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Evaluation of Hessenberg Determinants via Generating Function Approach

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Abstract. In this paper, we will present various results on computing of wide classes of Hessenberg matrices whose entries are the terms of any sequence. We present many new results on the subject as well as our results will cover and generalize earlier many results by using generating function method. Moreover, we will present a new approach on computing Hessenberg determinants, whose entries are general higher order linear recursions with arbitrary constant coefficients, based on finding an *adjacency-factor* matrix. We will give some interesting showcases to show how to use our new method.

1. Introduction

The $n \times n$ lower Hessenberg matrix H_n is defined as follows

$$H_n = \begin{bmatrix} h_{11} & h_{12} & & & & 0 \\ h_{21} & h_{22} & h_{23} & & & \\ h_{31} & h_{32} & h_{33} & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ h_{n-1,1} & h_{n-1,2} & h_{n-1,3} & \cdots & \ddots & h_{n-1,n} \\ h_{n1} & h_{n2} & h_{n3} & \cdots & \cdots & h_{nn} \end{bmatrix}.$$

Similarly, the $n \times n$ upper Hessenberg matrix is considered as transpose of the matrix H_n. Throughout the paper, we are interested in a lower Hessenberg matrix so in fact our results will be also valid for an upper Hessenberg matrix. Hessenberg matrices are one of the important matrices in numerical analysis [7, 9]. For example, the Hessenberg decomposition played an important role in the matrix eigenvalues computation [9].

The authors of [1, 3, 5, 13, 15, 17, 18, 24, 25] studied algebraic properties of some Hessenberg matrices such as inverses, determinants, permanents etc. For example, Cahill et al. [3] gave a recurrence relation for the determinant of the matrix H_n as follows

$$\det H_n = h_{nn} \det H_{n-1} + \sum_{r=1}^{n-1} \left((-1)^{n-r} h_{nr} \prod_{j=r}^{n-1} h_{j,j+1} \det H_{r-1} \right),$$

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where $H_0 = 1$ for $n > 0$.

Meanwhile some authors computed determinants and permanents of various tridiagonal matrices which are in fact Hessenberg matrices [4, 12, 14, 19–21]. For example, in [14], Kılıç et al. gave the following result

$$\begin{vmatrix} 2 & 1 & & & 0 \\ -1 & 2 & 1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & -1 & 2 \end{vmatrix} = P_{n+1},$$

where P_n is the nth Pell number.

Moreover the authors of [6, 7] gave closed formulas for the inverses of some Hessenberg matrices as well as algorithms to compute their inverses and determinants. The authors of [2, 11] gave *combinatorial approach* to compute the determinants of some Hessenberg matrices.

For $n \geq k$ and any reals c_i, $1 \leq i \leq k$, define the kth order linear recursive sequence $\{u_n\}$ with constant coefficients as

$$u_n = c_1 u_{n-1} + c_2 u_{n-2} + c_3 u_{n-3} + \cdots + c_k u_{n-k}, \tag{1}$$

with arbitrary initials u_t for $0 \leq t < k$ and assumed that at least one of them is different from zero.

We give the following table for some special cases of the sequence $\{u_n\}$:

Order	Coefficients	Initials	
2	$c_1 = c_2 = 1$	$u_0 = 0, u_1 = 1$	Fibonacci sequence $\{F_n\}$
2	$c_1 = p,\ c_2 = q$	$u_0 = 0, u_1 = 1$	Gen. Fibonacci sequence $\{U_n\}$
2	$c_1 = 2,\ c_2 = 1$	$u_0 = 0, u_1 = 1$	Pell sequence $\{P_n\}$
2	$c_1 = c_2 = 1$	$u_0 = 2, u_1 = 1$	Lucas sequence $\{L_n\}$
2	$c_1 = p,\ c_2 = q$	$u_0 = 2, u_1 = p$	Gen. Lucas sequence $\{V_n\}$
2	$c_1 = p,\ c_2 = -q$	$u_0 = a, u_1 = b$	Horadam sequence $\{W_n\}$
2	$c_1 = 1,\ c_2 = 2$	$u_0 = 0, u_1 = 1$	Jacobsthal sequence $\{J_n\}$
3	$c_1 = c_2 = c_3 = 1$	$u_0 = u_1 = 0, u_2 = 1$	Tribonacci sequence $\{T_n\}$

Table 1

Recently, Macfarlane [22] considered the following Hessenberg matrix whose entries consist of the terms of the sequence $\{W_n\}$:

$$A_n = \begin{bmatrix} W_1 & W_2 & W_3 & \cdots & W_{n-2} & W_{n-1} & W_n \\ -x & W_1 & W_2 & \cdots & W_{n-3} & W_{n-2} & W_{n-1} \\ & -x & W_1 & \cdots & W_{n-4} & W_{n-3} & W_{n-2} \\ & & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & \ddots & W_1 & W_2 & W_3 \\ & & & & -x & W_1 & W_2 \\ 0 & & & & & -x & W_1 \end{bmatrix},$$

where $\{W_n\}$ is the Horadam sequence as in Table 1. By using *the cofactor expansion* of the determinant, he showed that the sequence $\{\det A_n\}$ satisfies the recurrence for $n > 2$,

$$\det A_n = (b + px) \det A_{n-1} - qx\,(a + x) \det A_{n-2}.$$

For any sequence $\{a_n\}$, the generating function of $\{a_n\}$ is the power series [27]:

$$A(x) = \sum_{k \geq 0} a_k x^k.$$

For example, the generating function of the Fibonacci sequence $\{F_n\}$ is

$$F(x) = \sum_{k\geq 0} F_k x^k = \frac{x}{1-x-x^2}.$$

In general, the generating function of the sequence $\{u_n\}$ given in (1) is

$$U(x) = \sum_{k\geq 0} u_k x^k = \frac{p(x)}{1-c_1x-c_2x^2-\cdots-c_kx^k},$$

where the polynomial $p(x)$ is determined by the initial values of the sequence $\{u_n\}$.

Recently, by using *generating function* method, Merca [23] showed that determinant of an $n\times n$ Toeplitz-Hessenberg matrix is expressed as a sum over the integer partitions of n.

Getu [8] computed determinants of a class of Hessenberg matrices by using *generating function* method. He considered the infinite matrix

$$D = \begin{bmatrix} b_0 & 1 & 0 & 0 & \dots \\ b_1 & c_1 & 1 & 0 & \dots \\ b_2 & c_2 & c_1 & 1 & \dots \\ b_3 & c_3 & c_2 & c_1 & \dots \\ b_4 & c_4 & c_3 & c_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then he showed that if the following equation holds

$$A(x) = \frac{B(x)}{C(x)+1},$$

then

$$a_n = (-1)^n \det D_n,$$

where $A(x)$, $B(x)$ and $C(x)$ are the generating functions of the sequences $\{a_{k+1}\}$, $\{b_k\}$ and $\{c_{k+1}\}$, resp.

In this work, we use the generating function method to determine the relationships between determinants of three classes of Hessenberg matrices whose entries are terms of the certain sequences and generating functions of these sequences. So determinants of these Hessenberg matrices could be easily found by these relations. Some of our results will generalize the results of [8]. We show that earlier computed Hessenberg determinants in [12–16, 18, 21, 22, 25] with cofactor expansion could much easily be recomputed by our method. Moreover we compute two new classes of Hessenberg matrices whose determinants have not been computed before. Finally, we give an elegant method to compute the determinants of Hessenberg matrices whose entries consist of the terms of the recursive sequences: our approach is to find an adjacency-factor matrix and use the results of Section 2.

2. Evaluating Hessenberg Determinants via Generating Functions

Let $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 1}$ be any sequences. Denote their generating functions as $B(x) = \sum_{k\geq 0} b_k x^k$ and $C(x) = \sum_{k\geq 1} c_k x^k$, resp. To generalize the result of [8], we define the Hessenberg matrix $A_n(r,s)$ of order $n+1$:

$$A_n(r,s) := \begin{bmatrix} b_0 & r & & & & & 0 \\ b_1 & c_1 & s & & & & \\ b_2 & c_2 & c_1 & r & & & \\ b_3 & c_3 & c_2 & c_1 & s & & \\ \vdots & \vdots & \vdots & \dots & \ddots & \ddots & \\ b_{n-1} & c_{n-1} & c_{n-2} & \cdots & \cdots & c_1 & d_n(r,s) \\ b_n & c_n & c_{n-1} & \cdots & \cdots & c_2 & c_1 \end{bmatrix}, \tag{2}$$

where

$$d_n(r,s) = \begin{cases} r & \text{if } n \text{ is odd,} \\ s & \text{if } n \text{ is even,} \end{cases}$$

for arbitrary nonzero real numbers r and s. Briefly, we use A_n instead of $A_n(r,s)$ if there is no restrictions on r and s.

When $r = s = 1$, the matrix $A_n(1,1)$ is considered in [8] and the author computed its determinant via generating functions. To compute determinant of A_n via generating function method, we have the following result:

Theorem 2.1. *If*

$$A(x) = \frac{B(x)\left(C(-x) + \frac{r+s}{2}\right) - B(-x)\left(\frac{r-s}{2}\right)}{C(x)C(-x) + \left(\frac{r+s}{2}\right)(C(x) + C(-x)) + rs},$$

then (i) for even n such that $n = 2t$,

$$\det A_n = (-1)^n r^{t+1} s^t a_n,$$

(ii) for odd n such that $n = 2t+1$,

$$\det A_n = (-1)^n r^{t+1} s^{t+1} a_n,$$

where $A(x)$ is the generating function of $\{a_n\}$.

Proof. We consider the infinite linear system of equations

$$\begin{bmatrix} r & & & & & 0 \\ c_1x & sx & & & & \\ c_2x^2 & c_1x^2 & rx^2 & & & \\ c_3x^3 & c_2x^3 & c_1x^3 & sx^3 & & \\ c_4x^4 & c_3x^4 & c_2x^4 & c_1x^4 & rx^4 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1x \\ b_2x^2 \\ b_3x^3 \\ b_4x^4 \\ \vdots \end{bmatrix}. \tag{3}$$

Here we write

$$\begin{aligned} ra_0 &= b_0 \\ c_1a_0x + sa_1x &= b_1x \\ c_2a_0x^2 + c_1a_1x^2 + ra_2x^2 &= b_2x^2 \\ c_3a_0x^3 + c_2a_1x^3 + c_1a_2x^3 + sa_3x^3 &= b_3x^3 \\ \vdots &= \vdots \end{aligned}$$

By summing both sides of the above equalities, we obtain

$$A(x)C(x) + r\sum_{k\geq 0} a_{2k}x^{2k} + s\sum_{k\geq 0} a_{2k+1}x^{2k+1} = B(x). \tag{4}$$

Since

$$\sum_{k\geq 0} a_{2k}x^{2k} = \frac{A(x) + A(-x)}{2} \text{ and } \sum_{k\geq 0} a_{2k+1}x^{2k+1} = \frac{A(x) - A(-x)}{2},$$

Eq. (4) could be rewritten as

$$A(x)\left[C(x)+\frac{r+s}{2}\right]+A(-x)\left[\frac{r-s}{2}\right]=B(x).$$

Taking $(-x)$ instead of x, we get

$$A(-x)\left[C(-x)+\frac{r+s}{2}\right]+A(x)\left[\frac{r-s}{2}\right]=B(-x).$$

Solving two equations just above in terms of $A(x)$, we get

$$A(x)=\frac{B(x)\left(C(-x)+\frac{r+s}{2}\right)-B(-x)\left(\frac{r-s}{2}\right)}{C(x)C(-x)+\left(\frac{r+s}{2}\right)(C(x)+C(-x))+rs},$$

as desired.

We examine the relationship between the sequences $\{a_n\}$ and $\{\det(A_n)\}$. If we consider the system (3) for only first $n+1$ equations and take $x=1$, the system (3) turns to

$$\begin{bmatrix} r & & & & & 0 \\ c_1 & s & & & & \\ c_2 & c_1 & r & & & \\ c_3 & c_2 & c_1 & s & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ c_n & c_{n-1} & c_{n-2} & \cdots & \vdots & d_{n+1}(r,s) \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}=\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix},$$

where $d_n(r,s)$ is defined as before.

By Cramer's rule, we obtain $a_n=\dfrac{(-1)^n\det A_n}{r^{t+1}s^t}$ for even n such that $n=2t$ and $a_n=\dfrac{(-1)^n\det A_n}{r^{t+1}s^{t+1}}$ for odd n such that $n=2t+1$, which completes the proof. □

We want to note some important and useful special cases of Theorem 2.1 with the following corollaries:

Corollary 2.2. *For the matrix $A_n(1,1)$, we have that $a_n=(-1)^n\det A_n$ and the generating function of the sequence $\{\det A_n(1,1)\}$ is*

$$\mathcal{A}(x)=\frac{B(-x)}{1+C(-x)}.$$

This result was firstly given in [8].

Corollary 2.3. *For the matrix $A_n(-1,-1)$, we have that $a_n=-\det A_n$ and the generating function of the sequence $\{\det A_n(-1,-1)\}$ is*

$$\mathcal{A}(x)=\frac{B(x)}{1-C(x)}. \tag{5}$$

Let's give some examples.

Example 2.4. *For $n\geq 0$, we have that*

$$\begin{vmatrix} F_1 & -1 & & & & & 0 \\ F_2 & 1 & -1 & & & & \\ F_3 & 1 & 1 & -1 & & & \\ F_4 & 0 & 1 & 1 & -1 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ F_n & 0 & 0 & \cdots & \cdots & 1 & -1 \\ F_{n+1} & 0 & 0 & \cdots & \cdots & 1 & 1 \end{vmatrix}=\sum_{k=0}^{n}F_{k+1}F_{n+1-k}.$$

Proof. If $b_n = F_{n+1}$ and $\{c_n\}_1^\infty = \{1, 1, 0, \ldots\}$, then $B(x) = \frac{1}{1-x-x^2}$ and $C(x) = x + x^2$. So the generating function of $\{\det A_n(-1,-1)\}$ by Corollary 2.3 is $\frac{1}{(1-x-x^2)^2}$, which is the generating function of $\left\{\sum_{k=0}^{n} F_{k+1}F_{n+1-k}\right\}$, as well. □

Example 2.5. *For $n \geq 0$, we have that*

$$\begin{vmatrix} L_0 & -1 & & & & & 0 \\ L_1 & -F_1 & -1 & & & & \\ L_2 & -F_2 & -F_1 & -1 & & & \\ L_3 & -F_3 & -F_2 & -F_1 & -1 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ L_{n-1} & -F_{n-1} & -F_{n-2} & \cdots & \cdots & -F_1 & -1 \\ L_n & -F_n & -F_{n-1} & \cdots & \cdots & -F_2 & -F_1 \end{vmatrix} = \begin{cases} 2 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Since $b_n = L_n$ and $\{c_n\}_1^\infty = \{-F_n\}_1^\infty$, $B(x) = \frac{2-x}{1-x-x^2}$ and $C(x) = \frac{-x}{1-x-x^2}$. By Corollary 2.3, the generating function of $\{\det A_n(-1,-1)\}$ is

$$A(x) = \frac{B(x)}{1 - C(x)} = \frac{2-x}{1-x^2},$$

which gives the periodic sequence $\{2, -1, 2, -1, \ldots\}$. □

Let $\{b_n\}$ be any sequence and $\{c_n\}_1^\infty = \{1, 0, 0, \ldots\}$. Since $\frac{1}{1-x}B(x)$ is the generating function of the sum of the first nth term of $\{b_n\}$, by Corollary 2.3, we see that

$$\det A_n(-1,-1) = \sum_{k=0}^{n} b_k.$$

For example,

$$\begin{vmatrix} 1 & -1 & & & & & 0 \\ \frac{1}{2} & 1 & -1 & & & & \\ \frac{1}{3} & 0 & 1 & -1 & & & \\ \frac{1}{4} & 0 & 0 & 1 & -1 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ \frac{1}{n} & 0 & 0 & \cdots & \cdots & 1 & -1 \\ \frac{1}{n+1} & 0 & 0 & \cdots & \cdots & 0 & 1 \end{vmatrix} = H_{n+1},$$

where H_n stands for nth Harmonic number.

Since permenantal and determinantal relationships between the matrices $A_n(1,1)$ and $A_n(-1,-1)$ are

$$\det A_n(1,1) = perA_n(-1,-1) \text{ and } perA_n(1,1) = \det A_n(-1,-1),$$

the corollaries given above include the results of [12, 25].

Corollary 2.6. *If*

$$A(x) = \frac{C(-x)B(x) - B(-x)}{C(x)C(-x) - 1},$$

then we have

$$\det A_n(1,-1) = (-1)^{\frac{1}{2}n(n-1)} a_n.$$

We will give an example:

Example 2.7. *If we take* $\{c_n\} = \{(-1)^n F_{n-1}\}$ *and define the sequence* $\{b_n\}$ *as* $b_{2n} = -b_{2n+1} = F_{2n+2}$*, then for even* n *such that* $n = 2k$*, the matrix* $A_n(1,-1)$ *takes the form*

$$A_{2k}(1,-1) = \begin{bmatrix} F_2 & 1 & & & & & 0 \\ -F_2 & 0 & -1 & & & & \\ F_4 & F_1 & 0 & 1 & & & \\ -F_4 & -F_2 & F_1 & 0 & -1 & & \\ \vdots & \vdots & \vdots & \dots & \ddots & \ddots & \\ -F_{2k} & -F_{2k-2} & F_{2k-3} & \cdots & \cdots & 0 & -1 \\ F_{2k+2} & F_{2k-1} & -F_{2k-2} & \cdots & \cdots & F_1 & 0 \end{bmatrix}$$

and so

$$\det A_{2k}(1,-1) = (-1)^k F_{2k+1}.$$

Proof. The generating functions of $\{b_n\}$ and $\{c_n\}$ are $B(x) = \frac{1-x}{(1+x-x^2)(1-x-x^2)}$ and $C(x) = \frac{x^2}{1+x-x^2}$, resp. So we get $A(x) = \frac{1}{1-x-x^2}$ which means $\det A_{2k} = (-1)^k F_{2k+1}$ by Corollary 2.6. □

The example just above could be also given for odd n. Here we leave it.

Corollary 2.8. *If*

$$A(x) = \frac{B(x)}{C(x)+d},$$

then

$$\det A_n(d,d) = (-1)^n d^{n+1} a_n$$

and the generating function of $\{\det A_n(d,d)\}$ *is*

$$\mathcal{A}(x) = d \cdot A(-dx).$$

The result of [22] could be derived by using Corollary 2.8 and the properties of the generating functions.

Example 2.9. *If* $b_n = -(H_n+1)$ *with* $b_0 = -1$ *and* $c_n = \frac{2}{n}$*, then*

$$\begin{vmatrix} -1 & 2 & & & & & 0 \\ -(H_1+1) & 2 & 2 & & & & \\ -(H_2+1) & 1 & 2 & 2 & & & \\ -(H_3+1) & \frac{2}{3} & 1 & 2 & 2 & & \\ \vdots & \vdots & \vdots & \dots & \ddots & \ddots & \\ -(H_{n-1}+1) & \frac{2}{n-1} & \frac{2}{n-2} & \cdots & \cdots & 2 & 2 \\ -(H_n+1) & \frac{2}{n} & \frac{2}{n-1} & \cdots & \cdots & 1 & 2 \end{vmatrix} = (-1)^n 2^{n-1}.$$

Proof. If we take $d = 2$, $b_n = -(H_n+1)$ with $b_0 = -1$ and $c_n = \frac{2}{n}$ in Corollary 2.8, then we get

$$B(x) = \frac{\ln(1-x)-1}{1-x} \quad \text{and} \quad C(x) = \ln(1-x)^{-2}.$$

Thus $A(x) = \frac{1}{2x-2}$ and $\det A_n = 2(-2)^n a_n$, which gives us $\det A_n = (-1)^n 2^{n-1}$, as claimed. □

When $c_0 = d$, by Corollary 2.8, we obtain $A(x) = \frac{B(x)}{C(x)}$, where $C(x) = \sum_{k\geq 0} c_k x^k$. For example, if we choose $B(x) = x + 4x^2 + x^3$ and $C(x) = (1-x)^4$, then

$$\begin{vmatrix} 0 & 1 & & & & & & 0 \\ 1 & -4 & 1 & & & & & \\ 4 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ 0 & 1 & -4 & 6 & \ddots & \ddots & & \\ 0 & 0 & 1 & -4 & \ddots & -4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \cdots & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -4 & 6 & -4 \end{vmatrix}_{(n+1)\times(n+1)} = (-1)^n n^3.$$

Now we recall an already known result given in [23]. But we will give an alternative and much simple proof for it.

Corollary 2.10. *If $\{c_n\}$ is any sequence such that $c_0 \neq 0$, then we have*

$$\begin{vmatrix} c_1 & c_0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \end{vmatrix}_{n\times n} = [x^n]\,\frac{c_0}{C(-c_0 x)},$$

where $C(x) = \sum_{k\geq 0} c_k x^k$ and $[\circ]$ is the coefficient extraction operator.

Proof. To prove it by our result, Corollary 2.8, first we consider an equal determinant to the claimed determinant by the following equality

$$\begin{vmatrix} c_1 & c_0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \end{vmatrix}_{n\times n} = \begin{vmatrix} 1 & c_0 & 0 & 0 & \cdots & 0 \\ 0 & c_1 & c_0 & 0 & \cdots & 0 \\ 0 & c_2 & c_1 & c_0 & \cdots & 0 \\ 0 & c_3 & c_2 & c_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \end{vmatrix}_{(n+1)\times(n+1)}.$$

The value of the determinant on the RHS of the above equation could be easily found by Corollary 2.8. So the claimed result directly follows. □

Let's give an example related to Theorem 2.1.

Example 2.11. *Let $\{b_n\}$ be the alternating of the sequence A135491 in [26]. Then for $n = 2k$,*

$$\begin{vmatrix} b_0 & 1 & & & & & 0 \\ b_1 & 1 & -3 & & & & \\ b_2 & 1 & 1 & 1 & & & \\ b_3 & 1 & 1 & 1 & -3 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ b_{2k-1} & 0 & 0 & \cdots & \cdots & 1 & -3 \\ b_{2k} & 0 & 0 & \cdots & \cdots & 1 & 1 \end{vmatrix} = T_{2k+2}(-3)^k.$$

Similarly, for $n = 2k+1$, determinant of the corresponding Hessenberg matrix is equal to $-T_{2k+3}(-3)^{k+1}$, where T_n is the nth Tribonacci number.

Proof. The generating functions of $\{b_n\}$ and $\{c_n\}$ are $B(x) = \frac{1-x+x^2-x^3}{1+x-x^2+x^3}$ and $C(x) = x + x^2 + x^3$, resp. By Theorem 2.1, when $r = 1$ and $s = -3$, we obtain $\det A_n = T_{n+2}(-3)^t$ for $n = 2t$ and $\det A_n = -T_{n+2}(-3)^{t+1}$ for $n = 2t+1$, as desired. □

Let $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ be any number sequences. Their generating functions are $B(x) = \sum_{k\geq 0} b_k x^k$, $C(x) = \sum_{k\geq 1} c_k x^k$ and $D(x) = \sum_{k\geq 1} d_k x^k$, respectively.

Now we consider two classes of Hessenberg determinants, which are not considered before. We start with the first one: For any nonzero real d, we define a Hessenberg matrix of order $n+1$ as follows:

$$A_n = \begin{bmatrix} b_0 & d & & & & & 0 \\ b_1 & c_1 & d & & & & \\ b_2 & c_2 & d_1 & d & & & \\ b_3 & c_3 & d_2 & d_1 & d & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ b_{n-1} & c_{n-1} & d_{n-2} & \cdots & \cdots & d_1 & d \\ b_n & c_n & d_{n-1} & \cdots & \cdots & d_2 & d_1 \end{bmatrix}.$$

Theorem 2.12. *If*

$$A(x) = \frac{B(x) + a_0 D(x) - a_0 C(x)}{D(x) + d} \quad \text{with } a_0 = b_0/d, \tag{6}$$

then

$$\det A_n = (-1)^n d^{n+1} a_n$$

and the generating function of $\{\det A_n\}$ *is*

$$\mathcal{A}(x) = d \cdot A(-dx).$$

Proof. Similar to the proof of Theorem 2.1, we have the following infinite linear system of equations

$$\begin{bmatrix} d & & & & & 0 \\ c_1 x & dx & & & & \\ c_2 x^2 & d_1 x^2 & dx^2 & & & \\ c_3 x^3 & d_2 x^3 & d_1 x^3 & dx^3 & & \\ c_4 x^4 & d_3 x^4 & d_2 x^4 & d_1 x^4 & dx^4 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 x \\ b_2 x^2 \\ b_3 x^3 \\ b_4 x^4 \\ \vdots \end{bmatrix}.$$

By summing the equations come from the infinite linear system of equations just above and adding $a_0 D(x)$ to both sides of it, we obtain

$$a_0 C(x) + A(x) D(x) + a_0 A(x) = B(x) + a_0 D(x),$$

which gives

$$A(x) = \frac{B(x) + a_0 D(x) - a_0 C(x)}{D(x) + d},$$

as desired. Finally, if we restrict the linear system of equations to the fist $(n+1)$ equations and take $x = 1$, then by Cramer's rule, we get $a_n = \frac{(-1)^n \det A_n}{d^{n+1}}$, as claimed. □

Example 2.13. *For $n > 0$,*

$$\begin{vmatrix} P_1 & 1 & & & & & 0 \\ P_2 & F_2 & 1 & & & & \\ P_3 & F_3 & P_2 & 1 & & & \\ P_4 & F_4 & P_3 & P_2 & 1 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ P_n & F_n & P_{n-1} & \cdots & \cdots & P_2 & 1 \\ P_{n+1} & F_{n+1} & P_n & \cdots & \cdots & P_3 & P_2 \end{vmatrix} = (-1)^n F_{n-1},$$

where F_n and P_n are the nth Fibonacci and Pell number, resp.

Proof. It is a consequence of Theorem 2.12. When $d = 1$, $B(x) = \sum_{k\geq 0} P_{k+1}x^k = \frac{1}{1-2x-x^2}$, $C(x) = \sum_{k\geq 1} F_{k+1}x^k = \frac{x+x^2}{1-x-x^2}$ and $D(x) = \sum_{k\geq 1} P_{k+1}x^k = \frac{2x+x^2}{1-x-x^2}$, the proof follows. □

If $d = c_0 = d_0$, then we rewrite the equation (6) as

$$A(x) = \frac{B(x) + a_0 D(x) - a_0 C(x)}{D(x)},$$

where $C(x) = \sum_{k\geq 0} c_k x^k$ and $D(x) = \sum_{k\geq 0} d_k x^k$ and $B(x)$ is same to before.

Similarly, let $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ be any sequences, whose generating functions are denoted as before.

Now we define the second class of Hessenberg matrices of order $n+1$, whose columns are periodic after first column, as follows:

$$A_n = \begin{bmatrix} b_0 & d & & & & & & 0 \\ b_1 & c_1 & d & & & & & \\ b_2 & c_2 & d_1 & d & & & & \\ b_3 & c_3 & d_2 & c_1 & d & & & \\ b_4 & c_4 & d_3 & c_2 & d_1 & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & d & \\ b_{n-1} & c_{n-1} & d_{n-2} & c_{n-3} & d_{n-4} & \cdots & s(n,1) & d \\ b_n & c_n & d_{n-1} & c_{n-2} & d_{n-3} & \cdots & s(n,2) & s(n+1,1) \end{bmatrix},$$

where

$$s(n,k) = \begin{cases} c_k & \text{if } n \text{ is even,} \\ d_k & \text{if } n \text{ is odd.} \end{cases}$$

We have the following result for the generating function of the determinant of the just above matrix.

Theorem 2.14. *If*

$$A(x) = \frac{B(x)(C(-x) + D(-x) + 2) - B(-x)(C(x) - D(x))}{C(x)(1 + D(-x)) + D(x)(1 + C(-x)) + (C(-x) + D(-x)) + 2d},$$

then

$$\det A_n = (-1)^n d^{n+1} a_n$$

and the generating function of $\{\det A_n\}$ *is*

$$\mathcal{A}(x) = d \cdot A(-dx).$$

Proof. Similar to the previous theorems, if we consider the infinite linear system of equations, then we obtain

$$C(x)\sum_{k\geq 0} a_{2k}x^{2k} + D(x)\sum_{k\geq 0} a_{2k+1}x^{2k+1} + dA(x) = B(x). \tag{7}$$

Since $\sum_{k\geq 0} a_{2k}x^{2k} = \frac{A(x)+A(-x)}{2}$ and $\sum_{k\geq 0} a_{2k+1}x^{2k+1} = \frac{A(x)-A(-x)}{2}$, the equation (7) is written as

$$A(x)\left(\frac{C(x)+D(x)}{2}+1\right) + A(-x)\left(\frac{C(x)-D(x)}{2}\right) = B(x),$$

which, by solving in terms of $A(x)$, gives us

$$A(x) = \frac{B(x)(C(-x)+D(-x)+2) - B(-x)(C(x)-D(x))}{C(x)(1+D(-x)) + D(x)(1+C(-x)) + (C(-x)+D(-x)) + 2d},$$

as desired. When we restricted the infinite system of equations to the first $n+1$ equations with $x=1$, we complete the proof by Cramer's rule. □

Example 2.15. *For even n, we have*

$$\begin{vmatrix} L_0 & 1 & & & & & & 0 \\ L_1 & F_1 & 1 & & & & & \\ L_2 & F_2 & L_0 & 1 & & & & \\ L_3 & F_3 & L_1 & F_1 & 1 & & & \\ L_4 & F_4 & L_2 & F_2 & L_0 & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 1 & \\ L_{n-1} & F_{n-1} & L_{n-3} & F_{n-3} & L_{n-5} & \cdots & F_1 & 1 \\ L_n & F_n & L_{n-2} & F_{n-2} & L_{n-4} & \cdots & F_2 & L_0 \end{vmatrix} = 2^{\frac{n}{2}} + 1.$$

If $n = 2k+1$, the determinant of corresponding matrix is equal to 2^k.

Proof. Since $b_n = L_n$, $c_n = F_n$ and $d_n = L_{n-1}$, we have $B(x) = \frac{2-x}{1-x-x^2}$, $C(x) = \frac{x}{1-x-x^2}$ and $D(x) = \frac{2x-x^2}{1-x-x^2}$. Hence, for $d = 1$ by Theorem 2.14, we obtain

$$\begin{aligned} A(x) &= \frac{-x-3x^2+x^3+2}{(x-1)(x+1)(2x^2-1)} = \frac{1}{1-x^2} + \frac{1-x}{1-2x^2} \\ &= \sum_{k=0}^{\infty} x^{2k} + \sum_{k=0}^{\infty} 2^k x^{2k} - \sum_{k=0}^{\infty} 2^k x^{2k+1} \\ &= \sum_{k=0}^{\infty} \left(2^k+1\right) x^{2k} - \sum_{k=0}^{\infty} 2^k x^{2k+1}, \end{aligned}$$

as claimed. □

We consider certain Hessenberg matrices whose superdiagonal are constant or two periodic. Now we give a general idea for Hessenberg matrices with arbitrary superdiagonal entries. To show how this idea will be applied, we present two Hessenberg matrices whose superdiagonals now consist of the terms of two special sequences, $\{n+1\}$ and $\{2^n\}$, resp.

Let $\{b_n\}$, $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}$ such that $d_n \neq 0$ for all $n \in \mathbb{N}$ be any sequences. First define the Hessenberg matrix A_n of order $n+1$ of the form

$$A_n := \begin{bmatrix} b_0 & d_0 & & & & & 0 \\ b_1 & c_1 & d_1 & & & & \\ b_2 & c_2 & c_1 & d_2 & & & \\ b_3 & c_3 & c_2 & c_1 & d_3 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ b_{n-1} & c_{n-1} & c_{n-2} & \cdots & \cdots & c_1 & d_{n-1} \\ b_n & c_n & c_{n-1} & \cdots & \cdots & c_2 & c_1 \end{bmatrix}.$$

Consider the following infinite linear system of equations

$$\begin{bmatrix} d_0 & & & & & 0 \\ c_1x & d_1x & & & & \\ c_2x^2 & c_1x^2 & d_2x^2 & & & \\ c_3x^3 & c_2x^3 & c_1x^3 & d_3x^3 & & \\ c_4x^4 & c_3x^4 & c_2x^4 & c_1x^4 & d_4x^4 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1x \\ b_2x^2 \\ b_3x^3 \\ b_4x^4 \\ \vdots \end{bmatrix},$$

which gives us the relation

$$A(x)C(x) + \sum_{k=0}^{\infty} a_k d_k x^k = B(x), \tag{8}$$

where $C(x) = \sum_{k\geq 1} c_k x^k$. If we restricted this infinite system to the first $n+1$ equations with $x = 1$, then by Cramer's rule we have

$$a_n = \frac{(-1)^n \det A_n}{\prod_{k=0}^{n} d_k}.$$

Now we present two special cases of the idea mentioned above.

Theorem 2.16. *If* $\{d_n\} = \{n+1\}$, *then*

$$xA(x)\left(e^{\int \frac{C(x)}{x}dx}\right) = \int e^{\int \frac{C(x)}{x}dx} B(x)\,dx + C,$$

with

$$\det A_n = (-1)^n (n+1)! a_n,$$

where C is a constant.

Proof. By (8), we have

$$A(x)C(x) + \sum_{k=0}^{\infty} a_k (k+1) x^k = B(x),$$

which, equivalently, gives us

$$A(x)C(x) + (xA(x))' = B(x).$$

By taking $y = x \cdot A(x)$, we get the first order linear differential equation

$$y\frac{C(x)}{x} + y' = B(x).$$

The solution of this differential equation is

$$y = \left(e^{\int \frac{C(x)}{x}dx}\right)^{-1}\left(\int e^{\int \frac{C(x)}{x}dx} B(x)\,dx + C\right),$$

which completes the proof. Note that the constant C is determined by the initial $y(0) = 0$. □

Example 2.17. *For $n \geq 0$, we have*

$$\begin{vmatrix} 1 & 1 & & & & & 0 \\ 3 & 1 & 2 & & & & \\ 5 & 1 & 1 & 3 & & & \\ 7 & 1 & 1 & 1 & 4 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ 2n-1 & 1 & 1 & \cdots & \cdots & 1 & n \\ 2n+1 & 1 & 1 & \cdots & \cdots & 1 & 1 \end{vmatrix} = (-1)^n (n+1)!.$$

Proof. Since $b_n = 2n + 1$ and $c_n = 1$, we obtain $B(x) = \frac{x+1}{(x-1)^2}$ and $C(x) = \frac{x}{1-x}$. So we get

$$\int \frac{1}{1-x}dx = -\ln(x-1) \text{ and } e^{\int \frac{C(x)}{x}dx} = \frac{1}{x-1}.$$

By Theorem 2.16, we have that

$$\begin{aligned} xA(x)\frac{1}{x-1} &= \int \frac{x+1}{(x-1)^3}dx + C \\ xA(x)\frac{1}{x-1} &= -\frac{x}{(x-1)^2} + C. \end{aligned}$$

For $x = 0$, we find that $C = 0$ and so

$$A(x) = \frac{1}{1-x},$$

which gives $\det A_n = (-1)^n (n+1)!$. □

For the case $b_n = c_{n+1}$, i.e. $B(x) = \frac{C(x)}{x}$, the relation given in Theorem 2.16 turns

$$xA(x) = 1 + C\left(e^{\int \frac{C(x)}{x}dx}\right)^{-1}.$$

Now we present the other special case with an example which could be produced by (8).

Example 2.18. *For $n \geq 0$,*

$$\begin{vmatrix} 1 & 1 & & & & & 0 \\ 3 & 1 & 2 & & & & \\ 4 & 1 & 1 & 4 & & & \\ \frac{10}{3} & \frac{1}{2} & 1 & 1 & 8 & & \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \\ \frac{2^{n-2}(n+1)}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \cdots & \cdots & 1 & 2^{n-1} \\ \frac{2^{n-1}(n+2)}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & \cdots & 1 & 1 \end{vmatrix} = \frac{(-1)^n 2^{\binom{n+1}{2}}}{n!}.$$

Proof. Since $b_n = \frac{2^{n-1}(n+2)}{n!}$ and $c_n = \frac{1}{(n-1)!}$, their generating functions are $B(x) = e^{2x}(x+1)$ and $C(x) = xe^x$, resp. By (8), we have

$$xe^x A(x) + A(2x) = e^{2x}(x+1).$$

Hence we find that $A(x) = e^x$, which gives $a_n = \frac{1}{n!}$. Finally, from the relation $a_n = \frac{(-1)^n \det A_n}{2^{\binom{n+1}{2}}}$, we obtain claimed result. □

3. A Matrix Method to Compute a Class of Hessenberg Determinants

In this section, we give a new method to compute a class of Hessenberg determinants in which the entries of each matrix in the class are terms of a general linear recurrence relation.

Consider the following lower Hessenberg matrix of order n for nonzero real r:

$$E_n(r) = \begin{bmatrix} u_1 & r & & & & & 0 \\ u_2 & u_1 & r & & & & \\ u_3 & u_2 & u_1 & r & & & \\ u_4 & u_3 & u_2 & u_1 & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ u_{n-1} & u_{n-2} & u_{n-3} & u_{n-4} & \cdots & u_1 & r \\ u_n & u_{n-1} & u_{n-2} & u_{-3} & \cdots & u_2 & u_1 \end{bmatrix},$$

where the terms u_n's are defined as in (1).

We only consider the matrix $E_n(r)$ with case $r = -1$, briefly denoted by E_n, while giving our method but one could follow whole steps will be given above for the matrix $E_n(r)$ with any nonzero r.

Indeed one could compute determinant of the matrix E_n by using the results of Section 2. Here we will present a new and easy method to compute $\det(E_n)$. For this, we define an adjacency-factor matrix related with the matrix E_n : Define a $n \times n$ lower triangular adjacency-factor matrix M as

$$M_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -c_{i-j} & \text{if } 1 \leq i - j \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly the matrix M has the form

$$M = \begin{bmatrix} 1 & & & & & & 0 \\ -c_1 & 1 & & & & & \\ -c_2 & -c_1 & 1 & & & & \\ \vdots & -c_2 & \ddots & \ddots & & & \\ -c_k & & \ddots & & & & \\ & \ddots & & \ddots & \ddots & \ddots & \\ 0 & & -c_k & \cdots & -c_2 & -c_1 & 1 \end{bmatrix}.$$

From a matrix multiplication, we obtain that

$$ME_n = \widehat{E}_n,$$

where

$$\widehat{E}_n = \begin{cases} -1 & \text{if } j = i+1, \\ b_i & \text{if } j = 1 \text{ and } i \leq k, \\ d_{i-j+1} & \text{if } i \geq j > 1 \text{ and } i - j \leq k-1, \\ 0 & \text{otherwise,} \end{cases}$$

with

$$b_m = u_m - \sum_{l=1}^{m-1} u_{m-l} c_l \text{ and } d_m = u_m - \sum_{l=1}^{m-1} u_{m-l} c_l + c_m,$$

for $1 \le m \le k$.

Here since $\det M = 1$, we have $\det E_n = \det_n \widehat{E}$. Afterwards, we prefer to compute the value of the determinant of $\widehat{E_n}$ instead of the matrix E_n because the matrix $\widehat{E_n}$ is a banded matrix with bandwidth $k+1$ and includes *many zeros* and so it gives us to advantage to choose the matrix $\widehat{E_n}$ rather than E_n regard to use of the results of the previous section to compute determinants of Hessenberg matrices.

By Corollary 2.3, we have that

$$\sum_{i \ge 0} \det E_{i+1} x^i = \sum_{i \ge 0} \det \widehat{E}_{i+1} x^i = \frac{\sum_{i=1}^{k} b_i x^{i-1}}{1 - \sum_{i=1}^{k} d_i x^i}. \tag{9}$$

As a special case, if we consider the recurrence relation $\{u_n\}$ defined in (1) with the initials $u_{-k+2} = u_{-k+3} = \cdots = u_{-1} = u_0 = 0$ and $u_1 = 1$, then we have

$b_1 = 1$ and $b_i = 0$ for $1 < i \le k$,
$d_1 = 1 + c_1$ and $d_i = c_i$ for $1 < i \le k$.

Hence the generating function of the determinant of the matrix E_{n+1} is written as

$$\frac{1}{1 - (1 + c_1)x - c_2 x^2 - \cdots - c_k x^k}. \tag{10}$$

Now we give an example to show how to use the method described above.

Example 3.1. *For positive integer m, define the sequence $\{u_n\}$ with $u_n = \binom{m+n-1}{m}$ and construct the following $n \times n$ matrix $A_n(m)$*

$$A_n(m) := \begin{bmatrix} \binom{m}{m} & -1 & & & & & 0 \\ \binom{m+1}{m} & \binom{m}{m} & -1 & & & & \\ \binom{m+2}{m} & \binom{m+1}{m} & \binom{m}{m} & \ddots & & & \\ \binom{m+3}{m} & \binom{m+2}{m} & \binom{m+1}{m} & \cdots & -1 & & \\ \cdots & \cdots & \cdots & \cdots & \binom{m}{m} & -1 & \\ \binom{m+n-2}{m} & \binom{m+n-3}{m} & \binom{m+n-4}{m} & \vdots & \binom{m+1}{m} & \binom{m}{m} & -1 \\ \binom{m+n-1}{m} & \binom{m+n-2}{m} & \binom{m+n-3}{m} & \vdots & \binom{m+2}{m} & \binom{m+1}{m} & \binom{m}{m} \end{bmatrix},$$

where $\binom{n}{k}$ is the usual binomial coefficient. Then

$$\det A_{n+1}(m) = \sum_{k=0}^{n} \binom{(m+1)n + m(1-k)}{k}.$$

Proof. We should find the recursion relation for the sequence $\{u_n\}$. From [10], we recall the Equation 5.24: For $l \ge 0$ and integers m, n,

$$\sum_{k} \binom{l}{m+k} \binom{s+k}{n} (-1)^k = (-1)^{l+m} \binom{s-m}{n-l}.$$

If we choose $l \to m+1$, $m \to 1$, $s \to m-n$ and $n \to m$ in the equation above, then we obtain

$$\begin{aligned}\sum_{k=-1}^{m} (-1)^k \binom{m+1}{k+1}\binom{n-k-1}{m} &= \sum_{k=-1}^{m} (-1)^{k+m} \binom{m+1}{k+1}\binom{m-n+k}{m} \\ &= \binom{n-m-1}{-1} = 0.\end{aligned}$$

By the above equation, we could deduce

$$\sum_{k=0}^{m} (-1)^k \binom{m+1}{k+1}\binom{n-k-1}{m} = \binom{n}{m}.$$

If we take $n = n+m-1$, then we get the recurrence relation of order $m+1$ for the sequence $\{u_n\}$:

$$u_n = \sum_{k=0}^{m} (-1)^k \binom{m+1}{k+1} u_{n-k-1},$$

with $u_{-m+1} = u_{-m+2} = \cdots = u_{-1} = u_0 = 0$ and $u_1 = 1$. By our method, we see that the adjacency-factor matrix for the matrix $A_n(m)$ is

$$M_{ij} = (-1)^{i-j} \binom{m+1}{i-j},$$

which is also equal to

$$M_{ij} = \begin{bmatrix} 1 & & & & 0 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & -1 & 1 \end{bmatrix}^{m+1}.$$

Thus by (10), we find the generating function of the sequence $\{\det A_{n+1}(m)\}$ as follows

$$\frac{1}{1-\left(1+\binom{m+1}{1}\right)x+\binom{m+1}{2}x^2-\cdots-(-1)^m\binom{m+1}{m+1}x^{m+1}} = \frac{1}{(1-x)^{m+1}-x}.$$

In other words, we have that

$$[x^n]\,\frac{1}{(1-x)^{m+1}-x} = \det A_{n+1}(m)\,. \tag{11}$$

To prove this claim, it is sufficient to show that

$$\sum_{n\geq 0}\sum_{k=0}^{n} \binom{(m+1)\,n+m\,(1-k)}{k} x^n = \frac{1}{(1-x)^{m+1}-x}.$$

Consider,

$$\begin{aligned}\sum_{n\geq 0}\sum_{k=0}^{n}\binom{(m+1)\,n+m\,(1-k)}{k}x^n &= \sum_{k\geq 0}\sum_{n\geq k}\binom{(m+1)\,n+m\,(1-k)}{k}x^n\\ &= \sum_{n\geq 0}x^n\sum_{k\geq 0}\binom{m+n+mn+k}{k}x^k\\ &= \frac{1}{(1-x)^{m+1}}\sum_{n\geq 0}\left(\frac{x}{(1-x)^{m+1}}\right)^n\\ &= \frac{1}{(1-x)^{m+1}-x},\end{aligned}$$

which completes the proof. □

Finally, we obtain

$$\begin{vmatrix} \binom{m}{m} & -1 & & & & 0\\ \binom{m+1}{m} & \binom{m}{m} & -1 & & & \\ \binom{m+2}{m} & \binom{m+1}{m} & \ddots & \ddots & & \\ \cdots & \cdots & \cdots & \binom{m}{m} & -1 & \\ \binom{m+n-2}{m} & \binom{m+n-3}{m} & \vdots & \binom{m+1}{m} & \binom{m}{m} & -1\\ \binom{m+n-1}{m} & \binom{m+n-2}{m} & \vdots & \binom{m+2}{m} & \binom{m+1}{m} & \binom{m}{m} \end{vmatrix} = \sum_{k=0}^{n-1}\binom{(m+1)\,n-mk-1}{k}.$$

As a special case for $m = 1$, we get

$$\begin{vmatrix} 1 & -1 & & & & 0\\ 2 & 1 & -1 & & & \\ 3 & 2 & \ddots & \ddots & & \\ \cdots & \cdots & \cdots & 1 & -1 & 0\\ n-1 & n-2 & \vdots & 2 & 1 & -1\\ n & n-1 & \vdots & 3 & 2 & 1 \end{vmatrix} = \sum_{k=0}^{n-1}\binom{2n-k-1}{k} = F_{2n},$$

which could be also found in [22].

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