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Global Exponential Stability of Multi-Group Models with Multiple Dispersal and Stochastic Perturbation Based on Graph-Theoretic Approach

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Abstract. This paper is concerned with a more general model of multi-group models with multiple dispersal and stochastic perturbation, in which dispersal among multiple groups and stochastic perturbation are considered at the same time. By combining graph theory with Lyapunov method, we derive two types of sufficient criteria which are in the form of Lyapunov-type and coefficient-type respectively, to guarantee the global exponential stability of the model. Furthermore, coefficient-type criterion is successfully applied to stochastic coupled oscillators system. Finally, we offer a numerical example to illustrate the effectiveness and feasibility of the main results.

1. Introduction

In recent years, multi-group models (MGM) have been extensively researched due to their potential applications in various areas, among which ecology and epidemiology are the most active ones (see [1–10]). However, as the communication between people from different regions becomes more and more frequent, there is no absolutely independent place any more. In order to be more realistic, the dispersal among groups should be included into MGM. As a result, MGM considering dispersal have started to become a hot research topic. For instance, in [11–14], the dispersal of a single component in each group (we call it single dispersal) was considered in some kinds of MGM to research infectious diseases, which is of great importance in disease control. Furthermore, in [15], multi-group models with multiple dispersal (MGMMD), in which the dispersal of every component in each group was allowed, were researched and applied to n-patch predator-prey systems successfully.

In the real world, stochastic perturbation exists everywhere, which has been mentioned in [16–19]. In addition, a system could be stable or unstable in response to stochastic perturbation. As a consequence, stochastic perturbation should not be ignored when we research the stability of MGM. From the application

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point of view, it is of great practical significance to take stochastic perturbation into account in MGM. In the past few years, MGM with stochastic perturbation have attracted many researchers' attention and many results, which can supply a theoretical basis for investigating ecology, epidemiology, and so on, have been reported in the literature (see [20–24]). However, to the best of the authors' knowledge, stochastic perturbation has rarely been considered in MGM with single dispersal, let alone MGMMD. To fill this gap, we shall focus on multi-group models with multiple dispersal and stochastic perturbation (MGMMDS) in this paper.

It is widely accepted that most applications of MGM mainly depend on their dynamic properties, especially their stability. Currently, criteria on the stability of MGM are obtained mainly based on Lyapunov method. However, a well-known disadvantage of Lyapunov method is how to construct a proper Lyapunov function for the complicated MGM. Fortunately, Li et al. proposed a new method to build global Lyapunov functions for a complex system by including graph theory into Lyapunov method in [25]. Therein, a system can be described by a digraph, in which each vertex represents an individual subsystem called vertex system and the directed arcs stand for the inter-connections and interactions among vertex systems. Until now, lots of results based on this approach have arisen. Here, we only refer the reader to [26–31].

Inspired by the above discussion, we propose a class of more general MGMMDS in this paper. Based on graph theory and Lyapunov method, two kinds of sufficient criteria which are expressed in forms of Lyapunov-type and coefficient-type respectively, are derived to assure the global exponential stability of the model. Moreover, we apply the coefficient-type criterion to stochastic coupled oscillators system successfully and give a numerical example to support our theoretical results. The main contributions of this paper are in the following aspects:

1. We first propose the model of MGMMDS via drawing dispersal and stochastic perturbation into MGM. Compared with the MGM in [1, 3, 4, 8], the dispersal of every component in each group and stochastic perturbation are both considered in the model.
2. The approach we use to discuss MGMMDS is combined graph theory with Lyapunov method, which is motivated by [25]. The novel sufficient conditions are obtained to guarantee the global exponential stability of MGMMDS. The results of this paper also show that the global exponential stability of MGMMDS is closely related to the topological structure of n digraphs.

The rest of this paper is structured as follows. In Section 2, some preliminaries are presented and our model is constructed. In Section 3, sufficient criteria that guarantee the global exponential stability of MGMMDS are derived. Stochastic coupled oscillators system is employed to show the applicability of our main results in Section 4. To test the validity and effectiveness of the main results, in section 5, we give a numerical example. Finally, we put the paper to an end with concluding remarks in Section 6.

2. Preliminaries and Model Formulation

In this section, first of all, some necessary notations are given and some knowledge on graph theory is also introduced. Then, the structure of MGMMDS is described, after which an example is offered to make the model more understandable.

2.1. Notations and graph theory

In this paper, unless otherwise specified, we use the following notations. Write $\mathbb{R}$, $\mathbb{R}^n$ for the set of real numbers and n-dimensional Euclidean space, respectively. Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0, i = 1, 2, ..., n\}$, $\mathbb{Z}^+ = \{1, 2, \ldots\}$, $\mathbb{L} = \{1, 2, ..., l\}$, and $\mathbb{N} = \{1, 2, ..., n\}$. Denote $\mathbb{R}^1_+ = [0, \infty)$ and $|\cdot|$ be the Euclidean norm for vectors in $\mathbb{R}^n$. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^TA)}$, where A^T denotes the transpose of A. We use $C^{2,1}(\mathbb{R}^m \times \mathbb{R}^1_+; \mathbb{R}^1_+)$ for the family of all nonnegative functions $V(x, t)$ on $\mathbb{R}^m \times \mathbb{R}^1_+$ that are continuously twice differentiable in x and once in t. Let $(\Omega, \mathcal{F}, \mathfrak{F}, P)$ be a complete probability space with filtration $\mathfrak{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous and $\mathcal{F}_0$-contains all P-null sets) and $W(\cdot)$ be a Brownian motion defined on the space. The mathematical expectation with respect to the given probability measure P is denoted by $\mathbb{E}(\cdot)$.

In the sequel, we recall some basic concepts and a lemma on graph theory which can be found in [32] and [27], respectively. A digraph $\mathcal{G} = (\mathbb{L}, E)$ contains a set $\mathbb{L} = \{1, 2, ..., l\}$ of vertices and a set E of arcs (k, h) leading from initial vertex k to terminal vertex h. A subgraph $\mathcal{L}$ of $\mathcal{G}$ is said to be spanning if $\mathcal{L}$ and $\mathcal{G}$ have the same vertex set. A digraph $\mathcal{G}$ is weighted if each arc (h, k) is assigned a positive weight a_{kh}, where $a_{kh} > 0$ if and only if there exists an arc from vertex h to vertex k in $\mathcal{G}$. The weight $W(\mathcal{G})$ of $\mathcal{G}$ is the product of the weights on all its arcs. A directed path $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with distinct vertices $\{k_1, k_2, ..., k_s\}$ such that its set of arcs is $\{(k_i, k_{i+1}) : i = 1, 2, ..., s-1\}$. We call C a directed cycle, if $k_s = k_1$. A connected subgraph $\mathcal{T}$ is a tree if it contains no cycles. A tree $\mathcal{T}$ is rooted at vertex k, called the root, if k is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A digraph $\mathcal{G}$ is strongly connected if there exists a directed path from one to the other for any pair of distinct vertices. Denote the digraph with weight matrix A as $(\mathcal{G}, A)$. A weighted digraph $(\mathcal{G}, A)$ is said to be balanced if $W(C) = W(-C)$ for all directed cycles C. Here, $-C$ denotes the reverse of C and is constructed by reversing the direction of all arcs in C. For a unicyclic graph Q with cycle C_Q, let $\tilde{Q}$ be the unicyclic graph obtained by replacing C_Q with $-C_Q$. Suppose that $(\mathcal{G}, A)$ is balanced, then $W(Q) = W(\tilde{Q})$. The Laplacian matrix of $(\mathcal{G}, A)$ is defined as

$$L = \begin{pmatrix} \sum_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1l} \\ -a_{21} & \sum_{k\neq 2} a_{2k} & \cdots & -a_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{l1} & -a_{l2} & \cdots & \sum_{k\neq l} a_{lk} \end{pmatrix}.$$

Lemma 2.1. *[27] Assume $n \geq 2$. Then the following identity holds:*

$$\sum_{k,h=1}^{n} c_k a_{kh} F_{kh}(x_k, x_h) = \sum_{Q\in\mathbb{Q}} W(Q) \sum_{(k,h)\in E(C_Q)} F_{kh}(x_h, x_h).$$

Here c_k denotes the cofactor of the k-th diagonal element of Laplacian matrix of $(\mathcal{G}, A)$ and $F_{kh}(x_k, x_h)$, where $k, h \in \mathbb{L}$, are arbitrary functions, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A)$, $W(Q)$ is the weight of Q, and C_Q denotes the directed cycle of Q. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_k > 0$ for $k \in \mathbb{L}$.

2.2. Model formulation

For our model, we first assume that n dynamical systems in the k-th group ($k \in \mathbb{L}$) are described by

$$\frac{\mathrm{d}x_k^{(i)}(t)}{\mathrm{d}t} = f_k^{(i)}(x_k(t), t), \quad t \geq 0, \ i \in \mathbb{N}, \tag{1}$$

where $f_k^{(i)} = f_k^{(i)}(x_k(t), t) : \mathbb{R}^m \times \mathbb{R}_+^1 \to \mathbb{R}^{m_i}$ and $x_k(t) = (x_k^{(1)}(t), x_k^{(2)}(t), \ldots, x_k^{(n)}(t))^T \in \mathbb{R}^m$, represents the performance of the i-th component in the k-th group and state of the k-th group, respectively. Hereafter, $m = \sum_{i=1}^{n} m_i$ for $m_i \in \mathbb{Z}^+$.

Secondly, to be more realistic, we introduce the dispersal and stochastic perturbation into system (1). The form of MGMMDS can be described as below.

$$\mathrm{d}x_k^{(i)}(t) = \left[f_k^{(i)}(x_k(t), t) + \sum_{h=1}^{l} H_{kh}^{(i)}(x_k^{(i)}(t), x_h^{(i)}(t)) \right] \mathrm{d}t + g_k^{(i)}(x_k^{(i)}(t), t)\mathrm{d}W(t), \quad t \geq 0, \ i \in \mathbb{N}, \ k \in \mathbb{L}, \tag{2}$$

in which $W(t)$ is the one-dimensional Brownian motion and $g_k^{(i)} = g_k^{(i)}(x_k^{(i)}(t), t) : \mathbb{R}^{m_i} \times \mathbb{R}_+^1 \to \mathbb{R}^{m_i}$ reflects the intense of stochastic perturbation, function $H_{kh}^{(i)} = H_{kh}^{(i)}(x_k^{(i)}(t), x_h^{(i)}(t)) : \mathbb{R}^{m_i} \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$ is the dispersal of the i-th component from the h-th group to the k-th group. $H_{kh}^{(i)} = 0$ if and only if there is no dispersal from the h-th group to the k-th group for the i-th component. In addition, the initial condition for system (2) is given by $((x_1(t_0))^T, (x_2(t_0))^T, \ldots, (x_l(t_0))^T)^T = x_0$, where $x_k(t_0), k \in \mathbb{L}$ is a vector in $\mathbb{R}^m$.

We now construct n digraphs $(\mathcal{G}, A_i)$ for system (2). In each digraph, there is l vertices. Let the k-th vertex in the i-th digraph $(\mathcal{G}, A_i)$ represent the i-th component in the k-th group and call it a vertex system. Each dispersal function $H_{kh}^{(i)}$ is described by the arc from the h-th vertex to the k-th vertex in $(\mathcal{G}, A_i)$. Then MGMMDS (2) can be described as a system on multi-digraph in this paper.

For the aim of this paper, functions $f_k^{(i)}$, $H_{kh}^{(i)}$ and $g_k^{(i)}$ are supposed to guarantee that initial value problem (2) has a unique trivial solution

$$x(t) = ((x_1(t))^T, ..., (x_l(t))^T)^T = (x_1^{(1)}(t), x_1^{(2)}(t), ..., x_1^{(n)}(t), ..., x_l^{(1)}(t), x_l^{(2)}(t), ..., x_l^{(n)}(t))^T = 0.$$

To make our model more intelligible, an example is included here.

Example. It is common that a oscillator system can be described by a differential equation

$$\ddot{x}(t) + \varphi(x(t))\dot{x}(t) + x(t) = 0, \ t \geq 0, \tag{3}$$

in which $\varphi(x)$ is a damping coefficient. To match the system that we propose, l oscillator systems are taken into account and the k-th oscillator system can be described as

$$\ddot{x}_k(t) + \varphi_k(x_k(t))\dot{x}_k(t) + x_k(t) = 0, \ t \geq 0. \tag{4}$$

Let $y_k(t) = \dot{x}_k(t) + \eta x_k(t)$, in which η is a constant. Then system (4) can be rewritten as

$$\begin{cases} \mathrm{d}x_k(t) = (y_k(t) - \eta x_k(t))\mathrm{d}t, \\ \mathrm{d}y_k(t) = \Big((\eta - \varphi_k(x_k(t)))y_k(t) + (\eta\varphi_k(x_k(t)) - \eta^2 - 1)x_k(t)\Big)\mathrm{d}t, \quad t \geq 0. \end{cases}$$

To explicate our system, we construct a stochastic coupled oscillators system by introducing coupling terms as well as stochastic perturbation into the above oscillators system. The form of stochastic coupled oscillators system is in the following:

$$\begin{cases} \mathrm{d}x_k(t) = \left(y_k(t) - \eta x_k(t) + \sum\limits_{h=1}^{l} a_{kh}(x_h(t) - x_k(t))\right)\mathrm{d}t + g_k^{(1)}(x_k(t), t)\mathrm{d}W(t), \\ \mathrm{d}y_k(t) = \left((\eta - \varphi_k(x_k(t)))y_k(t) + (\eta\varphi_k(x_k(t)) - \eta^2 - 1)x_k(t) + \sum\limits_{h=1}^{l} b_{kh}(y_h(t) - y_k(t))\right)\mathrm{d}t + g_k^{(2)}(y_k(t), t)\mathrm{d}W(t), \\ \qquad t \geq 0, k \in \mathbb{L}, \end{cases} \tag{5}$$

in which coupled matrix $A = (a_{kh})_{l\times l}$, $B = (b_{kh})_{l\times l}$, the intensity of stochastic perturbation on x-component and y-component are denoted by $g_k^{(1)}$ and $g_k^{(2)}$, respectively.

Then, following system (2), we can construct two digraphs $(\mathcal{G}, A)$ and $(\mathcal{G}, B)$ for system (5).

Remark 2.2. *Compared with system (2), it is easy to find that there are only two parts x-component and y-component, i.e., $n = 2$ in each group. In fact, stochastic coupled oscillators system has been applied in [26], but the coupling terms are on y-component alone. In this paper, we investigate stochastic coupled oscillator systems with coupling terms both on x-component and y-component.*

Let $X_k(t) = (x_k(t), y_k(t))^T$, then

$$f_k^{(1)}(X_k(t), t) = y_k(t) - \eta x_k(t)$$

and

$$f_k^{(2)}(X_k(t), t) = (\eta - \varphi_k(x_k(t)))y_k(t) + (\eta\varphi_k(x_k(t)) - \eta^2 - 1)x_k(t).$$

Meanwhile, $x_k^{(1)} = x_k$, $x_k^{(2)} = y_k$, $H_{kh}^{(1)}(x_k^{(1)}, x_h^{(1)}) = a_{kh}(x_h - x_k)$, $H_{kh}^{(2)}(x_k^{(2)}, x_h^{(2)}) = b_{kh}(y_h - y_k)$, $a_{kh} \geq 0$ and $b_{kh} \geq 0$. It should be pointed out that $a_{kh} = 0$ if and only if there is no influence from x_h to x_k and $b_{kh} = 0$ if and only if there is no influence from y_h to y_k. For better understanding, we draw a figure for system (5) when $l = 5$, see Figure 1. In this case, $a_{11} = a_{14} = a_{15} = a_{24} = a_{25} = a_{35} = a_{41} = a_{42} = a_{51} = a_{52} = a_{53} = a_{22} = a_{33} = a_{44} = a_{55} = 0$ and $b_{11} = b_{13} = b_{14} = b_{15} = b_{24} = b_{25} = b_{31} = b_{41} = b_{42} = b_{42} = b_{51} = b_{52} = b_{22} = b_{33} = b_{44} = b_{55} = 0$. The details to prove that system (5) is exponentially stable are given in Section 4.

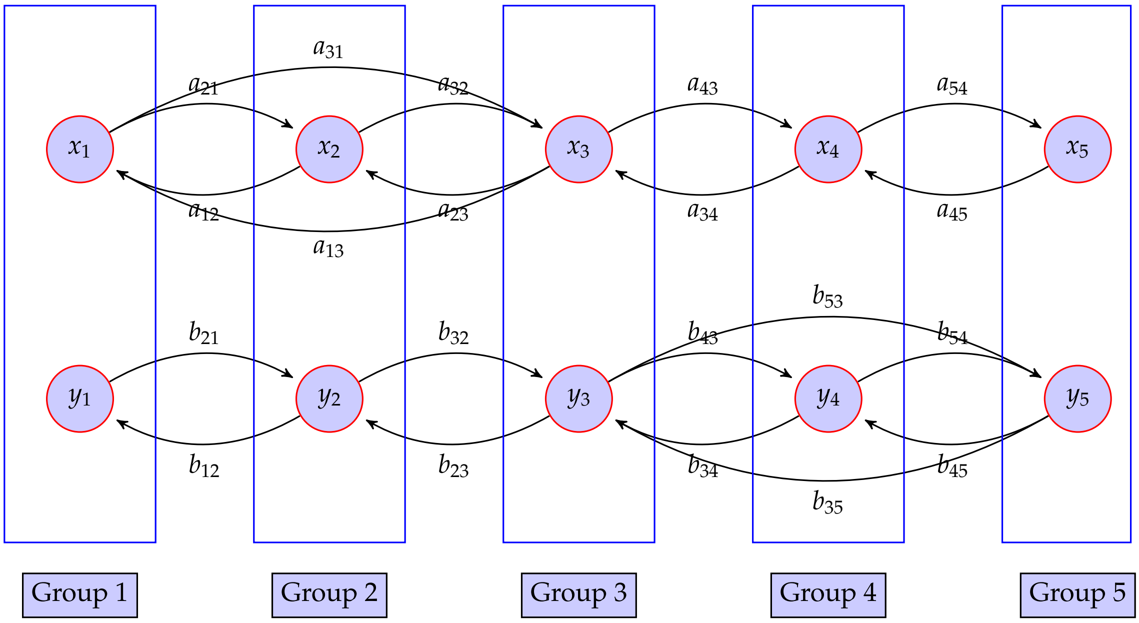

Figure 1: A sample description of the structure of system (5) with $l = 5$.

To be more precise, we now give two definitions here.

Definition 2.3. *[33] The trivial solution to system (2) is said to be p-th moment exponentially stable (ME-stable) if the p-th moment Lyapunov exponent*

$$\limsup_{t\to\infty} \frac{1}{t} \ln(\mathbb{E}|x(t,x_0)|^p) < 0$$

for some $p > 0$ and $x_0 \in \mathbb{R}^{m\times l}$. When $p = 2$, it is said to be exponentially stable in mean square.

Definition 2.4. *For $V_k^{(i)}(x_k^{(i)}, t) \in C^{2,1}(\mathbb{R}^{m_i} \times \mathbb{R}_+^1; \mathbb{R}_+^1)$, $i \in \mathbb{N}$, $k \in \mathbb{L}$, differential operator $\mathcal{L}V_k^{(i)}(x_k^{(i)}, t)$ related to the i-th system in the k-th group is defined by*

$$\mathcal{L}V_k^{(i)}(x_k^{(i)}, t) \triangleq \frac{\partial V_k^{(i)}(x_k^{(i)}, t)}{\partial t} + \frac{\partial V_k^{(i)}(x_k^{(i)}, t)}{\partial x_k^{(i)}} \left(f_k^{(i)} + \sum_{h=1}^{l} H_{kh}^{(i)} \right) + \frac{1}{2} trace\left(\left(g_k^{(i)}\right)^T \frac{\partial^2 V_k^{(i)}(x_k^{(i)}, t)}{\partial (x_k^{(i)})^2} g_k^{(i)} \right),$$

where

$$\frac{\partial V_k^{(i)}(x_k^{(i)}, t)}{\partial x_k^{(i)}} = \left(\frac{\partial V_k^{(i)}(x_k^{(i)}, t)}{\partial x_k^{(i_1)}}, \ldots, \frac{\partial V_k^{(i)}(x_k^{(i)}, t)}{\partial x_k^{(i_{m_i})}} \right), \quad \frac{\partial^2 V_k^{(i)}(x_k^{(i)}, t)}{\partial (x_k^{(i)})^2} = \left(\frac{\partial^2 V_k^{(i)}(x_k^{(i)}, t)}{\partial x_k^{(i_j)} \partial x_k^{(i_s)}} \right)_{m_i \times m_i}.$$

3. Stability Analysis

In this section, based on graph theory and Lyapunov method, a theoretical framework for constructing Lyapunov functions for system (2) is established. As main results in this paper, two kinds of sufficient criteria are gotten to guarantee the p-th moment exponential stability of the trivial solution to system (2).

Theorem 3.1. *Let $p \geq 2$. Assume that the following conditions hold.*

A1. *For each $k, h \in \mathbb{L}$, $i \in \mathbb{N}$, there exists positive-definite function $V_k^{(i)}(x_k^{(i)}, t) \in C^{2,1}(\mathbb{R}^{m_i} \times \mathbb{R}_+^1; \mathbb{R}_+^1)$ and positive constants $\alpha_k^{(i)}$, $\beta_k^{(i)}$ and $\sigma_k^{(i)}$ satisfying*

$$\alpha_k^{(i)}|x_k^{(i)}|^p \leq V_k^{(i)}(x_k^{(i)}, t) \leq \beta_k^{(i)}|x_k^{(i)}|^p \tag{6}$$

and

$$\sum_{i=1}^{n} c_k^{(i)} \mathcal{L}V_k^{(i)}(x_k^{(i)}, t) \leq -\sum_{i=1}^{n} c_k^{(i)}\sigma_k^{(i)}|x_k^{(i)}|^p + \sum_{i=1}^{n} c_k^{(i)} \sum_{h=1}^{l} a_{kh}^{(i)} F_{kh}^{(i)}(x_k^{(i)}, x_h^{(i)}), \tag{7}$$

where $F_{kh}^{(i)} : \mathbb{R}^{m_i} \times \mathbb{R}^{m_i} \to \mathbb{R}$, $a_{kh}^{(i)} \geq 0$, and $c_k^{(i)}$ is the cofactor of the k-th diagonal element of Laplacian matrix of digraph $(\mathcal{G}, A_i)$.

A2. *For every $i \in \mathbb{N}$, let digraph $(\mathcal{G}, A_i)$ be strongly connected, where matrix $A_i = (a_{kh}^{(i)})_{l\times l}$, and along each directed cycle $\mathcal{C}_{\mathcal{Q}_i}$ of digraph $(\mathcal{G}, A_i)$, there is*

$$\sum_{(k,h)\in E(\mathcal{C}_{\mathcal{Q}_i})} F_{kh}^{(i)}(x_k^{(i)}, x_h^{(i)}) \leq 0. \tag{8}$$

Then the trivial solution to system (2) is ME-stable.

Proof. Set $V(x, t) = \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)} V_k^{(i)}(x_k^{(i)}, t)$. From (6) we can obtain that

$$V(x,t) = \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)} V_k^{(i)}(x_k^{(i)}, t) \leq \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\beta_k^{(i)}|x_k^{(i)}|^p$$

and

$$\begin{aligned}
V(x,t) &= \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)} V_k^{(i)}(x_k^{(i)}, t) \\
&\geq \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)}|x_k^{(i)}|^p \\
&= \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)} \sum_{i=1}^{n}\left[\frac{c_k^{(i)}\alpha_k^{(i)}}{\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)}}(|x_k^{(i)}|^2)^{\frac{p}{2}}\right] \\
&\geq \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)}\left[\sum_{i=1}^{n}\frac{c_k^{(i)}\alpha_k^{(i)}}{\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)}}(|x_k^{(i)}|^2)\right]^{\frac{p}{2}} \\
&= \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)}\left\{\sum_{k=1}^{l}\frac{c_k^{(i)}\alpha_k^{(i)}}{\sum_{k=1}^{l} c_k^{(i)}\alpha_k^{(i)}}\left[\sum_{i=1}^{n}\frac{c_k^{(i)}\alpha_k^{(i)}}{\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)}}(|x_k^{(i)}|^2)\right]^{\frac{p}{2}}\right\} \\
&\geq \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)}\left[\sum_{k=1}^{l}\frac{c_k^{(i)}\alpha_k^{(i)}}{\sum_{k=1}^{l} c_k^{(i)}\alpha_k^{(i)}}\sum_{i=1}^{n}\frac{c_k^{(i)}\alpha_k^{(i)}}{\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)}}(|x_k^{(i)}|^2)\right]^{\frac{p}{2}} \\
&\geq \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)}\left[\min_{k\in\mathbb{L},i\in\mathbb{N}}\left(\frac{c_k^{(i)}\alpha_k^{(i)}}{\sum_{k=1}^{l} c_k^{(i)}\alpha_k^{(i)}}\right)\min_{k\in\mathbb{L},i\in\mathbb{N}}\left(\frac{c_k^{(i)}\alpha_k^{(i)}}{\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)}}\right)\sum_{k=1}^{l}\sum_{i=1}^{n}|x_k^{(i)}|^2\right]^{\frac{p}{2}} \\
&= \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)}\left[\min_{k\in\mathbb{L},i\in\mathbb{N}}\left(\frac{c_k^{(i)}\alpha_k^{(i)}}{\sum_{k=1}^{l} c_k^{(i)}\alpha_k^{(i)}}\right)\min_{k\in\mathbb{L},i\in\mathbb{N}}\left(\frac{c_k^{(i)}\alpha_k^{(i)}}{\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)}}\right)\right]^{\frac{p}{2}}|x|^p.
\end{aligned}$$

Write

$$\alpha \triangleq \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\alpha_k^{(i)}\left[\min_{k\in\mathbb{L},i\in\mathbb{N}}\left(\frac{c_k^{(i)}\alpha_k^{(i)}}{\sum_{k=1}^{l}c_k^{(i)}\alpha_k^{(i)}}\right)\min_{k\in\mathbb{L},i\in\mathbb{N}}\left(\frac{c_k^{(i)}\alpha_k^{(i)}}{\sum_{i=1}^{n}c_k^{(i)}\alpha_k^{(i)}}\right)\right]^{\frac{p}{2}}$$

and $\beta \triangleq \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\beta_k^{(i)}$. Then we obtain

$$\alpha|x|^p \le V(x,t) \le \beta|x|^p. \tag{9}$$

For the sake of simplicity, fix any x_0, and write $x(t) \triangleq x(t, x_0)$. From (7) and Lemma 2.1, it follows that

$$\begin{aligned}
\mathcal{L}V(x,t) &= \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\mathcal{L}V_k^{(i)}(x_k^{(i)},t)\\
&\le \sum_{k=1}^{l}\left[-\sum_{i=1}^{n} c_k^{(i)}\sigma_k^{(i)}|x_k^{(i)}|^p + \sum_{i=1}^{n} c_k^{(i)}\sum_{h=1}^{l} a_{kh}^{(i)}F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)})\right]\\
&= -\sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\sigma_k^{(i)}|x_k^{(i)}|^p + \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\sum_{h=1}^{l} a_{kh}^{(i)}F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)})\\
&\le -\sigma|x|^p + \sum_{k=1}^{l}\sum_{i=1}^{n}\sum_{h=1}^{l} c_k^{(i)}a_{kh}^{(i)}F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)})\\
&= -\sigma|x|^p + \sum_{i=1}^{n}\sum_{Q_i\in\mathbb{Q}_i} W(Q_i)\sum_{(k,h)\in E(C_{Q_i})} F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)}),
\end{aligned} \tag{10}$$

where

$$\sigma = \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\sigma_k^{(i)}\left[\min_{k\in\mathbb{L},i\in\mathbb{N}}\left(\frac{c_k^{(i)}\sigma_k^{(i)}}{\sum_{k=1}^{l}c_k^{(i)}\sigma_k^{(i)}}\right)\min_{k\in\mathbb{L},i\in\mathbb{N}}\left(\frac{c_k^{(i)}\sigma_k^{(i)}}{\sum_{i=1}^{n}c_k^{(i)}\sigma_k^{(i)}}\right)\right]^{\frac{p}{2}}.$$

In view of (10), condition $A2$ and the fact $W(Q_i) \ge 0$, we derive

$$\mathcal{L}V(x,t) \le -\sigma|x|^p. \tag{11}$$

It is easy to check that there exists a constant $\gamma > 0$, which assures that $\gamma\beta \le \sigma$. Write $x_0 \triangleq x(0)$. Then from (9) and (11) we can see

$$\begin{aligned}
\mathbb{E}[e^{\gamma t}V(x(t),t)] &= V(x(0),0) + \mathbb{E}\int_0^t e^{\gamma s}(\gamma V(x(s),s) + \mathcal{L}V(x(s),s))\mathrm{d}s\\
&\le V(x(0),0) + \mathbb{E}\int_0^t e^{\gamma s}(\gamma V(x(s),s) - \sigma|x(s)|^p)\mathrm{d}s\\
&\le V(x(0),0) + \mathbb{E}\int_0^t e^{\gamma s}(\gamma\beta|x(s)|^p - \sigma|x(s)|^p)\mathrm{d}s\\
&\le V(x(0),0)\\
&\le \beta|x_0|^p.
\end{aligned} \tag{12}$$

So we have

$$\mathbb{E}[e^{\gamma t}\alpha|x(t)|^p] \le \mathbb{E}[e^{\gamma t}V(x(t),t)] \le \beta|x_0|^p,$$

which means that

$$\mathbb{E}|x(t)|^p \le \frac{\beta}{\alpha}|x_0|^p e^{-\gamma t}.$$

Then it yields that

$$\limsup_{t\to\infty}\frac{1}{t}\ln(\mathbb{E}|x(t)|^p) \le -\gamma < 0.$$

That is, the trivial solution to system (2) is ME-stable and the p-th moment Lyapunov exponent is not greater than $-\gamma$. The proof is complete. □

Remark 3.2. *It is not difficult to see that constructing a global Lyapunov function $V(x,t)$ for system (2) directly is a challenging task. However, Theorem 3.1 offers a method of constructing a global Lyapunov function $V(x,t)$ for system (2) by using the Lyapunov function $V_k^{(i)}(x_k^{(i)},t)$ of each vertex system (see the definition of $V(x,t)$ in the proof of Theorem 3.1), in which $V_k^{(i)}(x_k^{(i)},t)$ is supposed to be available in practical applications. This method is effective, which can be seen from the proof of Theorem 3.1, and can be used in the stability analysis of many complex systems. What's more, Theorem 3.1 shows that there is a close relationship between the exponential stability of system (2) and the topological structure of n digraphs $(\mathcal{G},A_i)$, $i\in\mathbb{L}$.*

Remark 3.3. *It has been mentioned that the dispersal was considered in MGM in the literature [11–14], but the dispersal was along a single component in each group. In other words, the dispersal was allowed only on a single digraph. Though the dispersal was allowed on multiple digraphs, the model in [15] did not contain stochastic perturbation. More widely, we first introduce stochastic perturbation into MGMMD in our paper, and our model can be regarded as a stochastic model on multi-graph.*

The method that Theorem 3.1 offers is powerful, but the conditions of Theorem 3.1 cannot be verified easily. As a result, this method is not adaptive enough in actual applications, which inspires us to consider some simple and easy-verifiable conditions for the exponential stability of system (2). Suppose that for every $i\in\mathbb{N}$, $(\mathcal{G},A_i)$ is balanced. Then we have

$$\sum_{k,h=1}^{n} c_k^{(i)}a_{kh}^{(i)}F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)}) = \frac{1}{2}\sum_{\mathcal{Q}_i\in\mathbb{Q}_i} W(\mathcal{Q}_i)\sum_{(k,h)\in E(C_{\mathcal{Q}_i})}[F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)}) + F_{hk}^{(i)}(x_h^{(i)},x_k^{(i)})].$$

As a result, we can put the following inequality in the place of condition $A2$.

$$\sum_{(k,h)\in E(C_{\mathcal{Q}_i})}[F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)}) + F_{hk}^{(i)}(x_h^{(i)},x_k^{(i)})] \le 0. \tag{13}$$

Consequently, an easier stability criterion is derived as below.

Corollary 3.4. *Assume that $(\mathcal{G},A_i)$, $i\in\mathbb{N}$, is balanced. Then the conclusion of Theorem 3.1 holds with inequality (8) replaced by inequality (13).*

Further, it is noted that if for every $k,h\in\mathbb{L}$, there exist functions $Q_k^{(i)}(x_k^{(i)})$ and $Q_h^{(i)}(x_h^{(i)})$, such that

$$F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)}) \le Q_k^{(i)}(x_k^{(i)}) - Q_h^{(i)}(x_h^{(i)}), \tag{14}$$

then one has inequality (8) naturally. In this case, we obtain another corollary.

Corollary 3.5. *The conclusion of Theorem 3.1 holds if inequality (8) is replaced by inequality (14).*

Since the results obtained above are based on Lyapunov function of each vertex of system (2), it is not very convenient to verify whether a given system satisfies the exponential stability criteria. We now derive some sufficient conditions for the exponential stability of system (2) by using its coefficients.

Theorem 3.6. *Suppose that the following conditions hold for each $k,h\in\mathbb{L}$, $i\in\mathbb{N}$.*

B1. *There are constants* $(\beta_k^{(i)})_j$ *and* $\delta_k^{(i)}$, $1 \le j \le n$, *such that*

$$(x_k^{(i)})^T f_k^{(i)}(x_k, t) \le \sum_{j=1}^{n} (\beta_k^{(i)})_j |x_k^{(j)}|^2, \ \ |g_k^{(i)}(x_k^{(i)}, t)|^2 \le \delta_k^{(i)} |x_k^{(i)}|^2.$$

B2. *There exists a constant* $A_{kh}^{(i)} > 0$, *such that*

$$|H_{kh}^{(i)}(x_k^{(i)}, x_h^{(i)})| \le A_{kh}^{(i)}(|x_k^{(i)}| + |x_h^{(i)}|).$$

B3. *Let* $p \ge 2$ *and*

$$(p-2)\sum_{j=1}^{n}(\beta_k^{(i)})_j + 2p\sum_{h=1}^{l} A_{kh}^{(i)} + \frac{1}{2}p(p-1)\delta_k^{(i)} + \frac{2}{c_k^{(i)}}\sum_{j=1}^{n} c_k^{(j)}(\beta_k^{(j)})_i < 0.$$

B4. *Digraph* $(\mathcal{G}, A_i)$, *in which* $A_i = (A_{kh}^{(i)})_{l\times l}$, *is strongly connected.*
Then the trivial solution to system (2) is ME-stable.

Proof. For every $k \in \mathbb{L}$, $i \in \mathbb{N}$, define a function $V_k^{(i)}(x_k^{(i)}, t) = |x_k^{(i)}|^p$. Making use of condition *B1*, we can get that

$$\begin{aligned}
\mathcal{L}V_k^{(i)}(x_k^{(i)}, t) &= p|x_k^{(i)}|^{p-2}(x_k^{(i)})^T\left[f_k^{(i)}(x_k, t) + \sum_{h=1}^{l} H_{kh}^{(i)}\right] + \frac{1}{2}trace\left[\left(g_k^{(i)}\right)^T (p|x_k^{(i)}|^{p-2}I + p(p-2)|x_k^{(i)}|^{p-4}x_k^{(i)}(x_k^{(i)})^T)g_k^{(i)}\right] \\
&\le p\left[|x_k^{(i)}|^{p-2}\left(\sum_{j=1}^{n}(\beta_k^{(i)})_j|x_k^{(j)}|^2\right) + \sum_{h=1}^{l}|x_k^{(i)}|^{p-1}|H_{kh}^{(i)}|\right] + \frac{1}{2}p(p-1)|x_k^{(i)}|^{p-2}|g_k^{(i)}|^2 \\
&\le p\left[|x_k^{(i)}|^{p-2}\left(\sum_{j=1}^{n}(\beta_k^{(i)})_j|x_k^{(j)}|^2\right) + \sum_{h=1}^{l}|x_k^{(i)}|^{p-1}|H_{kh}^{(i)}|\right] + \frac{1}{2}p(p-1)\delta_k^{(i)}|x_k^{(i)}|^p \qquad (15) \\
&= p\left[\sum_{j=1}^{n}(\beta_k^{(i)})_j|x_k^{(j)}|^2|x_k^{(i)}|^{p-2} + \sum_{h=1}^{l}|x_k^{(i)}|^{p-1}|H_{kh}^{(i)}|\right] + \frac{1}{2}p(p-1)\delta_k^{(i)}|x_k^{(i)}|^p.
\end{aligned}$$

By using inequality (see [33, p.52])

$$|a|^r|b|^w \le \frac{r}{r+w}|a|^{r+w} + \frac{w}{r+w}|b|^{r+w}, \qquad (16)$$

where $a, b \in \mathbb{R}$, $r, w > 0$, we can obtain

$$\sum_{j=1}^{n}(\beta_k^{(i)})_j|x_k^{(j)}|^2|x_k^{(i)}|^{p-2} \le \frac{p-2}{p}\sum_{j=1}^{n}(\beta_k^{(i)})_j|x_k^{(i)}|^p + \frac{2}{p}\sum_{j=1}^{n}(\beta_k^{(i)})_j|x_k^{(j)}|^p. \qquad (17)$$

Meanwhile, from condition *B2* and inequality (16), we derive

$$\sum_{h=1}^{l}|x_k^{(i)}|^{p-1}|H_{kh}^{(i)}| \le \sum_{h=1}^{l}|x_k^{(i)}|^{p-1}A_{kh}^{(i)}(|x_k^{(i)}| + |x_h^{(i)}|) \le \frac{2p-1}{p}\sum_{h=1}^{l}A_{kh}^{(i)}|x_k^{(i)}|^p + \frac{1}{p}\sum_{h=1}^{l}A_{kh}^{(i)}|x_h^{(i)}|^p. \qquad (18)$$

Substituting (17) and (18) into (15), one has

$$
\begin{aligned}
\mathcal{L}V_k^{(i)}(x_k^{(i)},t) \le& p\left[\sum_{j=1}^{n}(\beta_k^{(i)})_j|x_k^{(j)}|^2|x_k^{(i)}|^{p-2}+\sum_{h=1}^{l}|x_k^{(i)}|^{p-1}|H_{kh}^{(i)}(x_k^{(i)},x_h^{(i)})|\right]+\frac{1}{2}p(p-1)\delta_k^{(i)}|x_k^{(i)}|^p\\
\le& \left[(p-2)\sum_{j=1}^{n}(\beta_k^{(i)})_j+(2p-1)\sum_{h=1}^{l}A_{kh}^{(i)}+\frac{1}{2}p(p-1)\delta_k^{(i)}\right]|x_k^{(i)}|^p+\sum_{h=1}^{l}A_{kh}^{(i)}|x_h^{(i)}|^p+\sum_{j=1}^{n}2(\beta_k^{(i)})_j|x_k^{(j)}|^p \qquad (19)\\
=& \left[(p-2)\sum_{j=1}^{n}(\beta_k^{(i)})_j+2p\sum_{h=1}^{l}A_{kh}^{(i)}+\frac{1}{2}p(p-1)\delta_k^{(i)}\right]|x_k^{(i)}|^p+\sum_{j=1}^{n}2(\beta_k^{(i)})_j|x_k^{(j)}|^p+\sum_{h=1}^{l}A_{kh}^{(i)}F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)}),
\end{aligned}
$$

where

$$F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)})=|x_h^{(i)}|^p-|x_k^{(i)}|^p.$$

In view of (19) and condition *B4*, we have

$$
\begin{aligned}
&\sum_{i=1}^{n}c_k^{(i)}\mathcal{L}V_k^{(i)}(x_k^{(i)},t)\\
\le&\sum_{i=1}^{n}c_k^{(i)}\Bigg[\left((p-2)\sum_{j=1}^{n}(\beta_k^{(i)})_j+2p\sum_{h=1}^{l}A_{kh}^{(i)}+\frac{1}{2}p(p-1)\delta_k^{(i)}\right)|x_k^{(i)}|^p+\sum_{j=1}^{n}2(\beta_k^{(i)})_j|x_k^{(j)}|^p+\sum_{h=1}^{l}A_{kh}^{(i)}F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)})\Bigg]\\
=&\sum_{i=1}^{n}c_k^{(i)}\left[(p-2)\sum_{j=1}^{n}(\beta_k^{(i)})_j+2p\sum_{h=1}^{l}A_{kh}^{(i)}+\frac{1}{2}p(p-1)\delta_k^{(i)}\right]|x_k^{(i)}|^p+\sum_{j=1}^{n}c_k^{(j)}\sum_{i=1}^{n}2(\beta_k^{(j)})_i|x_k^{(i)}|^p \qquad (20)\\
&+\sum_{i=1}^{n}c_k^{(i)}\sum_{h=1}^{l}A_{kh}^{(i)}F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)})\\
=&\sum_{i=1}^{n}c_k^{(i)}\left[(p-2)\sum_{j=1}^{n}(\beta_k^{(i)})_j+2p\sum_{h=1}^{l}A_{kh}^{(i)}+\frac{1}{2}p(p-1)\delta_k^{(i)}+\frac{2}{c_k^{(i)}}\sum_{j=1}^{n}c_k^{(j)}(\beta_k^{(j)})_i\right]|x_k^{(i)}|^p+\sum_{i=1}^{n}\sum_{h=1}^{l}c_k^{(i)}A_{kh}^{(i)}F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)})
\end{aligned}
$$

Note that *B3* holds. Then we can easily check that there exists a constant $\lambda_k^{(i)}>0$, $i\in\mathbb{N}$, $k\in\mathbb{L}$, such that

$$\sum_{i=1}^{n}c_k^{(i)}\mathcal{L}V_k^{(i)}(x_k^{(i)},t)\le-\sum_{i=1}^{n}c_k^{(i)}\lambda_k^{(i)}|x_k^{(i)}|^p+\sum_{i=1}^{n}\sum_{h=1}^{l}c_k^{(i)}A_{kh}^{(i)}F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)}).$$

Let $\alpha_k^{(i)}=\beta_k^{(i)}\equiv 1$, $\sigma_k^{(i)}=\lambda_k^{(i)}$, $a_{kh}^{(i)}=A_{kh}^{(i)}$ and $F_{kh}^{(i)}(x_k^{(i)},x_h^{(i)})=|x_h^{(i)}|^p-|x_k^{(i)}|^p$. From Corollary 3.5, we can see that the trivial solution to system (2) is ME-stable. This completes the proof. □

Remark 3.7. *The conditions in Theorem 3.6 are more convenient to be verified compared with those in Theorem 3.1. This is because the conditions in Theorem 3.6 are established in terms of coefficients, instead of the Lyapunov functions of vertex systems. Besides, in view of conditions B2, B3 and B4, we can see that the exponential stability of system (2) is closely related with the topological structure of n digraphs* $(\mathcal{G},A_i)$, $i\in\mathbb{N}$.

4. Application

In this section, the example mentioned in Section 2 is employed to show the applicability of our theoretical results.

Theorem 4.1. *The trivial solution to system (5) is exponentially stable in mean square if the following conditions hold.*

C1. *For each* $k\in\mathbb{L}$, *there are constants* $M_k\ge m_k>0$ *such that* $m_k\le\varphi_k(x_k)\le M_k$. *Functions* $|g_k^{(1)}|^2\le\theta_k^{(1)}|x_k|^2$

and $|g_k^{(2)}|^2 \le \theta_k^{(2)}|y_k|^2$, in which $\theta_k^{(1)}$ and $\theta_k^{(2)}$ are positive constants. Scalar $\eta > 1$ satisfies $\eta(m_k - \eta) \ge 1$.
C2. *For any $k \in \mathbb{L}$, $i = 1, 2$,*

$$4\sum_{h=1}^{l} d_{kh}^{(i)} + \theta_k^{(i)} + \frac{2\sum_{j=1}^{2} z_k^{(j)}(\varepsilon_k^{(j)})_i}{z_k^{(i)}} < 0,$$

in which $d_{kh}^{(1)} = a_{kh}$, $d_{kh}^{(2)} = b_{kh}$, $(\varepsilon_k^{(1)})_1 = \frac{1}{2} - \eta$, $(\varepsilon_k^{(1)})_2 = \frac{1}{2}$, $(\varepsilon_k^{(2)})_1 = \frac{M_k\eta-\eta^2-1}{2}$, and $(\varepsilon_k^{(2)})_2 = \frac{M_k\eta-\eta^2-1}{2} - m_k + \eta$.
C3. *Digraphs $(\mathcal{G}, A)$ and $(\mathcal{G}, B)$ are strongly connected.*

Proof. Let $X_k = (x_k, y_k)^T$, $f_k^{(1)}(X_k, t) = y_k - \eta x_k$, and $f_k^{(2)}(X_k, t) = (\eta - \varphi_k(x_k))y_k + (\eta\varphi_k(x_k) - \eta^2 - 1)x_k$. Then we have

$$x_k^T f_k^{(1)}(X_k, t) = x_k(y_k - \eta x_k) \le \left(\frac{1}{2} - \eta\right)x_k^2 + \frac{1}{2}y_k^2$$

and

$$\begin{aligned} y_k^T f_k^{(2)}(X_k, t) &\le (\eta - \varphi_k(x_k))y_k^2 + |\varphi_k(x_k)\eta - \eta^2 - 1||x_k y_k| \\ &\le \frac{|\eta(\varphi_k(x_k) - \eta) - 1|}{2}x_k^2 + \left(\frac{|\eta(\varphi_k(x_k) - \eta) - 1|}{2} - \varphi_k(x_k) + \eta\right)y_k^2 \\ &\le \frac{M_k\eta - \eta^2 - 1}{2}x_k^2 + \left(\frac{M_k\eta - \eta^2 - 1}{2} - m_k + \eta\right)y_k^2. \end{aligned}$$

Noting that $H_{kh}^{(1)}(x_k^{(1)}, x_h^{(1)}) = a_{kh}(x_h - x_k)$ and $H_{kh}^{(2)}(x_k^{(2)}, x_h^{(2)}) = b_{kh}(y_h - y_k)$, we have

$$\left|H_{kh}^{(1)}(x_k, x_h)\right| \le a_{kh}(|x_k| + |x_h|)$$

and

$$\left|H_{kh}^{(2)}(y_k, y_h)\right| \le b_{kh}(|y_k| + |y_h|).$$

Let $A_k^{(i)} = d_{kh}^{(i)}$, $\delta_k^{(i)} = \theta_k^{(i)}$, $c_k^{(i)} = z_k^{(i)}$ and $\beta_k^{(i)} = \varepsilon_k^{(i)}$, $i = 1, 2$. Then the conditions in Theorem 3.6 are satisfied with $p = 2$. Hence, the trivial solution to system (5) is exponentially stable in mean square. This completes the proof. □

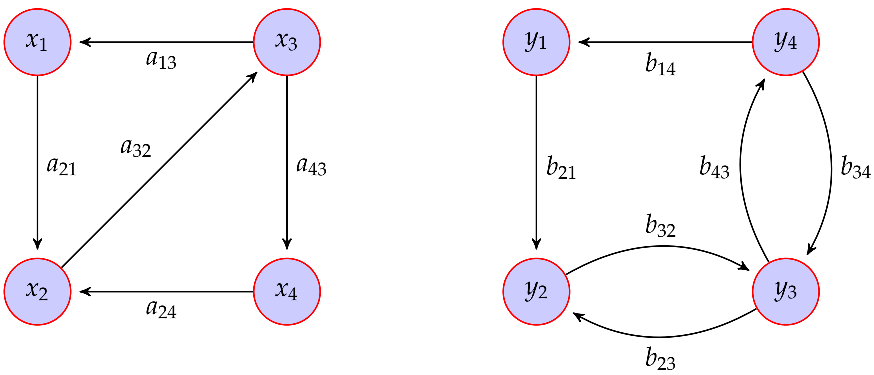

Figure 2: Two digraphs of system (5) with $l = 4$: $(\mathcal{G}, A)$ (left) and $(\mathcal{G}, B)$ (right).

5. Numerical Simulation

In this section, a numerical example is given to show the validity and effectiveness of our results. Consider system (5) on two digraphs $(\mathcal{G}, A)$ and $(\mathcal{G}, B)$, and set $l = 4$ (see Figure 2). The weighted matrices A and B are given as follows.

$$A = \begin{pmatrix} 0 & 0 & 0.02 & 0 \\ 0.01 & 0 & 0 & 0.02 \\ 0 & 0.02 & 0 & 0 \\ 0 & 0 & 0.02 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0.02 \\ 0.02 & 0 & 0.01 & 0 \\ 0 & 0.02 & 0 & 0.01 \\ 0 & 0 & 0.02 & 0 \end{pmatrix}.$$

Let $m = m_k = 2.5$, $M = M_k = 2.6$, $k = 1, 2, 3, 4$, and $\eta = 1.5$. Choose

$$\varphi_1(x_1) = \frac{|\sin(x_1)|}{10} + 2.5, \quad \varphi_2(x_2) = \frac{|\cos(x_2)|}{10} + 2.5,$$
$$\varphi_3(x_3) = \frac{\sin(x_3)}{20} + 2.55, \quad \varphi_4(x_4) = \frac{\cos(x_4)}{20} + 2.55.$$

Set $\theta_1^{(1)} = 0.01$, $\theta_1^{(2)} = 0.02$, $\theta_2^{(1)} = 0.03$, $\theta_2^{(2)} = 0.02$, $\theta_3^{(1)} = 0.01$, $\theta_3^{(2)} = 0.02$, $\theta_4^{(1)} = 0.03$, $\theta_4^{(2)} = 0.02$. With $l = 4$, the trivial solution to system (5) is $\gamma^* = (0, 0, 0, 0, 0, 0, 0, 0)^T$. From the offered conditions, we can see that $\eta(m_k - \eta) > 1$, $m_k \leq \varphi_k(x_k) \leq M_k$, $k = 1, 2, 3, 4$. Let $\rho_k^{(i)}$, $k = 1, 2, 3, 4$, $i = 1, 2$, stand for the results of the algebraic expression in **C2** of Theorem 4.1 respectively. For example,

$$\rho_1^{(1)} = 4 \sum_{h=1}^{4} a_{1h} + \theta_1^{(1)} + \frac{2 \sum_{j=1}^{2} z_1^{(j)} (\varepsilon_1^{(j)})_1}{z_1^{(1)}}$$

and

$$\rho_1^{(2)} = 4 \sum_{h=1}^{4} b_{1h} + \theta_1^{(2)} + \frac{2 \sum_{j=1}^{2} z_1^{(j)} (\varepsilon_1^{(j)})_2}{z_1^{(2)}}.$$

Then we can obtain that $\rho_1^{(1)} = -0.69$, $\rho_1^{(2)} = -1.26$, $\rho_2^{(1)} = -0.85$, $\rho_2^{(2)} = -0.21$, $\rho_3^{(1)} = -1.2$, $\rho_3^{(2)} = -0.75$, $\rho_4^{(1)} = -0.21$, and $\rho_4^{(2)} = -0.67$, which means that the condition **C2** in Theorem 4.1 is successfully checked. In view of Figure 2, we can easily find that digraphs $(\mathcal{G}, A)$ and $(\mathcal{G}, B)$ are both strongly connected. To sum up, all conditions in Theorem 4.1 are satisfied. As a result, the trivial solution to system (5) with $l = 4$ is exponentially stable in mean square. Here, we take $(0.32, 0.45, 0.7, 0.6, 0.4, 0.4, 0.3, 0.8)^T$ as the initial value and simulation results are showed in Figures 3 and 4.

6. Conclusions and further discussions

In this paper, the exponential stability of MGMMDS was discussed and a systematic method of constructing global Lyapunov functions for the considered model was provided by combining graph theory with Lyapunov method. Also, we derived two types of sufficient criteria to determine the exponential stability of the model. One is Lyapunov-type theorem, and the other is coefficient-type theorem. Coefficient-type theorem was applied to stochastic coupled oscillators system and a numerical example was given, which demonstrated the applicability and effectiveness of our main results. What's more, the results of this paper also showed that there is a close relationship between the exponential stability of our model and the topology structure of n digraphs.

In our model, dispersal is only taken into account in the drift coefficients. In fact, following this idea, we can also introduce the dispersal to the diffusion coefficients. On the other hand, as another important factor, time delays can also be included in our model in the near future. In addition, our main results hold required that digraph $(\mathcal{G}, A_i)(i \in \mathbb{N})$ is strongly connected. However, if digraph $(\mathcal{G}, A_i)(i \in \mathbb{N})$ is not strongly connected, then we cannot guarantee the stability of MGMMDS. Without the connectedness of $(\mathcal{G}, A_i)(i \in \mathbb{N})$, analyzing the stability of MGMMDS will be an interesting and challenging topic.

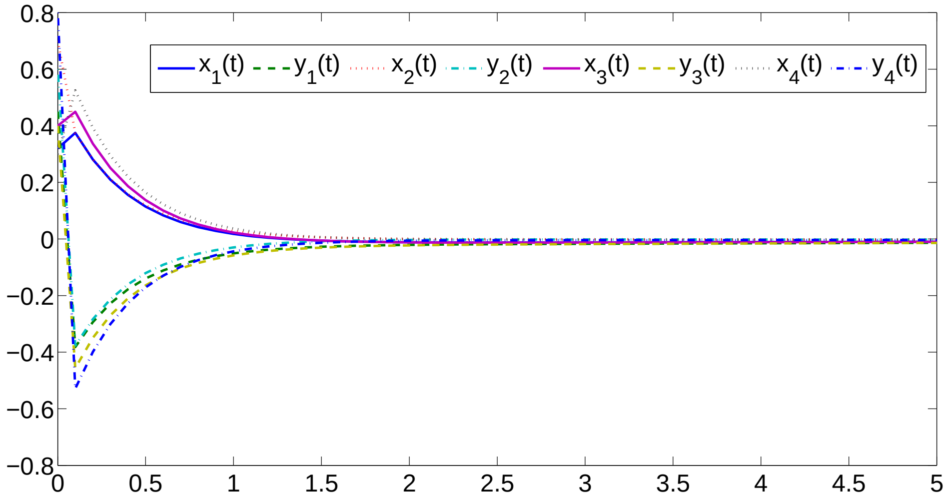

Figure 3: A sample path of the solution to system (5) with $l = 4$.

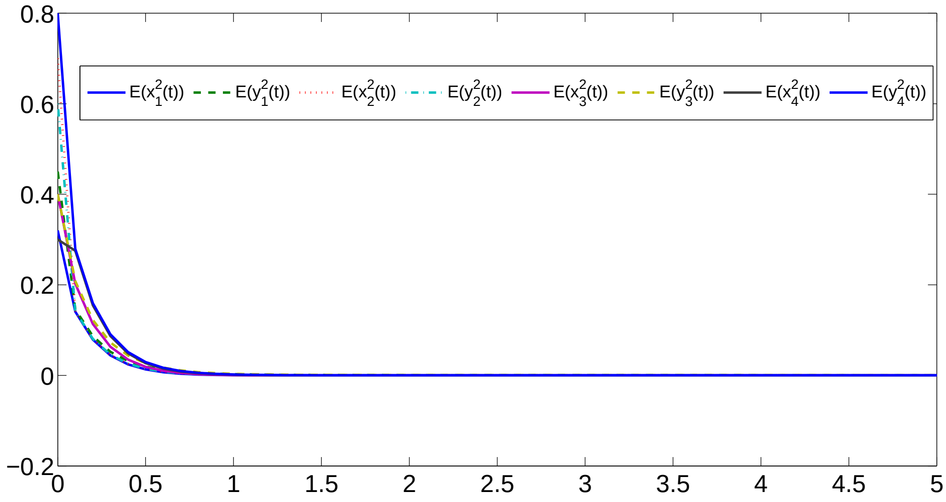

Figure 4: The 2-th moment of the solution to system (5) with $l = 4$.

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