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Integer Powers of Certain Complex Tridiagonal Matrices and Some Complex Factorizations

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Abstract. In this paper, we obtain a general expression for the entries of the rth power of a certain $n \times n$ complex tridiagonal matrix where if n is even, $r \in \mathbb{Z}$ or if n is odd, $r \in \mathbb{N}$. In addition, we get the complex factorizations of Fibonacci polynomials, Fibonacci and Pell numbers.

1. Introduction

Arbitrary integer powers of a square matrix is used to solve some difference equations, differential and delay differential equations and boundary value problems.

Recently, the calculations of integer powers and eigenvalues of tridiagonal matrices have been well studied in the literature. For instance, Rimas [1-4] obtained the positive integer powers of certain tridiagonal matrices of odd and even order in terms of the Chebyshev polynomials. Öteleş and Akbulak [6,7] generalized the results obtained in [1-4]. Gutiérrez [8,10] calculated the powers of tridiagonal matrices with costant diagonal. For details on the powers and the eigenvalues of tridiagonal matrices, see [5,9].

In [12], Cahill et al. considered the following tridagonal matrix

$$H(n) = \begin{pmatrix} h_{1,1} & h_{1,2} & & & 0 \\ h_{2,1} & h_{2,2} & h_{2,3} & & \\ & h_{3,2} & h_{3,3} & \ddots & \\ & & \ddots & \ddots & h_{n-1,n} \\ 0 & & & h_{n,n-1} & h_{n,n} \end{pmatrix}$$

and using the succesive determinants they computed determinant of $H(n)$ as

$$|H(n)| = h_{n,n} |H(n-1)| - h_{n-1,n} h_{n,n-1} |H(n-2)|$$

with initial conditions $|H(1)| = h_{1,1}$, $|H(2)| = h_{1,1}h_{2,2} - h_{1,2}h_{2,1}$.

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Let $\{H^{\dagger}(n), n = 1, 2, \ldots\}$ be the sequence of tridiagonal matrices as the following form

$$H^{\dagger}(n) = \begin{pmatrix} h_{1,1} & -h_{1,2} & & & \\ -h_{2,1} & h_{2,2} & -h_{2,3} & & \\ & -h_{3,2} & h_{3,3} & \ddots & \\ & & \ddots & \ddots & -h_{n-1,n} \\ & & & -h_{n,n-1} & h_{n,n} \end{pmatrix}.$$

Then

$$\det(H(n)) = \det(H^{\dagger}(n)). \tag{1}$$

Let T and $T^{\dagger}$ be $n \times n$ tridiagonal matrices as the following

$$T := \begin{pmatrix} 0 & 2 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 2 & 0 \end{pmatrix} [1],$$

$$T^{\dagger} := \begin{pmatrix} 1 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \end{pmatrix} [2].$$

By [1, p. 3] and [2, p. 2], the eigenvalues of T and $T^{\dagger}$ are obtained as

$$\mu_k = 2\cos\left(\frac{(k-1)\pi}{n-1}\right),\ k = 1, \ldots, n \ [1,\ p.3]$$

and

$$\mu_k^{\dagger} = -2\cos\left(\frac{k\pi}{n}\right),\ k = 1, \ldots, n \ [2,\ p.2]$$

respectively.

Let

$$A := \begin{pmatrix} a & 2b & & & & 0 \\ b & a & -b & & & \\ & -b & a & -b & & \\ & & \ddots & \ddots & \ddots & \\ & & & -b & a & b \\ 0 & & & & 2b & a \end{pmatrix} \tag{2}$$

and

$$A^{\dagger} := \begin{pmatrix} a+b & b & & & & 0 \\ b & a & -b & & & \\ & -b & a & -b & & \\ & & \ddots & \ddots & \ddots & \\ & & & -b & a & b \\ 0 & & & & b & a+b \end{pmatrix} \tag{3}$$

be the tridiagonal matrices, where $b \neq 0$ and $a, b \in \mathbb{C}$. In this paper, we obtain the eigenvalues and eigenvectors of the $n \times n$ complex tridiagonal matrices in (2) and (3). Also, we will calculate the all integer powers of the matrix in (2) for n is odd order and only will calculate the positive integer powers of the matrix in (2) for n is even order. We also get the complex factorizations of Fibonacci polynomials, Fibonacci and Pell numbers using the eigenvalues of the matrices A and $A^{\dagger}$.

2. Eigenvalues and Eigenvectors of A and $A^{\dagger}$

In this section, we obtain the eigenvalues and eigenvectors of the $n \times n$ complex tridiagonal matrices A and $A^{\dagger}$ in (2) and (3), respectively.

Theorem 2.1. *Let A be an $n \times n$ complex tridiagonal matrix given by (2). Then the eigenvalues and eigenvectors of A are*

$$\lambda_k = a + 2b\cos\left(\frac{(k-1)\pi}{n-1}\right),\ k = 1, \ldots, n \tag{4}$$

and

$$x_{jk} = \begin{cases} T_{j-1}(m_k), & j = 1, 2, n-1, n \\ (-1)^j T_{j-1}(m_k), & j = 3, \ldots, n-2 \end{cases} ;k = 1, \ldots, n$$

where $m_k = \frac{\lambda_k - a}{2b}$, $T_s(.)$ is the $s-$th degree Chebyshev polynomial of the first kind [11, p. 14].

Proof. Let B be the following $n \times n$ tridiagonal matrix

$$B := \begin{pmatrix} c & 2 & & & & \\ 1 & c & -1 & & & \\ & -1 & c & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & c & 1 \\ & & & & 2 & c \end{pmatrix} \tag{5}$$

where $c = \frac{a}{b}$. Then the characteristic polynomials of B are

$$p_n(t) = (t^2 - 4)\Delta_{n-2}(t) \tag{6}$$

where $t = \lambda - c$ and

$$\Delta_n(t) = t\Delta_{n-1}(t) - \Delta_{n-2}(t) \tag{7}$$

with initial conditions $\Delta_0(t) = 1, \Delta_1(t) = t$ and $\Delta_2(t) = t^2 - 1$. Note that the solution of the difference equation in (7) is obtained as $\Delta_n(t) = U_n(\frac{t}{2})$, where U_n is the $n-$th degree Chebyshev polynomial of the second kind [11, p.15]. i.e.

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)},\ \ -1 \leq x \leq 1.$$

All the roots of the polynomial $U_n(x)$ are included in the interval $[-1, 1]$ and can be found using the relation

$$x_{nk} = \cos\left(\frac{k\pi}{n+1}\right),\ \ k = 1, \ldots, n.$$

Therefore the characteristic polynomial in (6) can be written as

$$p_n(t) = (t^2 - 4)U_{n-2}(\tfrac{t}{2}).$$

From [6, p. 2], the eigenvalues of the matrix B are

$$t_k = 2\cos\left(\frac{(k-1)\pi}{n-1}\right),\ k = 1,\dots,n.$$

Then we get the eigenvalues of the matrix A as

$$\lambda_k = a + 2b\cos\left(\frac{(k-1)\pi}{n-1}\right).$$

Now we compute eigenvectors of the matrix A. All eigenvectors of A are obtained as the solutions of the following homogeneous linear equations system

$$(\lambda_k I_n - A)x = 0 \tag{8}$$

where λ_k is the kth eigenvalue of the matrix A ($k = \overline{1,n}$). The equations system (8) is clearly written as

$$\left.\begin{aligned} (\lambda_k - a)x_1 - 2bx_2 &= 0 \\ -bx_1 + (\lambda_k - a)x_2 + bx_3 &= 0 \\ bx_2 + (\lambda_k - a)x_3 + bx_4 &= 0 \\ &\vdots \\ bx_{n-2} + (\lambda_k - a)x_{n-1} - bx_n &= 0 \\ -2bx_{n-1} + (\lambda_k - a)x_n &= 0 \end{aligned}\right\} \tag{9}$$

Dividing all terms of the each equation in the system (9) by $b \neq 0$, substituting $m_k = \frac{\lambda_k - a}{2b}$, choosing $x_1 = 1$ and solving the sets of the system (9), we find the jth component of kth eigenvector of the matrix A as

$$x_{jk} = \begin{cases} T_{j-1}(m_k), & j = 1,2,n-1,n \\ (-1)^j T_{j-1}(m_k), & j = 3,\dots,n-2 \end{cases} ;k = 1,\dots,n \tag{10}$$

where $m_k = \frac{\lambda_k - a}{2b}$ and $T_s(.)$ is the $s-$th degree Chebyshev polynomial of the first kind. □

Theorem 2.2. *Let $A^\dagger$ be an $n \times n$ complex tridiagonal matrix given by (3). Then the eigenvalues and eigenvectors of $A^\dagger$ are*

$$\lambda_k^\dagger = a - 2b\cos\left(\frac{k\pi}{n}\right),\ k = 1,\dots,n$$

and

$$y_{jk}^\dagger = \begin{cases} T_{\frac{2j-1}{2}}(m_k^\dagger), & j = 1,2,n-1,n \\ (-1)^j T_{\frac{2j-1}{2}}(m_k^\dagger), & j = 3,\dots,n-2 \end{cases} ;k = 1,\dots,n$$

where $m_k^\dagger = \frac{\lambda_k^\dagger - a}{2b}$ and $T_s(.)$ is the $s-$th degree Chebyshev polynomial of the first kind [11, p. 14].

Proof. Let

$$S := \begin{pmatrix} \frac{a}{b}+1 & 1 & & & & 0 \\ 1 & \frac{a}{b} & -1 & & & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & \frac{a}{b} & -1 & \\ & & & -1 & \frac{a}{b} & 1 \\ 0 & & & & 1 & \frac{a}{b}+1 \end{pmatrix}.$$

From [7, Lemma 2, p. 65], we have the eigenvalues of the matrix S as

$$\delta_k = \frac{a}{b} - 2\cos\left(\frac{k\pi}{n}\right), \text{ for } k = 1, \ldots, n$$

Since the eigenvalues of $A^\dagger$ are equal to $\lambda_k^\dagger = b\delta_k$, the proof of theorem is completed.

The eigenvectors of $A^\dagger$ are obtained as the solution of the following linear homogeneous equations system:

$$(\lambda_j^\dagger I_n - A^\dagger) y_{jk} = 0 \tag{11}$$

where $\lambda_j^\dagger$ and y_{jk} $(1 \le j, k \le n)$ are the jth eigenvalues and kth eigenvectors of the matrix $A^\dagger$, respectively. Then the solution of the equations system in (11) is

$$y_{jk} = \begin{cases} T_{\frac{2j-1}{2}}(m_k^\dagger), & j = 1, 2, n-1, n \\ (-1)^j T_{\frac{2j-1}{2}}(m_k^\dagger), & j = 3, \ldots, n-2 \end{cases} ; k = 1, \ldots, n$$

where $m_k^\dagger = \frac{\lambda_k^\dagger - a}{2b}$ and $T_s(.)$ is the $s-$th degree Chebyshev polynomial of the first kind. □

3. The Integer Powers of A

In this section, we want to calculate the integer powers of the $n \times n$ complex matrix A in (2), where n is positive odd integer. We first recall that the well-known expression $A^r = PJ^rP^{-1}$, where J is the Jordan's form and P is the transforming matrix of A, respectively.

Since all the eigenvalues λ_k $(k = 1, \ldots, n)$ are simple, each eigenvalue λ_k corresponds single Jordan cells $J_1(\lambda_k)$ in the matrix J. Then, we write down the Jordan's form of the matrix A as

$$J = diag(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n).$$

If n is even, the matrix A hasn't inverse for some a and b (for example $a = b = 1$). Therefore there aren't the negative powers of the matrix A for n is even. Then, we get $r \in \mathbb{N}$ for n is even and $r \in \mathbb{Z}$ for n is odd.

In order to derive the expressions for the rth power of A, we also need to transforming matrix P and its inverse P^{-1}. From (10), we have the transforming matrix P as

$$P = [x_{jk}] = \begin{cases} T_{j-1}(m_k), & j = 1, 2, n-1, n \\ (-1)^j T_{j-1}(m_k), & j = 3, \ldots, n-2 \end{cases} ; k = 1, \ldots, n$$

where $T_s(.)$ is the $s-$th degree Chebyshev polynomial of the first kind.

We now obtain the inverse of transforming matrix. Denoting jth column of the matrix P^{-1} by p_j, we have

$$p_j = \begin{pmatrix} 2T_{j-1}(m_1) \\ T_{j-1}(m_2) \\ T_{j-1}(m_3) \\ 2T_{j-1}(m_4) \\ \vdots \\ 2T_{j-1}(m_n) \end{pmatrix}, \ j = 1, n$$

and

$$p_j = \begin{pmatrix} (-1)^j 4T_{j-1}(m_1) \\ (-1)^j 2T_{j-1}(m_2) \\ (-1)^j 2T_{j-1}(m_3) \\ (-1)^j 4T_{j-1}(m_4) \\ \vdots \\ (-1)^j 4T_{j-1}(m_n) \end{pmatrix}, j = 2, \ldots, n-1.$$

Therefore we obtain

$$P^{-1} = \frac{1}{2n-2}(p_1, p_2, \dots, p_n).$$

Let

$$A^r = PJ^rP^{-1} = U(r) = (u_{ij}(r)).$$

Then

$$PJ^r = \begin{cases} \lambda_k^r T_{j-1}(m_k), & j = 1, 2, n-1, n \\ (-1)^j \lambda_k^r T_{j-1}(m_k), & j = 3, \dots, n-2 \end{cases} ; k = 1, \dots, n.$$

Hence we get

$$\begin{aligned} u_{ij}(r) &= \frac{1}{2n-2}\Big(\lambda_2^r T_{i-1}(m_2)T_{j-1}(m_2) + \lambda_3^r T_{i-1}(m_3)T_{j-1}(m_3) \\ &\quad + 2\sum_{\substack{k=1\\k\neq 2,3}}^{n} \lambda_k^r T_{i-1}(m_k)T_{j-1}(m_k)\Big) \end{aligned} \tag{1}$$

where $i = 1, \dots, n;\ j = 1, n$ and

$$\begin{aligned} u_{ij}(r) &= \frac{1}{n-1}\Bigg((-1)^j(\lambda_2^r T_{i-1}(m_2)T_{j-1}(m_2) + \lambda_3^r T_{i-1}(m_3)T_{j-1}(m_3)) \\ &\quad + (-1)^j 2\sum_{\substack{k=1\\k\neq 2,3}}^{n} \lambda_k^r T_{i-1}(m_k)T_{j-1}(m_k)\Bigg) \end{aligned} \tag{2}$$

where $i = 1, \dots, n;\ j = 2, \dots, n-1$.

4. Numerical Examples

Considering the Eqs. (12) and (13), we can find the arbitrary integer powers of the $n \times n$ complex tridiagonal matrix A in (2), where n is positive odd integer.

Example 4.1. *Let $n = 3, r = 3, a = 1$ and $b = 3$. Then we have*

$$J = diag(\lambda_1, \lambda_2, \lambda_3) = diag(a, a+2b, a-2b) = diag(1, 7, -5).$$

Therefore we get

$$A^3 = (q_{ij}(r)) = (q_{ij}(3)) = \begin{pmatrix} 55 & 234 & 54 \\ 117 & 109 & 117 \\ 54 & 234 & 55 \end{pmatrix}.$$

Example 4.2. *For $n = 3, r = -2, a = 1$ and $b = 3$, we obtain*

$$A^{-2} = \frac{1}{1225}\begin{pmatrix} 631 & -12 & -594 \\ -6 & 37 & -6 \\ -594 & -12 & 631 \end{pmatrix}.$$

Example 4.3. *If $n=5, r=4, a=1$ and $b=3$, then*

$$\begin{aligned} J &= diag(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5) \\ &= diag(1,7,-5,1+3\sqrt{2},1-3\sqrt{2}). \end{aligned}$$

Therefore

$$A^4 = q_{ij}(4) = \begin{pmatrix} 595 & 672 & -756 & 216 & 162 \\ 336 & 973 & -444 & 540 & 108 \\ -378 & -444 & 757 & -444 & -378 \\ 108 & 540 & -444 & 973 & 336 \\ 162 & 216 & -756 & 672 & 595 \end{pmatrix}.$$

Example 4.4. *For $n=4, r=3, a=1$ and $b=1$, we obtain*

$$\begin{aligned} J &= diag(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \\ &= diag(-1,0,2,3). \end{aligned}$$

Then we have

$$A^3 = q_{ij}(3) = \begin{pmatrix} 7 & 12 & -6 & -2 \\ 6 & 10 & -8 & -3 \\ -3 & -8 & 10 & 6 \\ -2 & -6 & 12 & 7 \end{pmatrix}.$$

5. Complex Factorizations

Fibonacci and Pell numbers are defined by

$$F_n = F_{n-1} + F_{n-2}, (F_0 = 0, F_1 = 1, n \geq 2)$$

and

$$P_n = 2P_{n-1} + P_{n-2}, (P_0 = 0, P_1 = 1, n \geq 2),$$

respectively. The generalized order-k Fibonacci-Pell numbers are defined by

$$u_n^i = 2^m u_{n-1}^i + u_{n-2}^i + \cdots + u_{n-k}^i$$

with initial condition

$$u_n^i = \begin{cases} 1, \text{if } i = 1-n \\ 0, \text{ otherwise} \end{cases}$$

for $n > 0, m \geq 0$ and $1 \leq i \leq k$ [13, p. 135].

The well-known $F(x) = \{F_n(x)\}_{n=1}^{\infty}$ Fibonacci polynomials are defined by $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ with initial conditions $F_0(x) = 0$ and $F_1(x) = 1$. For instance, if $x = 1$ and $x = 2$, then we obtain the Fibonacci and Pell numbers sequences as

$$\{F_n\} = F_n(1) = \{0, 1, 1, 2, 3, 5, 8, \ldots\}$$

and

$$\{P_n\} = F_n(2) = \{0, 1, 2, 5, 12, 29, \ldots\},$$

respectively.

Theorem 5.1. *Let A be an $n \times n$ complex tridiagonal matrix given by (2). If $a := x$ and $b := \mathbf{i}$, then*

$$\det(A) = (x^2 + 4)F_{n-1}(x).$$

where $\mathbf{i} = \sqrt{-1}$.

Proof. Applying Laplace expansion according to the first two and last two rows of the determinant of A, we have

$$\det(A) = x^2 D_{n-2} + 4xD_{n-3} + 4D_{n-4}$$

where $D_n = \det(tridiag_n(-\mathbf{i}, x, -\mathbf{i}))$. Since

$$\det(tridiag_n(-\mathbf{i}, x, -\mathbf{i})) = F_{n+1}(x),$$

we arrive at

$$\begin{aligned} \det(A) &= x^2 F_{n-1}(x) + 4xF_{n-2}(x) + 4F_{n-3}(x) \\ &= x^2(xF_{n-2}(x) + F_{n-3}(x)) + 4xF_{n-2}(x) + 4F_{n-3}(x) \\ &= (x^2+4)(xF_{n-2}(x) + F_{n-3}(x)) = (x^2+4)F_{n-1}(x). \end{aligned}$$

Hence, the proof of theorem is completed. □

Corollary 5.2. *Let A be an $n \times n$ complex tridiagonal matrix given by (2). If $a := x$ and $b := \mathbf{i}$, then the complex factorization of generalized Fibonacci-Pell numbers is the following form:*

$$F_{n-1}(x) = \frac{1}{x^2+4} \prod_{k=1}^{n} \left(x + 2\mathbf{i} \cos\left(\frac{(k-1)\pi}{n-1} \right) \right)$$

Proof. Since the eigenvalues of the matrix A from (4)

$$\lambda_j = x + 2\mathbf{i} \cos\left(\frac{(j-1)\pi}{n-1} \right), \ j = \overline{1, n},$$

the determinant of the matrix A can be obtained as

$$\det(A) = \prod_{k=1}^{n} \left(x + 2\mathbf{i} \cos\left(\frac{(k-1)\pi}{n-1} \right) \right). \tag{14}$$

By considering (14) and Theorem 5, the complex factorization of generalized Fibonacci-Pell numbers is obtained. □

Theorem 5.3. *Let $A^\dagger$ be an $n \times n$ complex tridiagonal matrix given by (3). Then*

$$\det(A^\dagger) = \begin{cases} (1+2\mathbf{i})F_n, & \text{if } a = 1 \text{ and } b = \mathbf{i} \\ (2+2\mathbf{i})P_n, & \text{if } a = 2 \text{ and } b = \mathbf{i} \end{cases}$$

where $\mathbf{i} = \sqrt{-1}$ *and F_n and P_n denote the nth Fibonacci and Pell numbers, respectively.*

Proof. Applying Laplace expansion according to the first two and last two rows of the determinant of $A^\dagger$, we have

$$\begin{aligned} \det(A^\dagger) &= (a+b)^2 \det(tridiag_{n-2}(-b, a, -b)) \\ &\quad -2b^2(a+b) \det(tridiag_{n-3}(-b, a, -b)) \\ &\quad +b^4 \det(tridiag_{n-4}(-b, a, -b)). \end{aligned} \tag{3}$$

If we take $a = 1$ and $b = \mathbf{i}$ in (15), then we get

$$\begin{aligned} \det(A^{\dagger}) &= (1+\mathbf{i})^2 \det(tridiag_{n-2}(-\mathbf{i},1,-\mathbf{i})) \\ &\quad +2(1+\mathbf{i}) \det(tridiag_{n-3}(-\mathbf{i},1,-\mathbf{i})) \\ &\quad + \det(tridiag_{n-4}(-\mathbf{i},1,-\mathbf{i})). \end{aligned}$$

Considering the equality $\det(tridiag_n(\mathbf{i},1,\mathbf{i})) = \det(tridiag_n(-\mathbf{i},1,-\mathbf{i}))$ and Eq. (1), we obtain

$$\begin{aligned} \det(A^{\dagger}) &= (1+\mathbf{i})^2 F_{n-1} + 2(1+\mathbf{i})F_{n-2} + F_{n-3} \\ &= (1+2\mathbf{i})F_n. \end{aligned}$$

Similar to the above, we can easily obtain Pell numbers. □

Corollary 5.4. *Let $A^{\dagger}$ be an $n \times n$ complex tridiagonal matrix given by (3). If $a := 2$ and $b := \mathbf{i}$, then the complex factorizations of Fibonacci and Pell numbers are*

$$F_n = \prod_{k=1}^{n-1} \left(1 - 2\mathbf{i}\cos\left(\frac{k\pi}{n}\right)\right)$$

and

$$P_n = \prod_{k=1}^{n-1} \left(2 - 2\mathbf{i}\cos\left(\frac{k\pi}{n}\right)\right).$$

Proof. Since the eigenvalues of the matrix $A^{\dagger}$ are

$$\lambda_k = a - 2b\cos\cos\left(\frac{k\pi}{n}\right),\ k = \overline{1,n}$$

and the determinant of the matrix $A^{\dagger}$ is equal to multiplication of its eigenvalues, we get

$$\begin{aligned} F_n &= \frac{1}{1+2\mathbf{i}} \prod_{k=1}^{n} \left(1 - 2\mathbf{i}\cos\left(\frac{k\pi}{n}\right)\right) \\ &= \prod_{k=1}^{n-1} \left(1 - 2\mathbf{i}\cos\left(\frac{k\pi}{n}\right)\right) \end{aligned}$$

and

$$\begin{aligned} P_n &= \frac{1}{2+2\mathbf{i}} \prod_{k=1}^{n} \left(2 - 2\mathbf{i}\cos\left(\frac{k\pi}{n}\right)\right) \\ &= \prod_{k=1}^{n-1} \left(2 - 2\mathbf{i}\cos\left(\frac{k\pi}{n}\right)\right). \end{aligned}$$

Thus, the proof is completed. □

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