6$ and let $f : A \to M$ be a mapping with $f(0) = 0$ such that*

$$\|\Delta_\mu f(a,b)\| \le \|a\|^r\|b\|^r$$

$$\|\Delta f(x,y) + f(z^*) - f(z)^*\| \le \|x\|^r\|y\|^r\|z\|^r$$

*for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a,b,x,y,z \in A$. Then f is a sextic Lie ∗-derivation on A.*

*Proof.* On taking $\phi(a,b,x,y,z) = (\|a\|^r + \|x\|^r)(\|b\|^r + \|y\|^r\|z\|^r)$ in Theorem 4.3 for all $a,b,x,y,z \in A$, we have $\widetilde{\phi}(a,0,0,0,0) = 0$. Hence the inequality (26) implies that $f = S$, that is, $f$ is a sextic Lie ∗-derivation on $A$. □

**Corollary 4.6.** *Let r be positive real numbers with $r < 6$ and let $f : A \to M$ be a mapping with $f(0) = 0$ such that*

$$\|\Delta_\mu f(a,b)\| \le \|a\|^r\|b\|^r$$

$$\|\Delta f(x,y) + f(z^*) - f(z)^*\| \le \|x\|^r(\|y\|^r + \|z\|^r)$$

*for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a,b,x,y,z \in A$. Then f is a sextic Lie ∗-derivation on A.*

*Proof.* On taking $\phi(a,b,x,y,z) = (\|a\|^r + \|x\|^r)(\|b\|^r + \|y\|^r + \|z\|^r)$ in Theorem 4.3 for all $a,b,x,y,z \in A$, we have $\widetilde{\phi}(a,0,0,0,0) = 0$. Hence the inequality (26) implies that $f = S$, that is, $f$ is a sextic Lie ∗-derivation on $A$. □

## 5. Counterexample

In this section, we will present a counterexample to show that the functional equation (2) is not stable for $r = 6$ and $\mu = 1$ in Corollary 3.3.

**Example 5.1.** *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a mapping defined by*

$$\phi(x) = \begin{cases} \theta x^6 & \text{for } |x| < 1 \\ \theta & \text{otherwise} \end{cases}$$

*where $\theta > 0$ is a constant and a mapping $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \sum_{i=0}^{\infty} \frac{\phi(k^i x)}{k^{6i}} \tag{33}$$

*for all $x \in \mathbb{R}$. Then the mapping f satisfies the inequality*

$$|\Delta_1 f(x,y)| \le (2k^6 - k^4 - k^2 + 4)\frac{2k^{18}\theta}{k^6 - 1}(|x|^6 + |y|^6) \tag{34}$$

*for all $x \in \mathbb{R}$. Then there does not exist a sextic mapping $S : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that*

$$|f(x) - S(x)| \le \beta|x|^6 \tag{35}$$

*for all $x \in \mathbb{R}$.*

*Proof.* The definitions of $\phi$ and $f$ imply that

$$|f(x)| = \Big|\sum_{i=0}^{\infty}\frac{\phi(k^i x)}{k^{6i}}\Big| \le \sum_{i=0}^{\infty}\frac{\theta}{k^{6i}} = \frac{\theta k^6}{k^6-1}$$

for all $x \in \mathbb{R}$. Hence $f$ is bounded by $\frac{\theta k^6}{k^6-1}$. If $|x|^6+|y|^6 \ge 1$, then the inequality (34) holds. Now, we suppose that $0 < |x|^6+|y|^6 < 1$. Then there exists a positive integer $t$ such that

$$\frac{1}{k^{6(t+2)}} \le |x|^6+|y|^6 < \frac{1}{k^{6(t+1)}}\,. \tag{36}$$

Since $|x|^6+|y|^6 < \frac{1}{k^{6(t+1)}}$ we have

$$k^{6t}x^6 < \frac{1}{k^6} \text{ and } k^{6t}y^6 < \frac{1}{k^6}\,.$$

That is,

$$k^t x < \frac{1}{k} \text{ and } k^t y < \frac{1}{k}\,.$$

These imply that $k^{t-1}x, k^{t-1}y, k^{t-1}(x+y), k^{t-1}(x-y), k^{t-1}(kx+y), k^{t-1}(kx-y), k^{t-1}(x+ky), k^{t-1}(x-ky) \in (-1,1)$. Hence we obtain that $k^jx, k^jy, k^j(x+y), k^j(x-y), k^j(kx+y), k^j(kx-y), k^j(x+ky), k^j(x-ky) \in (-1,1)$ for each $j = 0,1,\cdots,t-1$. Also, for each $j = 0,1,\cdots,t-1$,

$$\begin{aligned}&\phi(k^j(kx+y)) + \phi(k^j(kx-y)) + \phi(k^j(x+ky)) + \phi(k^j(x-ky))\\ &\quad -k^2(k^2+1)[\phi(k^j(x+y)) + \phi(k^j(x-y))]\\ &\quad -2(k^2-1)(k^4-1)[\phi(k^jx)+\phi(k^jy)] = 0\,.\end{aligned}$$

From the definition of $f$ and the inequality (36), we have

$$\begin{aligned}|\Delta_1 f(x,y)| &\le \sum_{j=0}^{\infty}\Big\{\phi(k^j(kx+y)) + \phi(k^j(kx-y))\\ &\quad +\phi(k^j(x+ky)) + \phi(k^j(x-ky))\\ &\quad -k^2(k^2+1)[\phi(k^j(x+y)) + \phi(k^j(x-y))]\\ &\quad -2(k^2-1)(k^4-1)[\phi(k^jx)+\phi(k^jy)]\Big\}\\ &\le \sum_{j=t}^{\infty}\frac{2\theta(2k^6-k^4-k^2+4)}{k^{6j}}\\ &\le 2\theta k^{12}(2k^6-k^4-k^2+4)\frac{1}{k^{6(t+2)}}\frac{k^6}{k^6-1}\\ &\le (2k^6-k^4-k^2+4)\frac{2k^{18}\theta}{k^6-1}(|x|^6+|y|^6)\,.\end{aligned}$$

We claim that the sextic functional equation (2) is not stable for $r = 6$ and $\mu = 1$ in Corollary 3.3. Assume that there exists a sextic mapping $S : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying the inequality (35). Since $f$ is bounded and continuous for all $x \in \mathbb{R}$, $S$ is bounded on any open interval containing the origin and continuous at the origin. In view of Corollary 3.3, $S(x)$ must have the form $S(x) = \gamma x^6$ for all $x \in \mathbb{R}$. Hence we have that

$$|f(x)| \le (\beta + |\gamma|)|x|^6\,. \tag{37}$$

But we can choose a positive integer $m$ with $m\theta > \beta + |\gamma|$. If $x \in (0, \frac{1}{k^{6(m-1)}})$, then $k^{6t} \in (0,1)$ for all $t = 0,1,\cdots,m-1$. For this $x$, we have

$$f(x) = \sum_{i=0}^{\infty}\frac{\phi(k^ix)}{k^{6i}} \ge \sum_{i=0}^{m-1}\frac{\theta(k^ix)^6}{k^{6i}} = m\theta x^6 > (\beta+|\gamma|)x^6$$

This implies that it is a contradiction to the inequality (37). Therefore the sextic functional equation (2) is not stable when $r = 6$ and $\mu = 1$ . □

## References


[1] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan 2 (1950), 64–66.
[2] N. Brillouët-Belluot, J. Brzdęk and K. Ciepliński, Fixed point theory and the Ulam stability, Abstract and Applied Analysis 2014, Article ID 829419, 16 pages (2014).
[3] J. Brzdęk, L. Cădariu and K. Ciepliński, On some recent developments in Ulam's type stability, Abstract and Applied Analysis 2012, Article ID 716936, 41 pages (2012).
[4] St. Czerwik, On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg 62 (1992), 59–64.
[5] A. Fošner and M. Fošner, Approximate cubic Lie derivations, Abstract and Applied Analysis Article ID 425784, 5 pages (2013).
[6] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994), 431–436.
[7] D. H. Hyers, On the stability of the linear equation, Proc. Nat. Acad. Sci. U.S.A. 27 (1941), 222–224.
[8] D. H. Hyers, G. Isac, and T. M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Boston, USA, 1998.
[9] G. Isac and Th. M. Rassias, Stability of $\pi$-additive mappings: Applications to nonlinear analysis, Internat. J. Math. Math. Sci. 19 (1996), 219–228.
[10] S. Jang and C. Park, Approximate $*$-derivations and approximate quadratic $*$-derivations on $C^*$-algebra, J. Inequal. Appl. Articla ID 55 (2011).
[11] S.-M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis, vol. 48 of Springer Optimization and Its Applications, Springer, New York, USA, 2011.
[12] B. Margolis and J.B. Diaz, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc. 126 (1968), 305–309.
[13] C. Park, Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras, Fixed Point Theory Appl. (2007), Art. ID 50175.
[14] C. Park and A. Bodaghi, On the stability of $*$-derivations on Banach $*$-algebras, Adv. Diff. Equat. 2012:138 (2012).
[15] V. Radu, The fixed point alternative and the stability of functional equations, Fixed Point Theory 4 (2003), 91–96.
[16] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300.
[17] I. A. Rus, Principles and Appications of Fixed Point Theory, Ed. Dacia, Cluj-Napoca, 1979 (in Romanian).
[18] P. K. Sahoo, A generalized cubic functional equation, Acta Math. Sinica 21 (2005), 1159–1166.
[19] S. M. Ulam, Problems in Morden Mathematics, Wiley, New York, USA, 1960.
[20] T. Z. Xu, J. M. Rassias, M. J. Rassias and W. X. Xu, A Fixed Point Approach to the Stability of Quintic and Sextic Functional Equations in Quasi-$\beta$-Normed Spaces, Journal of Inequalities and Applications 2010.
[21] T. Z. Xu, J. M. Rassias and W. X. Xu, A generalized mixed quadratic-quartic functional equation, Bull. Malaysian Math. Scien. Soc. 35 (2012), 633–649.
[22] S. Y. Yang, A. Bodaghi, K. A. M. Atan, Approximate cubic *-derivations on Banach *-algebra, Abstract and Applied Analysis, Article ID 684179, 12 pages (2012).