Filomat 31:16 (2017), 5139–5150
https://doi.org/10.2298/FIL1716139K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Asymptotically Deferred Statistical Equivalence of Sequences

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Abstract. In this study, combining the definition of asymptotically equivalence of sequences and deferred density, the concepts of asymptotically deferred statistical equivalence and strong deferred asymptotically equivalence of nonnegative sequences are introduced. Besides, the main properties of asymptotically deferred statistical equivalence and strong deferred asymptotically equivalence, some inclusion and equivalence results are given.

1. Introduction and Some Definitions

The idea of statistical convergence was given by Zygmund [33] in the first edition of his monograph published in Warsaw in 1935. Later on the concept of statistical convergence was introduced by Steinhaus [30] and Fast [14] and later reintroduced by Schoenberg [31], independently. Over the years and under different names statistical convergence has been discussed different areas of mathematics such as Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Çakalli ([4],[5]), Caserta and Kočinac [6], Connor ([8],[9]), Esi [10], Et [11], Et and Şengül [12], Erdös and Tenenbaum [13], Fridy [15], Fridy and Miller [16], Işık [20], Mursaleen [24], Šalat [28] and many others.

Marouf in [23] introduced definition of asymptotically equivalent sequences and asymptotic regular matrices. Patterson [25] extend these concepts by presenting an asymptotically statistically equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Recently asymptotically equivalent sequences have been studied in ([2],[3],[27]).

In this work, we shall give a generalization of definition of asymptotically statistical equivalence of non negative sequences by considering deferred statistical density which is given and studied in ([22],[32]). The main goal of this work is to examine the relation between asymptotically deferred statistical convergence and strongly $r-$deferred Cesàro summability.

Let K be a subset of positive natural numbers $\mathbb{N}$ and $K(n)$ denotes the set $\{k \leq n : \; k \in K\}$. Asymptotic (or natural) density of the subset K is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |K(n)|,$$

2010 *Mathematics Subject Classification*. Primary 46A45; Secondary 40A99, 40C05
Keywords. Asympototically equivalent sequences, statistically equivalent sequences, deferred statistical convergence of sequences
Received: 08 September 2016; Revised: 28 November 2016; Accepted: 07 December 2016
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where $|K(n)|$ denotes the cardinality of the set $K(n)$, if this limit exists.

Definition 1.1. A real valued sequence $x = (x_k)$ is said to be statistically convergent to L, if for every $\varepsilon > 0$, the set $K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has zero asymptotic density. In symbol, we will write $st - \lim x = L$.

Definition 1.2. ([23]) Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1. \tag{1}$$

It is denoted by $x \sim y$. If the limit in (1) is L, we are going to use $x \overset{L}{\sim} y$.

By combination of Definition 1.1 and Definition 1.2 asymtotically statistical equivalence of nonnegative sequences of multiple L is defined by Patterson in [25] as follows:

Definition 1.3. ([25]) Two nonnegative sequence $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geqslant \varepsilon \right\} \right| = 0 \tag{2}$$

and it is denoted by $x \overset{S_L}{\sim} y$. If $L = 1$ in (2), then the sequences x and y are simply called asymptotically statistical equivalent and it is denoted by $x \overset{S}{\sim} y$.

The concept of asymptotically regular matrices which preserve the asymptotic equivalence of two non-negative number sequences has been investigated by Pobyvanets in [26]. Later on Marouf [23] continued this subject and gave some necessary and sufficient conditions for to be asymptotic regular matrices. The same problem in [23] is considered and under weak conditions some further results is given by Jinlu Li [21]. Recently, by considering different kind of convergence methods such as statistical convergence, ideal convergence, etc. this subject is studied by many different names in ([3],[7],[17],[19],[25], [29]).

Let us recall now some background about of deferred Cesàro mean. In 1932, Agnew [1] defined the deferred Cesàro mean $D_{p,q}$ of a sequence $x = (x_n)$ by

$$\left(D_{p,q}x\right)_n := \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k,$$

where $\{p(n)\}_{n\in\mathbb{N}}$ and $\{q(n)\}_{n\in\mathbb{N}}$ are sequences of non-negative integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = \infty. \tag{3}$$

Definition 1.4. A sequence $x = (x_n)$ is called; (i) deferred Cesàro convergent to L if deferred Cesàro mean of the sequence $\{x_n - L\}_{n\in\mathbb{N}}$ tends to zero,

(ii) strongly deferred Cesàro convergent to L if deferred Cesàro mean of the sequence $\{|x_n - L|\}_{n\in\mathbb{N}}$ tends to zero,

(iii) strongly $r-$deferred Cesàro convergent to L if deferred Cesàro mean of the sequence $\{|x_n - L|^r\}_{n\in\mathbb{N}}$ tends to zero.

Definition 1.5. (Deferred Density) Let $\{p(n)\}, \{q(n)\}$ as above and K be an arbitrary subset of $\mathbb{N}$. If the following limit

$$\delta_{p,q}(K) := \lim_{n\longrightarrow\infty} \frac{1}{q(n) - p(n)} \left|K_{p,q}(n)\right| \tag{4}$$

exists, then the limit $\delta_{p,q}(K)$ is called deferred density of K, where $K_{p,q}(n) := \{p(n) < k \leq q(n) : k \in K\}$. If p and q are well known, then it is denoted by $\delta_D(K)$.

Remark 1.6. If $q(n) = n$ and $p(n) = 0$, then deferred density coincides with asymptotic density of K.

Now, we can give definition of asymptotically deferred statistical equivalence of given two nonnegative sequences:

Definition 1.7. Two nonnegative sequence $x = (x_n)$ and $y = (y_n)$ are said to be; (i) asypmtotically deferred equivalent with multiple L provided that deferred Cesàro mean of the sequence $\left(\left|\frac{x_n}{y_n} - L\right|\right)_{n\in\mathbb{N}}$ tends to zero. It is denoted by $x \overset{D_L}{\sim} y$ and it is called simply asymptotically deferred equivalent (and denoted by $x \overset{D}{\sim} y$) when $L = 1$.

(ii) strongly $r-$deferred Cesàro asymptotically equivalent of multiple L provided that deferred Cesàro mean of the sequence $\left(\left|\frac{x_n}{y_n} - L\right|^r\right)_{n\in\mathbb{N}}$ tends to zero and simply strongly $r-$deferred Cesàro asymptotically equivalent if $L = 1$. It is denoted by $x \overset{D_L^r}{\sim} y$ and $x \overset{D^r}{\sim} y$ if $L = 1$. The case $q(n) = n, p(n) = 0$ will be denoted by $x \overset{C_L^r}{\sim} y$ and it is called strongly $r-$Cesàro asymptotically equivalence of sequences.

(iii) asymptotically deferred statistical equivalent with multiple L provided that for every $\varepsilon > 0$, the limit

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left| \left\{ p(n) < k \leq q(n) \ : \left|\frac{x_k}{y_k} - L\right| \geqslant \varepsilon \right\} \right| = 0 \tag{5}$$

holds. It is denoted by $x \overset{DS_L}{\sim} y$ and if $L = 1$, it is called simply asymptotically deferred statistically equivalent (and denoted by $x \overset{DS}{\sim} y$).

Remark 1.8. It is clear that; **(i)** If $q(n) = n$ and $p(n) = n - 1$, then (5) coincides with (1),

(ii) If $q(n) = n$ and $p(n) = 0$, then (5) coincides with (2),

(iii) If we consider $q(n) = k_n$ and $p(n) = k_{n-1}$, where (k_n) is a lacunary sequence of nonnegative integers with $k_n - k_{n-1} \to \infty$ as $n \to \infty$, then (5) coincides with the definition of asymptotically lacunary statistical equivalence which is given by Patterson-Savaş in [29] and Braha in [3],

(iv) If we consider $q(n) = \lambda_n$ and $p(n) = 0$ when λ_n is a strictly increasing sequence of natural numbers such that $\lim_{n\to\infty} \lambda_n = \infty$, then (5) coincides with $\lambda-$statistical equivalence of sequences,

(v) If we consider $q(n) = n$ and $p(n) = n - \lambda_n$ where (λ_n) is a nondecrasing sequence of natural numbers such that $\lambda_0 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$ satisfied then (5) coincides with the $\lambda-$density defined by Mursaleen in [24].

2. DS$_L$-Equivalence of Sequences

Let $x = (x_n)$ and $y = (y_n)$ be sequences of real numbers. The notation "$x \prec y$" will be used if "$x_n \leq y_n$" holds for all $n \in \mathbb{N}$.

Theorem 2.1. *Let $x = (x_n)$, $y = (y_n)$ and $z = (z_n)$ be sequences of non-negative real numbers. If $x \overset{DS_L}{\sim} y$ and $z \prec x$, then $z \overset{DS_L}{\sim} y$.*

Proof. Assume that $x \overset{DS_L}{\sim} y$ and $z \prec x$. Since the inequality $\left|\frac{z_k}{y_k} - L\right| \leq \left|\frac{x_k}{y_k} - L\right|$ holds for all $k \in \mathbb{N}$, then the inclusion

$$\left\{ p(n) < k \leq q(n) : \left|\frac{z_k}{y_k} - L\right| \geq \varepsilon \right\} \subseteq \left\{ p(n) < k \leq q(n) : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon \right\}$$

is satisfied. Hence, for any $\varepsilon > 0$, following inequality

$$\frac{1}{q(n) - p(n)} \left| \left\{ p(n) < k \leq q(n) : \left|\frac{z_k}{y_k} - L\right| \geq \varepsilon \right\} \right|$$

$$\leq \frac{1}{q(n)-p(n)}\left|\left\{p(n)<k\leq q(n):\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\right|$$

holds. If we take limit when $n \to \infty$, desired results is obtained. □

Theorem 2.2. *Let $x=(x_n)$, $y=(y_n)$ and $z=(z_n)$ be sequences of nonnegative real numbers. If $x \overset{DS_L}{\sim} y$ and $y \prec z$, then $x \overset{DS_L}{\sim} z$.*

Proof. The proof can be obtained by following the proof of Theorem 2.1. So, it is omitted here. □

Let us define $x \wedge y := (\min\{x_n, y_n\})$ and $x \vee y := (\max\{x_n, y_n\})$ for any real valued sequences $x=(x_n)$ and $y=(y_n)$.

Corollary 2.3. *Let $x=(x_n)$, $y=(y_n)$ and $z=(z_n)$ be sequence of nonnegative real numbers. If $x \overset{DS_L}{\sim} y$ then $x \wedge z \overset{DS_L}{\sim} z$ and $x \overset{DS_L}{\sim} y \vee z$ hold.*

Definition 2.4. If $x=(x_n)$ satisfies a property P for all $n \in \mathbb{N}$ except a set which has zero deferred density , then it is said that $x=(x_n)$ has the property P deferred almost all $n \in \mathbb{N}$ and it is denoted by "*d.a.a.e.*".

Following theorems are weak version of Theorem 2.1 and Theorem 2.2 .

Theorem 2.5. *Let $x=(x_n)$, $y=(y_n)$ and $z=(z_n)$ be any sequences of non negative real numbers. If $x \overset{DS_L}{\sim} y$ and $z \prec x$ (d.a.a.e.), then $z \overset{DS_L}{\sim} y$.*

Proof. Let $A := \{n : z_n > x_n\}$. From the assumption $\delta_{p,q}(A)=0$ holds. That is, the inequality $\left|\frac{z_k}{y_k}-L\right| \leq \left|\frac{x_k}{y_k}-L\right|$ holds (*d.a.a.e.*). Then, we have

$$\frac{1}{q(n)-p(n)}\left|\left\{p(n)<k\leq q(n):\left|\frac{z_k}{y_k}-L\right|\geq\varepsilon\right\}\right|$$

$$\leq \frac{1}{q(n)-p(n)}\left|\left\{p(n)<k\leq q(n):\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\right|+\frac{1}{q(n)-p(n)}|A|.$$

By taking limit when $n \to \infty$, we obtained $z \overset{DS_L}{\sim} y$. □

Theorem 2.6. *Let $x=(x_n)$, $y=(y_n)$ and $z=(z_n)$ be sequence of non negative real numbers. If $x \overset{DS_L}{\sim} y$ and $y \prec z$ (d.a.a.e.), then $x \overset{DS_L}{\sim} z$.*

Proof. The proof can be obtain easily by considering the proof of Theorem 2.5. So it is omitted here. □

Theorem 2.7. *Let $x=(x_n)$, $y=(y_n)$ and $z=(z_n)$ be sequence of non negative real numbers. If $x \overset{DS_L}{\sim} y$ and $x=z$ (d.a.a.e.), then $z \overset{DS_L}{\sim} y$.*

Proof. Take $A := \{n : x_n \neq z_n\}$. From the assumption we have $\delta_{p,q}(A)=0$. So, for any $\varepsilon>0$, the following inclusion

$$\left\{p(n)<k\leq q(n):\left|\frac{z_k}{y_k}-L\right|\geq\varepsilon\right\}$$

$$=\left\{p(n)<k\leq q(n):\left|\frac{z_k}{y_k}-L\right|\geq\varepsilon\right\}\cap\left(A^C\cup A\right)$$

$$\subseteq \left(\left\{ p(n) < k \leq q(n) : \left| \frac{z_k}{y_k} - L \right| \geq \varepsilon \right\} \cap A^C \right)$$
$$\cup \left(\left\{ p(n) < k \leq q(n) : \left| \frac{z_k}{y_k} - L \right| \geq \varepsilon \right\} \cap A \right)$$
$$\subseteq \left\{ p(n) < k \leq q(n) : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \cup A$$

holds. Hence,

$$\frac{1}{q(n) - p(n)} \left| \left\{ p(n) < k \leq q(n) : \left| \frac{z_k}{y_k} - L \right| \geq \varepsilon \right\} \right|$$
$$\leq \frac{1}{q(n) - p(n)} \left\{ p(n) < k \leq q(n) : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} + \frac{1}{q(n) - p(n)} |A|$$

holds. After taking limit when $n \to \infty$, desired result is obtained. □

3. Comparison of D_L^r with C_L^r and DS_L−Equivalence

In this section we give the relation between strongly r−deferred Cesàro asymptotically equivalence of sequences and asymptotically deferred statistical equivalence of sequences.

Theorem 3.1. *Let $x = (x_n)$ and $y = (y_n)$ be non negative real valued sequences. Then, $x \overset{D_L^r}{\sim} y$ implies $x \overset{DS_L}{\sim} y$.*

Proof. Assume that $x \overset{D_L^r}{\sim} y$ such that $\lim\limits_{n\to\infty} \frac{1}{q(n)-p(n)} \sum\limits_{k=p(n)}^{q(n)} \left| \frac{x_k}{y_k} - L \right|^r = 0$. For an arbitrary $\varepsilon > 0$, we have

$$\sum_{k=p(n)+1}^{q_n} \left| \frac{x_k}{y_k} - L \right|^r = \left(\sum_{\substack{k=p(n)+1 \\ \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon}}^{q(n)} + \sum_{\substack{k=p(n)+1 \\ \left|\frac{x_k}{y_k} - L\right| < \varepsilon}}^{q(n)} \right) \left| \frac{x_k}{y_k} - L \right|^r$$
$$\geq \sum_{\substack{k=p(n)+1 \\ \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon}}^{q(n)} \left| \frac{x_k}{y_k} - L \right|^r \geq \varepsilon^r . \left| \left\{ p(n) < k \leq q(n) : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right|$$

and

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q_n} \left| \frac{x_k}{y_k} - L \right|^r$$
$$\geq \varepsilon^r . \frac{1}{q(n) - p(n)} \left| \left\{ p(n) < k \leq q(n) : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| .$$

So, if we take limit when $n \to \infty$, the proof is obtained. □

Corollary 3.2. *Let $x = (x_n)$ and $y = (y_n)$ be non negative real valued sequences. Then, $x \overset{L}{\sim} y$ implies $x \overset{DS_L}{\sim} y$.*

Remark 3.3. The converse of Theorem 3.1 and Corollary 3.2 are not true, in general.

To see, this let us consider nonnegative two sequences $x = (x_n)$ and $y = (y_n)$ as follows:

$$x_n := \begin{cases} n, & \left[\sqrt{q(k)}\right] - m_0 < n \leq \left[\sqrt{q(k)}\right], \quad k = 1, 2, 3, .., \\ L, & otherwise, \end{cases}$$

and

$$y_n := \begin{cases} \frac{1}{n}, & \left[\sqrt{q(k)}\right] - m_0 < n \leq \left[\sqrt{q(k)}\right], \quad k = 1, 2, 3, ..., \\ L, & otherwise, \end{cases}$$

where $q(n)$ is a strictly monotone increasing sequence and m_0 is an arbitrary fixed natural number and $[x]$ denotes the Gauss bracket of x, the largest integer not exceeding x.

If we also consider a method $D[p, q]$ for any $p(n)$ satisfying $0 < p(n) < \left[\sqrt{q(k)}\right] - m_0$, then for an arbitrary $\varepsilon > 0$ we have

$$\frac{1}{q(n) - p(n)} \left| \left\{ p(n) < k \leq q(n) : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = \frac{m_0}{q(n) - p(n)} \to 0,$$

when $n \to \infty$. On the other hand,

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q_n} \left| \frac{x_k}{y_k} - L \right|^r \geq m_0 \frac{\left(\left[\sqrt{q(k)}\right] - m_0 \right)^2}{q(n) - p(n)} \to m_0$$

holds, when $n \to \infty$. That is, $x \overset{DS_L}{\sim} y$ but $x \overset{D_L^r}{\nsim} y$. It is also clear that $x \overset{L}{\nsim} y$.

Theorem 3.4. *If $x = (x_n)$ and $y = (y_n) \in l_\infty$, then $x \overset{DS_L}{\sim} y$ implies $x \overset{D_L^r}{\sim} y$, where l_∞ denotes the set of all bounded sequences.*

Proof. Assume that the sequences $x = (x_n)$ and $y = (y_n)$ from ℓ_∞ and they are satisfying $x \overset{DS_L}{\sim} y$. Then, there exists a positive real number $M > 0$ such that $\left| \frac{x_k}{y_k} - L \right| \leq M$ holds for all $k \in \mathbb{N}$. So, for any $\varepsilon > 0$, the following inequality

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q_n} \left| \frac{x_k}{y_k} - L \right|^r = \frac{1}{q(n) - p(n)} \left(\sum_{k \in A} + \sum_{k \in B} \right) \left| \frac{x_k}{y_k} - L \right|^r$$

$$\leq \frac{M^r}{q(n) - p(n)} \left| \left\{ p(n) < k \leq q(n) : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| + \varepsilon^r$$

is satisfied where A and B denote the following sets

$$\left\{ p(n) < k \leq q(n) : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \text{ and } \left\{ p(n) < k \leq q(n) : \left| \frac{x_k}{y_k} - L \right| < \varepsilon \right\}$$

respectively. Hence, after taking limit when $n \to \infty$ desired result is obtained. $\square$

Definition 3.5. A method $D[p, q]$ is called properly deferred when $\{p(n)\}$ and $\{q(n)\}$ satisfy in addition to (3), the sequence

$$\left\{ \frac{p(n)}{q(n) - p(n)} \right\}_{n \in \mathbb{N}}$$

is bounded for all $n \in \mathbb{N}$.

Theorem 3.6. *In order that $x \overset{S_L}{\sim} y$ implies $x \overset{DS_L}{\sim} y$ if and only if the method $D(p,q)$ is properly deferred.*

Before proof of Theorem 3.6, let us recall a simple result for sequences of positive real numbers:

Lemma 3.7. *Let $a = (a_n)$ and $b = (b_n)$ be sequences of positive real numbers. If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = \infty$, then $\lim_{n\to\infty} a_{b_n} = a$.*

Proof. (proof of Theorem 3.6) Since $x \overset{S_L}{\sim} y$, then we have

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\} \right| = 0.$$

If we consider $a_n := \frac{1}{n} \left| \left\{ k \le n : \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\} \right|$ and $b_n := q(n)$, then from Lemma 3.7 we have

$$\lim_{n\to\infty} \frac{1}{q(n)} \left| \left\{ k \le q(n) : \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\} \right| = 0.$$

Also, by set comparison the following inequality

$$\left\{ p(n) < k \le q(n) : \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\} \subseteq \left\{ k \le q(n) : \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\}.$$

holds for every $\varepsilon > 0$. Therefore, we have

$$\begin{aligned}
&\frac{1}{q(n) - p(n)} \left| \left\{ p(n) < k \le q(n) : \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\} \right| \\
&\le \frac{q(n)}{q(n) - p(n)} \frac{1}{q(n)} \left| \left\{ k \le q(n) : \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\} \right| \\
&= \left(1 + \frac{p(n)}{q(n) - p(n)} \right) \frac{1}{q(n)} \left| \left\{ k \le q(n) : \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\} \right|.
\end{aligned}$$

After taking limit when $n \to \infty$, we obtain desired result. □

Remark 3.8. The converse of Theorem 3.6 does not hold even if $D(p,q)$ is properly deferred.

For to see this, let us consider $x = (x_k)$ and $y = (y_k)$ as follows:

$$x_k = \begin{cases} \frac{k+1}{2}, & k \text{ is odd}, \\ -\frac{k}{2}, & k \text{ is even}, \end{cases} \quad \text{and } y_k = 1, \text{ for all } k \in \mathbb{N}.$$

Take $p(n) = 2n$ and $q(n) = 4n$. It is clear that $x \overset{DS_L}{\sim} y$ (by Theorem 3.1), but $x \overset{S_L}{\nsim} y$.

Theorem 3.9. *If $x \overset{DS_L}{\sim} y$ with respect to $q(n) = n$ and arbitrary $p(n)$, then $x \overset{S_L}{\sim} y$ hold.*

Proof. Let us assume that $x \overset{DS_L}{\sim} y$ with respect to $q(n) = n$ and arbitrary $p(n)$. For any $n \in \mathbb{N}$, there exists an $h \in \mathbb{N}$ such that $n^{h+1} = 0$ and the inequality

$$p(n) = n^{(1)} > p\left(n^{(1)}\right) = n^{(2)} > p\left(n^{(2)}\right) = n^{(3)} > ... > p\left(n^{(h-1)}\right) = n^{(h)} \ge 1$$

holds. Therefore, the set $\left\{k \le n : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}$ can be represented as

$$= \left\{k \le n^{(1)} : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\} \cup \left\{n^{(1)} < k \le n : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}$$

by the same way the first set in the union can be represented as

$$\left\{k \le n^{(2)} : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\} \cup \left\{n^{(2)} < k \le n^{(1)} : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}.$$

After finite step (at most h step)

$$\begin{aligned}&\left\{k \le n^{(h-1)} : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\\ &= \left\{k \le n^{(h)} : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\} \cup \left\{n^{(h)} < k \le n^{(h-1)} : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\end{aligned}$$

is obtained. Therefore;

$$\frac{1}{n}\left|\left\{k \le n : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| = \sum_{m=0}^{h} \frac{n^{(m)} - n^{(m+1)}}{n} T_m,$$

where

$$T_m := \frac{1}{n^{(m)} - n^{(m+1)}} \left|\left\{n^{(m+1)} < k \le n^{(m)} : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right|.$$

If we consider a matrix $A := (a_{n,m})$ as

$$a_{n,m} := \begin{cases} \frac{n^{(m)} - n^{(m+1)}}{n}, & m = 0, 1, 2, ..., h, \\ 0, & otherwise, \end{cases}$$

then the sequence $\left\{\frac{1}{n}\left|\left\{k \le n : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right|\right\}_{n\in\mathbb{N}}$ is the $(a_{n,m})$ transformation of the sequence (T_m) .Since the matrix $A = (a_{n,m})$ satisfies Silverman -Toeplitz Theorem (see in [18]) and from assumption on $x = (x_n)$ and $y = (y_n)$ then we have desired result. □

Combining Theorem 3.6 and Theorem 3.9 the following theorem is obtained:

Theorem 3.10. *An DS_L−equivalence with respect to any $p(n)$ and $q(n) = n$ is equivalent to S_L−equivalence if and only if $\left\{\frac{p(n)}{n-p(n)}\right\}_{n\in\mathbb{N}}$ is bounded.*

Also, as a corollary of Theorem 3.10, the following result can be given: If we consider the method as

$$D_n^\theta := \frac{S_{[\theta n]+1} + S_{[\theta n]+2} + ... + S_n}{n - [\theta n]}$$

where θ is a constant $0 \le \theta < 1$ and $[\theta n]$ is the greatest integer of $\le \theta n$. Then, we have following result:

Corollary 3.11. $x \overset{D_n^\theta S_L}{\sim} y$ *if and only if* $x \overset{S_L}{\sim} y$.

Theorem 3.12. *If the method $D[p, q]$ is properly deferred, then $x \overset{C_L^r}{\sim} y$ implies $x \overset{D_L^r}{\sim} y$.*

Proof. Let us assume that $x \overset{C_L^r}{\sim} y$ and the sequences $\left(\dfrac{p(n)}{q(n)-p(n)}\right)$ is bounded. Then, we have

$$\frac{1}{q(n)-p(n)}\sum_{p(n)+1}^{q(n)}\left|\frac{x_k}{y_k}-L\right|^r = \frac{1}{q(n)-p(n)}\left(\sum_{k=1}^{q(n)}-\sum_{k=1}^{p(n)}\right)\left|\frac{x_k}{y_k}-L\right|^r$$

$$= \frac{-p(n)}{q(n)-p(n)}\frac{1}{p(n)}\sum_{k=1}^{p(n)}\left|\frac{x_k}{y_k}-L\right|^r + \frac{q(n)}{q(n)-p(n)}\frac{1}{q(n)}\sum_{k=1}^{q(n)}\left|\frac{x_k}{y_k}-L\right|^r.$$

By taking limit when $n \to \infty$ desired result is obtained. □

4. Comparison of DS_L-Equivalence for any $p(n)$ and $q(n)$

Let us consider $p' = \left\{p'(n)\right\}$ and $q' = \{q'(n)\}$ which are satisfying

$$p(n) \le p'(n) < q'(n) \le q(n) \tag{6}$$

for all $n \epsilon \mathbb{N}$ besides (3). Denote by the associated set $E := \{p(n) : n \in \mathbb{N}\}$, $E' := \left\{p'(n) : n \in \mathbb{N}\right\}$, $F := \{q(n) : n \in \mathbb{N}\}$ and $F' := \left\{q'(n) : n \in \mathbb{N}\right\}$.

Theorem 4.1. *If the set $F' \setminus F$ is finite and $\lim_{n\to\infty}\frac{q(n)-q'(n)}{q'(n)-p(n)} < \infty$ holds. Then, $x \overset{DS_L}{\backsim} y$ w.r.t. p and q implies $x \overset{DS_L}{\backsim} y$ w.r.t. p and q'.*

Proof. Since $F' \setminus F$ is finite, then there is an $n_0 \in \mathbb{N}$ such that the inclusion $\left\{q'(n) : n > n_0\right\} \subset \{q(n) : n \in \mathbb{N}\}$ holds. So, there is a strictly increasing sequence $j = \{j(n)\}$ such that $q'(n) = q(j(n))$ for all $n \ge n_0$. Therefore, sufficiently large $n \in \mathbb{N}$, following inequality

$$\frac{1}{q'(n)-p(n)}\left|\left\{p(n) < k \le q'(n) \ : \ \left|\frac{x_k}{y_k}-L\right| \ge \varepsilon\right\}\right|$$

$$= \frac{1}{q(j(n))-p(n)}\left|\left\{p(n) < k \le q(j(n)) : \left|\frac{x_k}{y_k}-L\right| \ge \varepsilon\right\}\right|$$

$$\le \frac{q(n)-p(n)}{q'(n)-p(n)}.\frac{1}{q(n)-p(n)}\left|\left\{p(n) < k \le q(n) : \left|\frac{x_k}{y_k}-L\right| \ge \varepsilon\right\}\right|$$

$$= \left(\frac{q(n)-q'(n)}{q'(n)-p(n)}+1\right)\frac{1}{q(n)-p(n)}\left|\left\{p(n) < k \le q(n) : \left|\frac{x_k}{y_k}-L\right| \ge \varepsilon\right\}\right|$$

holds. Under the assumption we have desired result. □

Theorem 4.2. *If the set $F \setminus F'$ is finite and $\liminf_{n\to\infty}\frac{q'(n)-p(n)}{q(n)-p(n)} > 0$ hold. Then, $x \overset{DS}{\backsim} y$ w.r.t. p and q' implies $x \overset{DS_L}{\backsim} y$ w.r.t. p and q.*

Proof. It can be proved by following above Theorem 4.1. So, it is omitted here. □

Corollary 4.3. *If $F \vartriangle F'$ is finite, then $x \overset{DS_L}{\backsim} y$ w.r.t. p and q if and only if $x \overset{DS_L}{\backsim} y$ w.r.t. p and q'.*

Theorem 4.4. *If $E^{'} \setminus E$ is a finite set and $\liminf_{n\to\infty}\frac{q(n)-p^{'}(n)}{q(n)-p(n)} > 0$ hold. Then, $x \overset{DS_L}{\backsim} y$ w.r.t. p and q implies $x \overset{DS_L}{\backsim} y$ w.r.t. $p^{'}$ and q.*

Proof. It can be proved by using the same idea in Theorem 4.1. So, it is omitted here. $\square$

Theorem 4.5. *The sequence $p'(n)$ and $q'(n)$ are satisfying (6) such that the set $\{k : p(n) < k \le p'(n)\}$ and $\{k : q'(n) < k \le q(n)\}$ are finite. Then, $x \overset{DS_L}{\sim} y$ w.r.t. p' and q' implies $x \overset{DS_L}{\sim} y$ w.r.t. p and q.*

Proof. Assume that $x \overset{DS_L}{\sim} y$ w.r.t. p' and q'. So, for an arbitrary $\varepsilon > 0$, we have the following inequality

$$
\begin{aligned}
&\frac{1}{q(n)-p(n)}\left|\left\{p(n) < k \le q(n) : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| \\
&\le \frac{1}{q'(n)-p'(n)}\left|\left\{p(n) < k \le p'(n) : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| \\
&+\frac{1}{q'(n)-p'(n)}\left|\left\{p'(n) < k \le q'(n) : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| \\
&+\frac{1}{q'(n)-p'(n)}\left|\left\{q'(n) < k \le q(n) : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| \le \frac{m_1}{q'(n)-p'(n)} \\
&+\frac{1}{q'(n)-p'(n)}\left|\left\{p'(n) < k \le q'(n) : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| + \frac{m_2}{q'(n)-p'(n)}
\end{aligned}
$$

where $m_1 := \left|\{k : p(n) < k \le p'(n)\}\right|$ and $m_2 := \left|\{k : q'(n) < k \le q(n)\}\right|$. On taking limit when $n \to \infty$ we have

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}\left|\left\{p(n) < k \le q(n) : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| = 0,$$

thus $x \overset{DS_L}{\sim} y$ (w.r.t. p and q). $\square$

Theorem 4.6. *If the sequence $p'(n)$ and $q'(n)$ are satisfying (6) such that*

$$\lim_{n\to\infty}\frac{q(n)-p(n)}{q'(n)-p'(n)} = 0, \tag{7}$$

then, $x \overset{DS_L}{\sim} y$ w.r.t. p and q implies $x \overset{DS_L}{\sim} y$ w.r.t. p' and q'.

Proof. It is clear from (6) that the inclusion

$$\left\{p'(n) < k \le q'(n) : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\} \subset \left\{p(n) < k \le q(n) : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}$$

and the inequality

$$
\begin{aligned}
&\frac{1}{q'(n)-p'(n)}\left|\left\{p'(n) < k \le q'(n) : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| \\
&\le \frac{q(n)-p(n)}{q'(n)-p'(n)}.\frac{1}{q(n)-p(n)}\left|\left\{p(n) < k \le q(n) : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right|
\end{aligned}
$$

hold. After taking limit when $n \to \infty$ and (7) the desired result is obtained. $\square$

Theorem 4.7. *Under the assumption of Theorem 4.5, $x \overset{D_L^r}{\sim} y$ w.r.t. p' and q' implies $x \overset{D_L^r}{\sim} y$ w.r.t. p and q for any bounded x and y.*

Proof. Let x and y are bounded sequences, then there exists a positive real numbers M such that $\left|\frac{x_k}{y_k} - L\right| \le M$. Then, we can write

$$\frac{1}{q(n)-p(n)}\sum_{p(n)+1}^{q(n)}\left|\frac{x_k}{y_k}-L\right|^r = \frac{1}{q(n)-p(n)}\left[\begin{array}{c} \sum\limits_{p(n)+1}^{p'(n)}\left|\frac{x_k}{y_k}-L\right|^r + \\ +\sum\limits_{p'(n)+1}^{q'(n)}\left|\frac{x_k}{y_k}-L\right|^r + \\ +\sum\limits_{q'(n)+1}^{q(n)}\left|\frac{x_k}{y_k}-L\right|^r \end{array}\right]$$

$$\le \frac{2}{q'(n)-p'(n)}M^r O(1) + \frac{1}{q'(n)-p'(n)}\sum_{q'(n)+1}^{q(n)}\left|\frac{x_k}{y_k}-L\right|^r$$

So, we have $x \overset{D_L^r[p,q]}{\sim} y$. □

Theorem 4.8. *Let $\{p(n)\}, \{q(n)\}, \{p'(n)\}$ and $\{q'(n)\}$ be sequences of non-negative integers satisfying (6) and (7), then $x \overset{D_L^r}{\sim} y$ w.r.t. p and q implies $x \overset{D_L^r}{\sim} y$ w.r.t. p' and q'.*

Proof. It is easy to see that the inequality

$$\frac{1}{q(n)-p(n)}\sum_{p(n)+1}^{q(n)}\left|\frac{x_k}{y_k}-L\right|^r \ge \frac{1}{q(n)-p(n)}\sum_{p'(n)+1}^{q'(n)}\left|\frac{x_k}{y_k}-L\right|^r$$

$$\ge \frac{q'(n)-p'(n)}{q(n)-p(n)}\frac{1}{q'(n)-p'(n)}\sum_{p'(n)+1}^{q'(n)}\left|\frac{x_k}{y_k}-L\right|^r$$

holds. So, by taking limit when $n \to \infty$, desired result is obtained. □

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