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A Fixed Point Theorem for JS-contraction Type Mappings with Applications to Polynomial Approximations

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Abstract. A fixed point theorem is established for a new class of JS-contraction type mappings. As applications, some Kelisky-Rivlin type results are obtained for linear and nonlinear q-Bernstein-Stancu operators.

1. Introduction

Let Θ be the set of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:
(Θ_1) θ is non-decreasing;
(Θ_2) For each sequence $\{t_n\} \subset (0, \infty)$, we have

$$\lim_{n\to\infty} \theta(t_n) = 1 \iff \lim_{n\to\infty} t_n = 0^+;$$

(Θ_3) There exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$.

Recently, Jleli and Samet [4] introduced the class of JS-contraction mappings as follows.

Definition 1.1. *Let (X, d) be a metric space, and let $T : X \to X$ be a given mapping. The mapping T is said to be a JS-contraction if there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$(x, y) \in X \times X,\ d(Tx, Ty) > 0 \implies \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k.$$

In [4], the following generalization of Banach contraction principle was established.

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Theorem 1.2. *Let (X, d) be a complete metric space, and let $T : X \to X$ be JS-contraction. Then T has a unique fixed point.*

Observe that Banach contraction principle follows from Theorem 1.2 by taking $\theta(t) = e^{\sqrt{t}}$. For other related results, we refer the reader to [13, 16].

In this paper, a fixed point theorem for a new class of JS-contraction type mappings is presented. Next, this theorem is used to study the iterates properties of some polynomial operators including q-Bernstein-Stancu operators and q-Bernstein-Stancu operators of nonlinear type.

2. A Fixed Point Theorem

In this section, a new fixed poin theorem is established for a new class of JS-contraction type mappings. The obtained result is an extension of Theorem 1.2.

At first, let us introduce some notations. Let M be a nonempty set, and let $T : M \to M$ be a given mapping. We denote by $\mathrm{Fix}(T)$ the set of all the fixed points of T, that is,

$$\mathrm{Fix}(T) = \{x \in M : \ x = Tx\}.$$

Suppose that M is a group with respect to a certain operation $+$. For $x \in M$ and $N \subset M$, we denote by $x + N$ the subset of M defined by

$$x + N = \{x + y : \ y \in N\}.$$

We denote by $\mathbb{N}$ the set of positive integers, that is,

$$\mathbb{N} = \{0, 1, 2, \cdots\}.$$

We denote by $\mathbb{N}^*$ the set defined by

$$\mathbb{N}^* = \{1, 2, 3, \cdots\}.$$

Our fixed point theorem can be stated as follows.

Theorem 2.1. *Let E be a group with respect to a certain operation $+$. Let X be a subset of E endowed with a certain metric d such that (X, d) is complete. Let $X_0 \subset X$ be a closed subset of X such that X_0 is a subgroup of E. Let $T : X \to X$ be a given mapping satisfying*

$$(x, y) \in X \times X,\ x - y \in X_0,\ d(Tx, Ty) \neq 0 \implies \theta(d(Tx, Ty)) \leq [\theta(d(x, y)]^k, \tag{1}$$

where $k \in (0, 1)$ is a constant and $\theta \in \Theta$. Suppose that the operation mapping $\pm : X \times X \to X$ defined by

$$\pm(x, y) = x \pm y, \quad (x, y) \in X \times X$$

is continuous with respect to the metric d. Moreover, suppose that

$$x - Tx \in X_0, \quad x \in X. \tag{2}$$

Then we have

(i) For every $x \in X$, the Picard sequence $\{T^n x\}$ converges to a fixed point of T.

(ii) For every $x \in X$,

$$(x + X_0) \cap \mathrm{Fix}(T) = \left\{ \lim_{n\to\infty} T^n x \right\}.$$

Proof. Let $x \in X$ be an arbitrary point in X. If for some $p \in \mathbb{N}$, we have $T^p x = T^{p+1}x$, then $T^p x$ will be a fixed point of T. So, without restriction of the generality, we can suppose that $d(T^n x, T^{n+1}x) > 0$, for all $n \in \mathbb{N}$. From (2), we have

$$x - Tx \in X_0.$$

Using (1), we obtain

$$\theta(d(Tx, T^2x)) \leq [\theta(d(x, Tx))]^k.$$

Again, using (2), we obtain

$$Tx - T^2x = Tx - T(Tx) \in X_0,$$

which implies from (1) that

$$\theta(d(T^2x, T^3x)) \leq [\theta(d(Tx, T^2x))]^k \leq [\theta(d(x, Tx))]^{k^2}.$$

Therefore, by induction we obtain

$$T^n x - T^{n+1}x \in X_0, \quad n \in \mathbb{N}, \tag{3}$$

and

$$\theta(d(T^n x, T^{n+1}x)) \leq [\theta(d(x, Tx))]^{k^n}, \quad n \in \mathbb{N}.$$

Thus, we have

$$1 \leq \theta(d(T^n x, T^{n+1}x)) \leq [\theta(d(x, Tx))]^{k^n}, \quad n \in \mathbb{N}. \tag{4}$$

Passing to the limit as $n \to \infty$ in (4), we obtain

$$\lim_{n\to\infty} \theta(d(T^n x, T^{n+1}x)) = 1,$$

which implies from (Θ_2) that

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0. \tag{5}$$

From condition (Θ_3), there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(T^n x, T^{n+1}x)) - 1}{[d(T^n x, T^{n+1}x)]^r} = \ell.$$

Suppose that $\ell < \infty$. In this case, let $B = \ell/2 > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(d(T^n x, T^{n+1}x)) - 1}{[d(T^n x, T^{n+1}x)]^r} - \ell \right| \leq B, \quad n \geq n_0.$$

This implies that

$$\frac{\theta(d(T^n x, T^{n+1}x)) - 1}{[d(T^n x, T^{n+1}x)]^r} \geq \ell - B = B, \quad n \geq n_0.$$

Then,

$$n[d(T^n x, T^{n+1}x)]^r \leq An[\theta(d(T^n x, T^{n+1}x)) - 1], \quad n \geq n_0,$$

where $A = 1/B$.
Suppose now that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(d(T^n x, T^{n+1}x)) - 1}{[d(T^n x, T^{n+1}x)]^r} \geq B, \quad n \geq n_0.$$

This implies that

$$n[d(T^n x, T^{n+1} x)]^r \leq An[\theta(d(T^n x, T^{n+1} x)) - 1], \quad n \geq n_0,$$

where $A = 1/B$.
Thus, in all cases, there exists $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[d(T^n x, T^{n+1} x)]^r \leq An[\theta(d(T^n x, T^{n+1} x)) - 1], \quad n \geq n_0.$$

Using (4), we obtain

$$n[d(T^n x, T^{n+1} x)]^r \leq An\left([\theta(d(x, Tx))]^{k^n} - 1\right), \quad n \geq n_0.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n[d(T^n x, T^{n+1} x)]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d(T^n x, T^{n+1} x) \leq \frac{1}{n^{1/r}}, \quad n \geq n_1. \tag{6}$$

Using (6), we have

$$\begin{aligned} & d(T^n x, T^{n+m} x) \leq d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) + \cdots + d(T^{n+m-1} x, T^{n+m} x) \\ & \leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+m-1)^{1/r}} \\ & \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}, \end{aligned}$$

which implies that the Picard sequence $\{T^n x\}$ is Cauchy in the complete metric space (X, d) (since $r \in (0, 1)$). Then there is some $\omega \in X$ such that

$$\lim_{n\to\infty} d(T^n x, \omega) = 0. \tag{7}$$

On the other hand, observe that for $n, p \geq 1$,

$$T^n x - T^{n+p} x = (T^n x - T^{n+1} x) + (T^{n+1} x - T^{n+2} x) + \cdots + (T^{n+p-1} x - T^{n+p} x).$$

Therefore, by (3) and using the fact that $(X_0, +)$ is a group, we deduce that

$$T^n x - T^{n+p} x \in X_0, \quad n, p \geq 1.$$

Passing to the limit as $p \to \infty$, using (7), the continuity of the operation mapping $\pm$, and the closure of X_0, we obtain that

$$T^n x - \omega \in X_0, \quad n \in \mathbb{N}. \tag{8}$$

Without restriction of the generality, we may suppose that $d(T^n x, T\omega) > 0$, for all $n \in \mathbb{N}$. Therefore, using (8) and (1), we have

$$1 \leq \theta(d(T^{n+1} x, T\omega)) \leq [\theta(T^n x, \omega)]^k, \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$, using (7) and (Θ_2), we deduce that

$$\lim_{n\to\infty} d(T^{n+1} x, T\omega) = 0. \tag{9}$$

Next, (7), (9) and the uniqueness of the limit yield $\omega = T\omega$, that is, ω is a fixed point of T. Then (i) is proved. In order to prove (ii), let $x \in X$ be fixed. We know that the Picard sequence $\{T^n x\}$ converges to $\omega \in X$, a fixed point of T. Moreover, from (8), we have $\omega - x \in X_0$, that is, $\omega \in x + X_0$. Therefore, we have

$$\left\{\lim_{n\to\infty} T^n x\right\} \subset (x + X_0) \cap \mathrm{Fix}(T).$$

Now, let $z \in (x + X_0) \cap \mathrm{Fix}(T)$ be fixed. Then

$$Tz = z \quad \text{and} \quad z - x \in X_0.$$

Therefore, we have

$$z - Tx = Tz - Tx = (Tz - z) + (x - Tx) + (z - x) \in X_0.$$

Again,

$$z - T^2x = T^2z - T^2x = (T^2z - Tz) + (Tx - T^2x) + (z - Tx) \in X_0.$$

Hence, by induction we obtain

$$z - T^n x \in X_0, \quad n \in \mathbb{N}.$$

Without restriction of the generality, we may suppose that $z \neq T^n x$, for all $n \in \mathbb{N}$. Therefore, by (1) we have

$$1 \leq \theta(d(z, T^{n+1}x)) = \theta(d(Tz, T^{n+1}x)) \leq [\theta(d(z, T^n x))]^k \leq \cdots \leq [\theta(d(z, x))]^{k^{n+1}}, \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$ and using (Θ_2), we deduce that

$$\lim_{n\to\infty} d(T^n x, z) = 0,$$

which yields $z \in \left\{\lim_{n\to\infty} T^n x\right\}$. Then we proved that

$$(x + X_0) \cap \mathrm{Fix}(T) \subset \left\{\lim_{n\to\infty} T^n x\right\}.$$

The proof is complete. □

The following result follows immediately from Theorem 2.1 with $\theta(t) = e^{\sqrt{t}}$.

Corollary 2.2. *Let E be a group with respect to a certain operation $+$. Let X be a subset of E endowed with a certain metric d such that (X, d) is complete. Let $X_0 \subset X$ be a closed subset of X such that X_0 is a subgroup of E. Let $T : X \to X$ be a given mapping satisfying*

$$(x, y) \in X \times X,\ x - y \in X_0 \implies d(Tx, Ty) \leq k d(x, y),$$

where $k \in (0, 1)$ is a constant. Suppose that the operation mapping $\pm : X \times X \to X$ defined by

$$\pm(x, y) = x \pm y, \quad (x, y) \in X \times X$$

is continuous with respect to the metric d. Moreover, suppose that

$$x - Tx \in X_0, \quad x \in X.$$

Then we have

(i) For every $x \in X$, the Picard sequence $\{T^n x\}$ converges to a fixed point of T.

(ii) For every $x \in X$,

$$(x + X_0) \cap \mathrm{Fix}(T) = \left\{\lim_{n\to\infty} T^n x\right\}.$$

3. Applications: Iterates Properties of Some Polynomial Operators

In this section, as applications of Theorem 2.1, the iterates properties of some polynomial operators are investigated. Two types of polynomial operators are discussed: q-Bernstein-Stancu operators and q-Bernstein-Stancu operators of nonlinear type. For each kind of operators, a Kelisky-Rivlin type result is established. Let us mention some well known contributions in this topic. In [6], via some linear algebra tools, Kelisky and Rivlin studied the iterates properties of the class of Bernstein operators. Another proof of Kelisky-Rivlin theorem was presented by I.A. Rus [10] with the help of some trick with the Contraction principle. Another possibility to establish Kelisky-Rivlin theorem, which is based on a fixed point theorem for linear operators on a Banach space, was suggested by Jachymski [3]. For other related works, we refer to [1, 2, 8, 14, 15] and references therein.

The following basic notations in quantum calculus will be used. Let $q > 0$. For any $n \in \mathbb{N}$, the q-integer $[n]_q$ is defined by

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} \ (n \geq 1), \ [0]_q = 0.$$

The q-factorial $[n]_q!$ is defined by

$$[n]_q! = [1]_q[2]_q \cdots [n]_q \ (n \geq 1), \ [0]_q! = 1.$$

For integers $0 \leq k \leq n$, the q-binomial is defined by

$$\begin{pmatrix} n \\ k \end{pmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}.$$

It is clear that for $q = 1$, we have

$$[n]_1 = n, \ [n]_1! = n!, \ \begin{pmatrix} n \\ k \end{pmatrix}_1 = \begin{pmatrix} n \\ k \end{pmatrix}.$$

For more details on quantum calculus, we refer to [5].

3.1. A Kelisky-Rivlin type result for q-Bernstein-Stancu operators

Let $C([0,1];\mathbb{R})$ be the set of real valued and continuous functions $f : [0,1] \to \mathbb{R}$. For $f \in C([0,1];\mathbb{R})$, $q > 0$, $\alpha \geq 0$ and each $n \in \mathbb{N}^*$, the q-Bernstein-Stancu operator of order n is defined by [7]

$$B_n(q,\alpha)(f)(t) = \sum_{i=0}^{n} f\left(\frac{[i]_q}{[n]_q}\right) B_{n,i}^{q,\alpha}(t), \quad t \in [0,1],$$

where

$$B_{n,i}^{q,\alpha}(t) = \begin{pmatrix} n \\ i \end{pmatrix}_q \frac{\prod_{s=0}^{i-1}(t + \alpha[s]_q) \prod_{j=0}^{n-i-1}(1 - q^j t + \alpha[j]_q)}{\prod_{j=0}^{n-1}(1 + \alpha[j]_q)}.$$

From here on an empty product is taken to be equal to 1.

If $\alpha = 0$, $B_n(q,0)$ reduces to the q-Bernstein polynomial of order n introduced by Phillips [9]

$$B_n(q,0)(f)(t) = \sum_{i=0}^{n} f\left(\frac{[i]_q}{[n]_q}\right) \begin{pmatrix} n \\ i \end{pmatrix}_q t^i \prod_{j=0}^{n-1-i}(1 - q^j t), \quad t \in [0,1].$$

If $q = 1$, $B_n(1, \alpha)$ reduces to the Bernstein-Stancu polynomial of order n introduced by Stancu [11]

$$B_n(1,\alpha)(f)(t) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right)\begin{pmatrix} n \\ i \end{pmatrix} \frac{\prod_{s=0}^{i-1}(t+\alpha s)\prod_{j=0}^{n-i-1}(1-t+\alpha j)}{\prod_{j=0}^{n-1}(1+\alpha j)}, \quad t \in [0,1].$$

If $(\alpha, q) = (0, 1)$, we obtain the standard Bernstein polynomial of order n

$$B_n(1,0)(f)(t) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right)\begin{pmatrix} n \\ i \end{pmatrix} t^i(1-t)^{n-i}, \quad t \in [0,1].$$

The following lemmas will be useful later (see [2, 15]).

Lemma 3.1. *Let $n \in \mathbb{N}^*$, $q \in (0, 1)$ and $\alpha \geq 0$. Then*

$$\sum_{i=0}^{n} B_{n,i}^{q,\alpha}(t) = 1.$$

Lemma 3.2. *Let $n \in \mathbb{N}^*$, $q \in (0, 1)$ and $\alpha \geq 0$. Then*

$$\min\left\{B_{n,0}^{q,\alpha}(t) + B_{n,n}^{q,\alpha}(t) : t \in [0,1]\right\} > 0.$$

We have the following Kelisky-Rivlin type result for q-Bernstein-Stancu operators.

Theorem 3.3. *Let $n \in \mathbb{N}^*$, $\alpha \geq 0$ and $0 < q < 1$. Then, for every $f \in C([0, 1]; \mathbb{R})$,*

$$\lim_{N\to\infty} [B_n(q,\alpha)]^N(f)(t) = f(0) + [f(1) - f(0)]t, \quad t \in [0,1].$$

Proof. Let $X = E = C([0, 1]; \mathbb{R})$. We endow X with the metric d defined by

$$d(f,g) = \max\{|f(t) - g(t)| : t \in [0,1]\}, \quad (f,g) \in X \times X.$$

Then (X, d) is a complete metric space. Let X_0 be the subset of X defined by

$$X_0 = \{f \in X : f(0) = f(1) = 0\}.$$

Then X_0 is a closed linear subspace of X. Let $(f, g) \in X \times X$ be such that $f - g \in X_0$, that is,

$$(f,g) \in X \times X \quad \text{and} \quad f(0) = g(0),\ f(1) = g(1).$$

Let $t \in [0, 1]$ be fixed. Then we have

$$\begin{aligned}
&|B_n(q,\alpha)(f)(t) - B_n(q,\alpha)(g)(t)| \\
&= \left|\sum_{i=0}^{n} f\left(\frac{[i]_q}{[n]_q}\right)B_{n,i}^{q,\alpha}(t) - \sum_{i=0}^{n} g\left(\frac{[i]_q}{[n]_q}\right)B_{n,i}^{q,\alpha}(t)\right| \\
&= \left|\sum_{i=0}^{n}\left(f\left(\frac{[i]_q}{[n]_q}\right) - g\left(\frac{[i]_q}{[n]_q}\right)\right)B_{n,i}^{q,\alpha}(t)\right| \\
&\leq \sum_{i=0}^{n}\left|f\left(\frac{[i]_q}{[n]_q}\right) - g\left(\frac{[i]_q}{[n]_q}\right)\right|B_{n,i}^{q,\alpha}(t) \\
&= \sum_{i=1}^{n-1}\left|f\left(\frac{[i]_q}{[n]_q}\right) - g\left(\frac{[i]_q}{[n]_q}\right)\right|B_{n,i}^{q,\alpha}(t) \\
&\leq \left(\sum_{i=1}^{n-1} B_{n,i}^{q,\alpha}(t)\right)d(f,g).
\end{aligned}$$

On the other hand, using Lemmas 3.1 and 3.2, we get

$$\begin{aligned}\sum_{i=1}^{n-1} B_{n,i}^{q,\alpha}(t) &= 1-\left(B_{n,0}^{q,\alpha}(t)+B_{n,n}^{q,\alpha}(t)\right)\\ &\leq 1-\lambda,\end{aligned}$$

where

$$\lambda = \min\left\{B_{n,0}^{q,\alpha}(t)+B_{n,n}^{q,\alpha}(t) : t\in[0,1]\right\} > 0. \tag{10}$$

Terefore, we have

$$(f,g)\in X\times X,\ f-g\in X_0 \implies d(B_n(q,\alpha)(f),B_n(q,\alpha)(g))\leq kd(f,g),$$

where $k=1-\lambda\in(0,1)$. Next, using lemma 3.1, for every $f\in X$ we have

$$\gamma(t) := f(t)-B_n(q,\alpha)(f)(t) = \sum_{i=0}^{n}\left(f(t)-f\left(\frac{[i]_q}{[n]_q}\right)\right)B_{n,i}^{q,\alpha}(t),\ t\in[0,1].$$

We can check easily that

$$\gamma(0)=\gamma(1)=0,$$

which yields

$$f-B_n(q,\alpha)(f)\in X_0,\quad f\in X.$$

Applying Theorem 2.1 (or Corollary 2.2), we deduce that

$$(f+X_0)\cap \mathrm{Fix}(B_n(q,\alpha)) = \left\{\lim_{N\to\infty}[B_n(q,\alpha)]^N(f)\right\},\quad f\in X.$$

Let $f\in X$. It is not difficult to observe that the function $\omega:[0,1]\to\mathbb{R}$ defined by

$$\omega(t)=f(0)(1-t)+f(1)t,\quad t\in[0,1]$$

belongs to $\mathrm{Fix}(B_n(q,\alpha))$. Moreover, for all $t\in[0,1]$,

$$\mu(t) := \omega(t)-f(t) = f(0)(1-t)+f(1)t-f(t).$$

Observe that

$$\mu(0)=f(0)-f(0)=0$$

and

$$\mu(1)=f(1)-f(1)=0.$$

Therefore, $\omega\in f+X_0$. As consequence, we get

$$\lim_{N\to\infty} d([B_n(q,\alpha)]^N(f),\omega)=0,$$

which yields the desired result. □

Remark 3.4. *Another proof of Theorem 3.3 can be found in [15]. This proof is based on some linear algebra tools. In our opinion, the presented proof in this paper is more esay and more simplified.*

3.2. A Kelisky-Rivlin type result for nonlinear q-Bernstein-Stancu operators

For $f \in C([0,1];\mathbb{R})$, $q > 0$, $\alpha \geq 0$ and each $n \in \mathbb{N}^*$, we define the nonlinear q-Bernstein-Stancu operator of order n by

$$T_n(q,\alpha)(f)(t) = \sum_{i=0}^{n} \left| f\left(\frac{[i]_q}{[n]_q}\right)\right| B_{n,i}^{q,\alpha}(t), \quad t \in [0,1].$$

Using Theorem 2.1, we shall establish the following Kelisky-Rivlin type result.

Theorem 3.5. *Let $n \in \mathbb{N}^*$, $\alpha \geq 0$ and $0 < q < 1$. Then, for every $f \in C([0,1];\mathbb{R})$ such that $f(0) \geq 0$ and $f(1) \geq 0$,*

$$\lim_{N\to\infty} [T_n(q,\alpha)]^N(f)(t) = f(0) + [f(1) - f(0)]t, \quad t \in [0,1].$$

Proof. Let $E = C([0,1];\mathbb{R})$ and X be the subset if E defined by

$$X = \{f \in E : f(0) \geq 0, f(1) \geq 0\}.$$

We endow X with the metric d defined by

$$d(f,g) = \max\{|f(t) - g(t)| : t \in [0,1]\}, \quad (f,g) \in X \times X.$$

Then (X,d) is a complete metric space. Let X_0 be the subset of X defined by

$$X_0 = \{f \in E : f(0) = f(1) = 0\}.$$

Then X_0 is a closed subgroup of E. Let $(f,g) \in X \times X$ be such that $f - g \in X_0$, that is,

$$(f,g) \in X \times X \quad \text{and} \quad f(0) = g(0),\ f(1) = g(1).$$

Let $t \in [0,1]$ be fixed. Then we have

$$\begin{aligned}
&|T_n(q,\alpha)(f)(t) - T_n(q,\alpha)(g)(t)| \\
&= \left| \sum_{i=0}^{n} \left| f\left(\frac{[i]_q}{[n]_q}\right)\right| B_{n,i}^{q,\alpha}(t) - \sum_{i=0}^{n} \left| g\left(\frac{[i]_q}{[n]_q}\right)\right| B_{n,i}^{q,\alpha}(t) \right| \\
&= \left| \sum_{i=0}^{n} \left(\left| f\left(\frac{[i]_q}{[n]_q}\right)\right| - \left| g\left(\frac{[i]_q}{[n]_q}\right)\right| \right) B_{n,i}^{q,\alpha}(t) \right| \\
&\leq \sum_{i=0}^{n} \left| f\left(\frac{[i]_q}{[n]_q}\right) - g\left(\frac{[i]_q}{[n]_q}\right)\right| B_{n,i}^{q,\alpha}(t) \\
&= \sum_{i=1}^{n-1} \left| f\left(\frac{[i]_q}{[n]_q}\right) - g\left(\frac{[i]_q}{[n]_q}\right)\right| B_{n,i}^{q,\alpha}(t) \\
&\leq \left(\sum_{i=1}^{n-1} B_{n,i}^{q,\alpha}(t) \right) d(f,g) \\
&= (1-\lambda) d(f,g),
\end{aligned}$$

where λ is given by (10). Terefore, we have

$$(f,g) \in X \times X,\ f - g \in X_0 \implies d(T_n(q,\alpha)(f), T_n(q,\alpha)(g)) \leq k d(f,g),$$

where $k = 1 - \lambda \in (0,1)$. Next, for every $f \in X$ we have

$$\gamma'(t) := f(t) - T_n(q,\alpha)(f)(t) = \sum_{i=0}^{n} \left(f(t) - \left| f\left(\frac{[i]_q}{[n]_q}\right)\right| \right) B_{n,i}^{q,\alpha}(t),\ t \in [0,1].$$

Observe that

$$\gamma'(0) = f(0) - |f(0)| = f(0) - f(0) = 0$$

and

$$\gamma'(1) = f(1) - |f(1)| = f(1) - f(1) = 0.$$

Then

$$f - T_n(q,\alpha)(f) \in X_0, \quad f \in X.$$

Applying Theorem 2.1 (or Corollary 2.2), we deduce that

$$(f + X_0) \cap \mathrm{Fix}(T_n(q,\alpha)) = \left\{ \lim_{N\to\infty} [T_n(q,\alpha)]^N(f) \right\}, \quad f \in X.$$

Let $f \in X$. It is not difficult to observe that the function $\omega : [0,1] \to \mathbb{R}$ defined by

$$\omega(t) = f(0)(1-t) + f(1)t, \quad t \in [0,1]$$

belongs to $(f + X_0) \cap \mathrm{Fix}(T_n(q,\alpha))$. As consequence, we get

$$\lim_{N\to\infty} d([T_n(q,\alpha)]^N(f), \omega) = 0,$$

which yields the desired result. □

Remark 3.6. *Note that Theorem 4.1 in [3] cannot be applied in our case since it requires linear operators defined on a certain Banach space X. Observe that in our case, X is not a linear space.*

Remark 3.7. *The case* $(\alpha, q) = (0, 1)$ *was considered in [12]. The authors claimed that if* $n \in \mathbb{N}$, *for every* $f \in X = C([0,1];\mathbb{R})$, *the Picard sequence* $[T_n(0,1)]^N(f)$ *converges uniformly to a fixed point of* $T_n(0,1)$ *(see Corollary 4 in [12]). For the proof of this claim, the authors used that* $f - T_n(0,1)(f) \in X_0$ *for every* $f \in X$, *where* X_0 *is the set of functions* $u \in X$ *such that* $u(0) = u(1) = 0$. *Unfortunately, the above property is not true. To observe this fact, we have just to consider a function* $f \in X$ *such that* $f(0) < 0$ *or* $f(1) < 0$. *Our Theorem 3.5 for the case* $(\alpha, q) = (0, 1)$ *is a corrected version of Corollary 4 in [12].*

References

[1] O. Agratini, I. A. Rus, Iterates of a class of discrete linear operators via contraction principle, Commentationes Mathematicae Universitatis Carolinae 44 (2003) 555–563.

[2] O. Agratini, On a q-analogue of Stancu operators, Open Mathematics 8 (1) (2010) 191–198.

[3] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proceedings of the American Mathematical Society 136 (2008) 1359–1373.

[4] M. Jleli, B. Samet, A new generalization of the Banach contraction principle, Journal of Inequalities and Applications 2014:38 (2014).

[5] V. Kac, P. Cheung, Quantum Calculus, Universitext, Springer-Verlag, New York (2002)

[6] R. P. Kelisky, T. J. Rivlin, Iterates of Bernstein polynomials, Pacific Journal of Mathematics 21 (1967) 511–520.

[7] G. Nowak, Approximation properties for generalized q-Bernstein polynomials, Journal of Mathematical Analysis and Applications 350 (2009) 50–55.

[8] S. Ostrovska, q-Bernstein polynomials and their iterates, Journal of Approximation Theory 123 (2003) 232–255.

[9] G. M. Phillips, A generalization of the Bernstein polynomials based on the q-integers, The ANZIAM Journal 42 (2000) 79–86.

[10] I. A. Rus, Iterates of Bernstein operators, via contraction principle, Journal of Mathematical Analysis and Applications 292 (2004) 259–261.

[11] D. D. Stancu, Approximation of functions by a new class of linear polynomial operators, Rev. Roumaine Math. Pures Appl 13 (1968) 1173–1194.

[12] A. Sultana, V. Vetrivel, Fixed points of Mizoguchi-Takahashi contraction on a metric space with a graph and applications, Journal of Mathematical Analysis and Applications 417 (2014) 336–344.

[13] T. Suzuki, Comments on some recent generalization of the Banach contraction principle, Journal of Inequalities and Applications 2016:111 (2016).

[14] T. Vedi, M. A. Ozarslan, Chlodowsky-type q-Bernstein-Stancu-Kantorovich operators, Journal of Inequalities and Applications 2015:91 (2015).

[15] W. Yali, Z. Yinying, Iterates properties for q-Bernstein-Stancu operators, International Journal of Modeling and Optimization 3 (2013) 362–368.

[16] Li. Zhilong, J. Shujun, Fixed point theorems of JS-quasi-contractions, Fixed Point Theory and Applications 2016:40 (2016).