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Absolute Weighted Arithmetic Mean Summability Factors of Infinite Series and Trigonometric Fourier Series

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Abstract. In this paper, we generalized a known theorem dealing with absolute weighted arithmetic mean summability of infinite series by using a quasi-f-power increasing sequence instead of a quasi-σ-power increasing sequence. And we applied it to the trigonometric Fourier series

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence c_n and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [1]). A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where $f = \{f_n(\sigma, \beta)\} = \{n^\sigma (\log n)^\beta, \beta \geq 0, 0 < \sigma < 1\}$(see [13]). If we take β=0, then we get a quasi-σ-power increasing sequence. Every almost increasing sequence is a quasi-σ-power increasing sequence for any non-negative σ, but the converse is not true for $\sigma > 0$ (see [11]). For any sequence (λ_n) we write that $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. The sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty$. Let $\sum a_n$ be a given infinite series with the partial sums (s_n). By u_n^α and t_n^α we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, that is (see [6])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \quad ({t_n}^1 = t_n) \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \ \texttt{for} \ n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [8], [10])

$$\sum_{n=1}^{\infty} n^{k-1} \left|u_n^\alpha - u_{n-1}^\alpha\right|^k = \sum_{n=1}^{\infty} \frac{1}{n} \left|t_n^\alpha\right|^k < \infty. \tag{3}$$

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If we take $\alpha = 1$, then $|C, \alpha|_k$ summability reduces to $|C, 1|_k$ summability. Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{4}$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{5}$$

defines the sequence (w_n) of the weighted arithmetic mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [9]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |w_n - w_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$, (resp. $|\bar{N}, p_n|$) summability.

2. Known Results

The following theorems are known dealing with the $|\bar{N}, p_n|_k$ summability factors of infinite series.

Theorem 2.1 ([12]). Let (X_n) be an almost increasing sequence. If the sequences (X_n), (λ_n), and (p_n) satisfy the conditions

$$\lambda_m X_m = O(1) \quad as \quad m \to \infty, \tag{6}$$

$$\sum_{n=1}^{m} n X_n |\Delta^2 \lambda_n| = O(1) \quad as \quad m \to \infty, \tag{7}$$

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m) \quad as \quad m \to \infty, \tag{8}$$

$$\sum_{n=1}^{m} \frac{|\, t_n \,|^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{9}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

then the series $\sum a_n \lambda_n$ is summable $|\, \bar{N}, p_n \,|_k, k \geq 1$.

It should be remarked that Theorem 2.1 also implies the known result of Bor dealing with the absolute $|\bar{N}, p_n|_k$ summability factors of infinite series (see [3]).

Theorem 2.2 ([5]). Let (X_n) be a quasi-σ-power increasing sequence. If the sequences (X_n), (λ_n), and (p_n) satisfy the conditions (6), (7), (8), and

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n {X_n}^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{11}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{{X_n}^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{12}$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.
Remark. It should be noted that conditions (11) and (12) are the same as conditions (9) and (10), respectively, when k=1. When $k > 1$ conditions (11) and (12) are weaker than conditions (9) and (10), respectively. But the converses are not true. As in [14], we can show that if (9) is satisfied, then we get

$$\sum_{n=1}^{m} \frac{\mid t_n \mid^k}{n X_n^{k-1}} = O(\frac{1}{X_1^{k-1}}) \sum_{n=1}^{m} \frac{\mid t_n \mid^k}{n} = O(X_m).$$

To show that the converse is false when $k > 1$, as in [4], the following example is sufficient. We can take $X_n = n^\delta$, $0 < \delta < 1$, and then construct a sequence (u_n) such that

$$\frac{|t_n|^k}{n{X_n}^{k-1}} = X_n - X_{n-1},$$

hence

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n{X_n}^{k-1}} = \sum_{n=1}^{m} (X_n - X_{n-1}) = X_m = m^\delta,$$

and so

$$\begin{aligned} \sum_{n=1}^{m} \frac{|t_n|^k}{n} &= \sum_{n=1}^{m} (X_n - X_{n-1}) X_n^{k-1} = \sum_{n=1}^{m} (n^\delta - (n-1)^\delta) n^{\delta(k-1)} \\ &\geq \delta \sum_{n=1}^{m} n^{\delta-1} n^{\delta(k-1)} = \delta \sum_{n=1}^{m} n^{\delta k - 1} \sim \frac{m^{\delta k}}{k} \quad as \quad m \to \infty. \end{aligned}$$

It follows that

$$\frac{1}{X_m} \sum_{n=1}^{m} \frac{|t_n|^k}{n} \to \infty \quad as \quad m \to \infty$$

provided $k > 1$. This shows that (9) implies (11) but not conversely. The similar argument is also valid for the conditions (10) and (12).

3. Main Result

The aim of this paper is to generalize Theorem 2.2 by taking a quasi-f-power increasing sequence instead of a quasi-σ-power increasing sequence. Now, we shall prove the following theorem.
Theorem 3.1 Let (X_n) be a quasi-f-power increasing sequence. If the sequences (X_n), (λ_n), and (p_n) satisfy the all conditions of Theorem 2.2, then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.
Remark 3.2 It should be noted that if we take $\beta = 0$, then we obtain Theorem 2.2. Also if we take (X_n) as an almost increasing sequence, then we get a new result.
We need the following lemma for the proof of our theorem.
Lemma 3. 3 ([4]) Under the conditions (6) and (7) of Theorem 3.1, we have the following

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty, \tag{13}$$

$$n X_n |\Delta \lambda_n| = O(1) \quad as \quad n \to \infty. \tag{14}$$

4. Proof of Theorem 3.1

Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum a_n\lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\lambda_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v. \tag{15}$$

Then, for $n \geq 1$, we get

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}\lambda_v}{v}va_v. \tag{16}$$

Applying Abel's transformation to the right-hand side of (16), we have

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\Delta\left(\frac{P_{v-1}\lambda_v}{v}\right)\sum_{r=1}^{v} ra_r + \frac{p_n\lambda_n}{nP_n}\sum_{r=1}^{n} va_v \\ &= \frac{(n+1)p_nt_n\lambda_n}{nP_n} - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} p_vt_v\lambda_v\frac{v+1}{v} \\ &\quad + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v\Delta\lambda_vt_v\frac{v+1}{v} + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v\lambda_{v+1}t_v\frac{1}{v} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,r}\right|^k < \infty, \quad for \quad r = 1, 2, 3, 4.$$

Firstly, we have that

$$\begin{aligned} \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k &= O(1)\sum_{n=1}^{m}|\lambda_n|^{k-1}|\lambda_n|\frac{p_n}{P_n}|t_n|^k = O(1)\sum_{n=1}^{m}|\lambda_n|\frac{p_n}{P_n}\frac{|t_n|^k}{X_n^{k-1}} \\ &= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\frac{p_v}{P_v}\frac{|t_v|^k}{X_v^{k-1}} + O(1)|\lambda_m|\sum_{n=1}^{m}\frac{p_n}{P_n}\frac{|t_n|^k}{X_n^{k-1}} \\ &= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n + O(1)|\lambda_m|X_m = O(1) \quad as \quad m \to \infty, \end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3. Also, as in $T_{n,1}$, we have that

$$\begin{aligned} \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}p_v|t_v|^k|\lambda_v|^k\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1} \\ &= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} \\ &= O(1)\sum_{v=1}^{m}|\lambda_v|\frac{p_v}{P_v}\frac{|t_v|^k}{X_v^{k-1}} = O(1) \quad as \quad m \to \infty. \end{aligned}$$

Again, by using (8), we get that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,3}\right|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP^k_{n-1}}\left\{\sum_{v=1}^{n-1}P_v|\Delta\lambda_v||t_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP^k_{n-1}}\left(\sum_{v=1}^{n-1}\frac{P_v}{v}v|\Delta\lambda_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}\frac{P_v}{v}(v|\Delta\lambda_v|)^k|t_v|^k\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{v}\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\frac{P_v}{v}(v|\Delta\lambda_v|)^{k-1}v|\Delta\lambda_v|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{|t_v|^k}{vX_v{}^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta\left(v|\Delta\lambda_v|\right)\sum_{r=1}^{v}\frac{|t_r|^k}{rX_r{}^{k-1}}+O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{|t_v|^k}{vX_v{}^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\left|\Delta\left(v|\Delta\lambda_v|\right)\right|X_v+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}vX_v|\Delta^2\lambda_v|+O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v|+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and and Lemma 3.3. Finally, by using (8), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,4}\right|^k &\leq \sum_{n=2}^{m+1}\frac{p_n}{P_nP^k_{n-1}}\left(\sum_{v=1}^{n-1}\frac{P_v}{v}|\lambda_{v+1}||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}\frac{P_v}{v}|\lambda_{v+1}|^k|t_v|^k\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{v}\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\frac{P_v}{v}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}||t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{|t_v|^k}{vX_v{}^{k-1}}=O(1)\quad as\quad m\to\infty.
\end{aligned}$$

This completes the proof of Theorem 3.1.

5. An Application to Trigonometric Fourier Series

Let f be a periodic function with period 2π and Lebesgue integrable over $(-\pi,\pi)$. The trigonometric Fourier series of f is defined as

$$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}A_n(x).$$

Write $\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$ and $\phi_\alpha(t) = \frac{\alpha}{t^\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)du, \quad (\alpha > 0)$.

It is well know that if $\phi_1(t) \in \mathcal{BV}(0,\pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C,1)$ mean of the sequence $(nA_n(x))$ (see [7]). Using this fact, we have obtained the following main result dealing with the trigonometric Fourier series.

Theorem 5.1 ([5]) Let (X_n) be a quasi-σ-power increasing sequence. If $\phi_1(t) \in \mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 3.1, then the series $\sum A_n(x)\lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

Now, we have the following general theorem for the trigonometric Fourier series.

Theorem 5. 2 Let (X_n) be a quasi-f-power increasing sequence. If $\phi_1(t) \in \mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 3.1, then the series $\sum A_n(x)\lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

It should be noted that if we take $\beta = 0$, then we get Theorem 5.1. Also if we take $p_n = 1$ for all values of n, then we obtain a new result for the $|C,1|_k$ summability of trigonometric Fourier series.

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