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On the Classical Solvability of Mixed Problems for a Second–Order One–Dimensional Parabolic Equation

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Abstract.

We prove the existence and uniqueness of classical solutions to mixed problems for the equation

$$\frac{\partial u}{\partial t}(x,t) \; - \; \frac{\partial^2 u}{\partial x^2}(x,t) \; + \; q(x)\, u(x,t) \; = \; f(x,t)$$

on a rectangle $\overline{\Omega} = [a,b]\times[0,T]$, with arbitrary self–adjoint homogenous boundary conditions. We assume that q and f are continuous functions, that $f(x,\cdot)$ satisfies a Hölder condition uniformly with respect to x , and the initial function belongs to the class $\overset{\circ}{W}{}_p^{(1)}(a,b)$ ($1 < p \leq 2$) . Also, an upper–bound estimate for the solution and, as a consequence, a kind of stability of the solution with respect to the initial function are established. Moreover, some convergence rate estimates for the series defining solutions (and their first derivatives) are given. A modification of the Fourier method is used.

Based on the obtained results, we also study the mixed problems on an unbounded rectangle $\overline{\Omega_\infty} = [a,b] \times [0,+\infty)$. The existence and uniqueness of classical solutions are established, and some properties of the solutions are considered.

1. Introduction

Let $G = (a,b)$ be a bounded open interval of the real axis $\mathbb{R}$, and let $T > 0$ be an arbitrary number. In this paper we consider the problem of existence of a real–valued function $u = u(x,t)$ defined on the closed rectangle $\overline{\Omega} = [a,b]\times[0,T]$, and satisfying the following partial differential equation, initial condition and boundary conditions :

$$\frac{\partial u}{\partial t}(x,t) \; - \; \frac{\partial^2 u}{\partial x^2}(x,t) \; + \; q(x)\, u(x,t) \; = \; f(x,t)\,, \quad (x,t)\in\Omega\,, \tag{1.1}$$

$$u(x,0) \; = \; \varphi(x)\,, \qquad x\in\overline{G}\,, \tag{2.1}$$

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$$\begin{aligned} \alpha_{10}\, u(a,t) + \alpha_{11}\, u'_x(a,t) + \beta_{10}\, u(b,t) + \beta_{11}\, u'_x(b,t) &= 0\,, \\ \alpha_{20}\, u(a,t) + \alpha_{21}\, u'_x(a,t) + \beta_{20}\, u(b,t) + \beta_{21}\, u'_x(b,t) &= 0\,, \quad t \in [0,T]\,, \end{aligned} \tag{3.1}$$

where $(\alpha_{i0}, \alpha_{i1}, \beta_{i0}, \beta_{i1}) \in \mathbb{C}^4$ $(i = 1,2)$ are linearly independent vectors, and q, φ, f are given real–valued functions. We suppose conditions (3.1) are such that the formal Schrödinger operator

$$\mathcal{L}(v)(x) \;=\; -\,v''(x) + q(x)\,v(x)\,, \qquad x \in G\,, \tag{4.1}$$

and the boundary conditions

$$\begin{aligned} \alpha_{10}\, v(a) + \alpha_{11}\, v'(a) + \beta_{10}\, v(b) + \beta_{11}\, v'(b) &= 0\,, \\ \alpha_{20}\, v(a) + \alpha_{21}\, v'(a) + \beta_{20}\, v(b) + \beta_{21}\, v'(b) &= 0\,, \end{aligned} \tag{5.1}$$

generate an arbitrary self–adjoint operator L, with the discrete spectrum.

Definition 1.1 . A real–valued function $u = u(x,t)$ is called a *classical solution* of the mixed (i.e. *initial/boundary–value*) problem (1.1)–(3.1) if it has the following properties :

1_u) $u \in C(\overline{\Omega})$, u'_x exists on $\Omega \cup (\partial G \times [0,T])$, $u'_t,\, u''_{x^2} \in C(\Omega)$;

2_u) u satisfies equation (1.1) for all $(x,t) \in \Omega$, in the ordinary sense ;

3_u) u satisfies conditions (2.1)–(3.1) in the ordinary sense. •

We first prove the existence of a classical solution to the problem (1.1)–(3.1) , under certain smoothness conditions imposed on functions q, φ and f. Then we study the problem of stability of the solution, with respect to the initial data. The uniqueness of the solution is established under some less restrictive conditions then in the case of the existence. Finally, some convergence rate estimates for the series representing the solution (and its first derivatives) are obtained.

In the second part of this paper, we study a *mixed problem* $[(1.1) - (3.1)]_\infty$, defined as the problem (1.1)–(3.1) with $[0,T]$ and Ω replaced by $[0,+\infty)$ and $\Omega_\infty = (a,b) \times (0,+\infty)$ respectively. In accordance with this, we define a *classical solution* of the new problem as a function $u : \overline{\Omega}_\infty \to \mathbb{R}$ which has the following properties :

1_u^∞) $u \in C(\overline{\Omega}_\infty)$, u'_x exists on $\Omega_\infty \cup (\partial G \times [0,+\infty)])$, $u'_t,\, u''_{x^2} \in C(\Omega_\infty)$;

2_u^∞) u satisfies equation (1.1) for all $(x,t) \in \Omega_\infty$, in the ordinary sense ;

3_u^∞) u satisfies conditions (2.1) and (3.1) in the ordinary sense, the later ones for every $t \in [0,+\infty)$.

Most of the mentioned above results can be applied in proving the existence of such a solution, and in the analysis of its properties.

We started to investigate the problem (1.1)–(3.1) in [10] , working with functions satisfying certain monotonicity conditions. The existence and uniqueness of the solutions, and some upper–bound estimates for them were established therein. In the present paper further essential extensions and refinements are given, and the unbounded case is considered. The technique used in proofs of the theorems is based, on one side, on uniform and exact, with respect to the order, upper–bound estimates for the eigenfunctions (and their derivatives) of the operator (4.1)–(5.1) , and, on the other side, on the known asymptotics of its eigenvalues and eigenfunctions.

Note that the corresponding mixed problem for a second–order one–dimensional hyperbolic equation was considered in [8]–[9] and [12] .

There are three more sections in this paper.

In Section 2 our results concerning the existence (Theorem 1.2) and the uniqueness (Theorem 2.2) of the solution are established. Assuming that the functions q, f are continuous, that φ belongs to a subclass of $W_p^{(1)}(G)$ $(1 < p \le 2)$ and $f(t,\cdot)$ satisfies a Hölder condition, we prove the existence and we obtain some a priori estimate for the solution. (As in our previous papers, by *a priori estimate* we mean an estimation (from above) of the uniform norm of the solution, by the corresponding norms of φ and f.) Then we derive the stability of the solution with respect to the initial function φ (Corollary 1.2).

Theorem 1.2 is inspired by Chernyatin's paper [2], where the existence and uniqueness of an appropriately defined classical solution u of the equation (1.1), satisfying the conditions

$$u(x,0) = 0, \quad x \in [0,\pi]; \qquad u(0,t) = u(\pi,t) = 0, \quad t \in [0,T],$$

were proved. (Conditions 1) and 3) from Theorem 1.2 were supposed therein.) In the proof of Theorem 1.2 we use an appropriate modification of the method developed in [2]. This method contains only one differentiation (with respect to both x, t) of the series representing the solution, and it gives the possibility to impose a "minimal" smoothness condition on f. (As a price, the second partial derivative of the solution can not be represented as the sum of a series converging in the uniform metric.) Our approach is mostly based on the mentioned above estimates for the operator (4.1)–(5.1).

It is possible to "separate" the problem of the existence from the problem of the uniqueness, in the following sense. Having in mind the additional properties of the classical solution, claimed by the assertion (a) of Theorem 1.2, we appropriately modify 1_u)−conditions from Definition 1.1. For such subclass of classical solutions we can establish the uniqueness under less restrictive conditions on q, φ, and f. Theorem 2.2 is proved by a method used in paper [5].

In Section 3 we consider the series representing the solution and its first derivatives. Supposing that the conditions of Theorem 1.2 are fulfilled, we prove some convergence rate estimates for these series (Theorem 1.3).

The last section is devoted to the mixed problem $[(1.1)-(3.1)]_\infty$. Assuming that q, f are continuous on their (closed) domains, that $\varphi \in \overset{\circ}{W}{}_p^{(1)}(G)$, and that $f(x,\cdot)$ satisfies a Hölder condition locally on $[0,+\infty)$ (and uniformly with respect to x), we prove the existence and uniqueness of the classical solution (Theorem 1.4). Then, we establish an upper–bound estimate for of the solution (Theorem 2.4), and we discuss convergence rate estimates (subsection 3.4).

2. Existence and uniqueness

1.2. Main theorems

Let $AC(\overline{G})$ be the class of (real–valued) absolutely continuous functions on $\overline{G} = [a,b]$, and let $\overset{\circ}{W}{}_p^{(1)}(G) \overset{\text{def}}{=} \{h \in W_p^{(1)}(G) \mid h(a) = 0 = h(b)\}$, $p \in [1,+\infty)$. (Note, $h \in W_p^{(1)}(G)$ if $h \in AC(\overline{G})$ and $h^{(1)} \in L_p(G)$.)

Let $g = g(x,t)$ be a real function defined on the closed rectangle $\overline{\Omega}$. We say that this function *satisfies the Hölder condition on* $[0,T]$, with *an exponent* $\alpha \in (0,1]$, *uniformly with respect to* $x \in \overline{G}$ if there exists a constant $B > 0$ such that

$$(\forall x \in \overline{G})(\forall t,\, t' \in [0,T]) \qquad |g(x,t) - g(x,t')| \le B\,|t-t'|^\alpha. \tag{1.2}$$

If a series $\sum_{n=1}^{\infty} |h_n(y)|$ converges uniformly on a set $\tilde{\Omega} \subseteq \overline{\Omega}$, then we say that the series $\sum_{n=1}^{\infty} h_n(y)$ converges (or, is convergent) *absolutely and uniformly on* $\tilde{\Omega}$ (in abbreviation: *a.u. on* $\tilde{\Omega}$). Finally, for any $\epsilon \in (0,T)$ we set $\overline{\Omega}_\epsilon \overset{\text{def}}{=} \overline{G} \times [\epsilon, T]$.

We can now state our main results.

Theorem 1.2. *Let us assume*:

1) $q \in C(\overline{G})$.

2) $\varphi \in \overset{\circ}{W}{}_p^{(1)}(G)$ $(1 < p \le 2)$, *and* φ *satisfies the boundary conditions* (5.1).

3) $f \in C(\overline{\Omega})$, *and* f *satisfies the Hölder condition on* $[0, T]$, *with an exponent* $\alpha \in (1/2, 1]$, *uniformly with respect to* $x \in \overline{G}$.

Then, the following is valid:

(a) *There exists an unique classical solution* $u = u(x, t)$ *to the problem* (1.1)–(3.1). *It belongs to the class* $C^{(1)}(\overline{G} \times (0, T])$, *and* $u''_{x^2} \in C(\overline{G} \times (0, T])$.

(b) *The solution can be represented as a series converging absolutely and uniformly on* $\overline{\Omega}$. *This series can be differentiated once with respect to* x *or* t *on every closed rectangle* $\overline{\Omega}_\epsilon$. *The obtained series for the first derivatives of the solution converge absolutely and uniformly on* $\overline{\Omega}_\epsilon$.

(c) *The estimate*

$$\|u\|_{C(\overline{\Omega})} \leq C \left(\|\varphi\|_{C(\overline{G})} + \|\varphi'\|_{L_p(G)} + \|f\|_{C(\overline{\Omega})} \right) \tag{2.2}$$

holds, with a constant $C > 0$ *not depending on* φ, f.

Corollary 1.2. *Let* $w_i = w_i(x, t)$ $(1 \leq i \leq 2)$ *be the classical solution of the problem* (1.1)–(3.1), *with the initial function* $\varphi_i = \varphi_i(x)$. *There exists a constant* $C > 0$ *not depending on* φ_i *and* f, *such that*

$$\| w_1 - w_2 \|_{C(\overline{\Omega})} \leq C \left(\|\varphi_1 - \varphi_2 \|_{C(\overline{G})} + \|\varphi_1' - \varphi_2' \|_{L_p(G)}\| \right).$$

Remark 1.2. If the coefficients in (3.1) satisfy $\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11} \neq 0$, then the above condition $\varphi \in \overset{\circ}{W}{}_p^{(1)}(G)$ can be relaxed by $\varphi \in W_p^{(1)}(G)$ (see subsection 9.2). $\diamond$

Let us precise the statement (from Theorem 1.2) concerning the uniqueness of the solution. Replace the conditions $1_u)$ from Definition 1.1 by the following ones:

$1_u^*)$ $u \in C(\overline{\Omega})$, $u \in C^{(1)}(\overline{G} \times (0, T))$, $u''_{x^2} \in C(\Omega)$.

Definition 1.2. We say that a function $u : \overline{\Omega} \to \mathbb{R}$ is *a classical solution* to the problem (1.1)–(3.1) if the conditions $1_u^*)$, $2_u)$, and $3_u)$ are satisfied. •

For such functions the following is valid.

Theorem 2.2. *Let us assume*: 1) $q \in C(\overline{G})$; 2) $\varphi \in W_1^{(1)}(G)$, *and* φ *satisfies the boundary conditions* (5.1); 3) $f \in C(\overline{\Omega})$.

Then, there exists at most one classical solution to the problem (1.1)–(3.1).

Remark 2.2. The assertion (a) of Theorem 1.2 shows that the classical solution, "established" by the theorem, is a classical solution in the sense of Definition 1.2. So, this solution must be unique. It follows that every classical solution of the problem (1.3)–(3.3) has the form (11.2). As the proof of estimate (2.2) is based on the form, we conclude that Corollary 1.2 is valid (see subsection 7.2). $\diamond$

Proof of Theorem 1.2 or, generally, our approach to the justification of the Fourier method is based on a set of results obtained by several authors. In the next subsection we will formulate (almost all of) them in detail.

2.2. Preliminaries

Consider an arbitrary *non–negative* self–adjoint extension L of the (symmetric) minimal operator generated by (4.1), with the potential $q \in L_1(G)$. This extension is defined by the corresponding *self–adjoint* boundary conditions (5.1); its spectrum is discrete (see [13], §18 or [14], Chp. 10). Recall the definition of the operator L. Let $\mathcal{D}(L)$ be the set of functions $g \in L_2(G)$ such that $g, g' \in AC(\overline{G})$, $\mathcal{L}(g) \in L_2(G)$, and g satisfies (5.1). If $g \in \mathcal{D}(L)$, then $L(g)(x) = \mathcal{L}(g)(x)$. Denote by $\{v_n\}_1^\infty$ the orthonormal (and complete in $L_2(G)$) system of eigenfunctions of L, and by $\{\lambda_n\}_1^\infty$ the corresponding system of non–negative eigenvalues enumerated in the non–decreasing order. (By definition, $v_n \in \mathcal{D}(L)$, and v_n satisfies the differential equation

$$-\,v_n''(x) + q(x)\,v_n(x) = \lambda_n\,v_n(x) \tag{3.2}$$

almost everywhere on G.) Then, the following assertions are true.

Proposition 1.2 ([4], [6]). *If $q \in L_1(G)$, then there exist constants $C_0 > 0$ and $A > 0$, independent of $n \in \mathbb{N}$ and $t \geq 0$, such that*

$$\max_{x\in\overline{G}} |v_n(x)| \leq C_0\,; \tag{4.2}$$

$$\sum_{t\leq\sqrt{\lambda_n}\leq t+1} 1 \leq A\,. \tag{5.2}$$

Proposition 2.2 ([7]). *Suppose $q \in C(\overline{G})$. Then $v_n \in C^{(2)}(\overline{G})$, the equation* (3.2) *is satisfied everywhere on G, and there are constants $\mu_0(G) > 0$ and $C_j > 0$ ($j = 1, 2$), independent of $n \in \mathbb{N}$, such that*

$$\max_{x\in\overline{G}} |v_n^{(j)}(x)| \leq \begin{cases} C_j\,\lambda_n^{j/2} & \text{if } \lambda_n > \mu_0(G)\,, \\ C_j & \text{if } 0 \leq \lambda_n \leq \mu_0(G)\,. \end{cases} \tag{6.2}$$

Proposition 3.2 ([11]). (*a*) *Let $q \in L_1(G)$, $h \in \overset{\circ}{W}{}_p^{(1)}(G)$ ($1 < p \leq 2$). Then equality $h(x) = \sum_{n=1}^{\infty} h_n\,v_n(x)$ ($h_n \overset{\text{def}}{=} \int_a^b h(x)\,v_n(x)\,dx$) holds on $\overline{G}$, and the series is convergent a.u. on $\overline{G}$.*

(*b*) *Suppose $q \in L_1(G)$, $h \in \mathcal{D}(L)$. Then equalities $h^{(j)}(x) = \sum_{n=1}^{\infty} h_n\,v_n^{(j)}(x)$ ($j = 0, 1$) hold on $\overline{G}$, the series being convergent a.u. on $\overline{G}$.*

Note that Propositions 1.2–3.2 are also valid in the case of an *arbitrary* self–adjoint extension L of the operator (4.1). (Then, only a finite number of negative eigenvalues of L may exist; some obvious minor changes in formulation of Propositions 1.2–2.2 are needed.) For the sake of simplicity, we will work with a *non–negative* operator L, and estimates (6.2) will be used supposing that $\mu_0(G) = 1$.

We will also use an appropriate estimate for Fourier coefficients h_n of a function $h \in \overset{\circ}{W}{}_p^{(1)}(G)$. The estimate is based on the known asymptotic behavior of the eigenfunctions and eigenvalues of the operator L, and it was obtained in [11]. Suppose $q \in L_1(G)$, where $G = (-1, 1)$. There exists a number $n_0 \in \mathbb{N}$ such that for every $n > n_0$ the following holds

$$\sqrt{\lambda_n} = n\,\pi + \gamma + \frac{\rho_n}{n^{\beta-1}}\,, \tag{7.2}$$

where $\beta = 2$ if the boundary conditions (5.1) satisfy $\theta_0^2 - 4\,\theta_{-1}\,\theta_1 \neq 0$, and $\beta = 3/2$ if $\theta_0^2 - 4\,\theta_{-1}\,\theta_1 = 0$ (see [13], pp. 66–67, 74). Here, $\gamma > 0$ is a constant, and $\{\rho_n\}_{n=n_0}^{\infty}$ is a bounded sequence: $|\rho_n| \leq \rho$.

In order to formulate the estimate mentioned, we introduce the following functions (defined a.e. on G) and the following Fourier coefficients:

$$h_{jc}(x) \overset{\text{def}}{=} h'(x)\cdot x^j \cdot \cos\gamma x\,, \quad h_{js}(x) \overset{\text{def}}{=} h'(x)\cdot x^j \cdot \sin\gamma x\,, \qquad j = 0, 1, 2\,;$$

$$a_n(g) = \int_{-1}^{1} g(x)\,\cos n\pi x\,dx\,, \quad b_n(g) = \int_{-1}^{1} g(x)\,\sin n\pi x\,dx\,, \qquad n \in \mathbb{N} \cup \{0\},$$

where $g \in L_1(G)$ is an arbitrary function. Now, for any $n > n_0$ the estimate

$$|h_n| \leq \frac{D_1}{n} \cdot \Big(|a_n(h_{0c} + h_{0s})| + |b_n(h_{0c} + h_{0s})| \Big) + \frac{D_2}{n^\beta} \cdot \Big(|a_n(h_{1c} + h_{1s})) + |b_n(h_{1c} + g_{1s})| \Big) + \frac{D_3}{n^2} \quad (8.2)$$

holds, where the constants D_j have the values:

$$D_1 \overset{\text{def}}{=} \frac{\sqrt{c}}{\pi}, \qquad D_2 \overset{\text{def}}{=} \frac{\rho\,\sqrt{c}}{\pi},$$
$$D_3 \overset{\text{def}}{=} \left(\frac{4\rho^2\,\sqrt{c}}{\pi\, n_0^{2\beta-3}} + \frac{2\,\sqrt{c}\,D}{\pi^2} \right) \cdot \|h'\|_{L_1(G)} + \frac{C_0}{\pi^2} \cdot \|h\|_{C(\overline{G})}\,\|q\|_{L_1(G)}\,. \quad (9.2)$$

Here, numbers $D > 0$ and $c > 0$ are defined by asymptotic relations for the eigenfunctions.

In order to avoid technicalities, we will prove all of our results *supposing that* $G = (-1,1)$. Transition from the interval $(-1,1)$ to an arbitrary bounded interval (a,b) can be realized by the following change of variable:

$$x = \frac{b-a}{2}\,t + \frac{a+b}{2}, \quad -1 \leq t \leq 1\,.$$

Consequently, in the general case one should put $(2x - a - b)/(b-a)$ instead of x, and π should be replaced by $\pi/(b-a)$.

Finally, let us note that we will frequently use the known inequalities of Bessel, Hölder and Riesz, applied to the orthonormal (on $\overline{G}$) systems $\{v_n\}_{n=1}^\infty$ and $\{\cos n\pi x\,,\ \sin n\pi x \mid n \in \mathbb{N} \cup \{0\}\}$.

3.2. Existence of the solution

In this subsection we begin the proof of Theorem 1.2, establishing the existence of a classical solution.

Let $\{v_n\}_1^\infty$, $\{\lambda_n \geq 0\}_1^\infty$ be defined as before. (Then $\lambda_n \to +\infty$ as $n \to \infty$.) Denote

$$\varphi_n = \int_{-1}^{1} \varphi(x)\,v_n(x)\,dx\,, \qquad f_n(t) = \int_{-1}^{1} f(x,t)\,v_n(x)\,dx\,, \quad t \in [0,T]\,,$$

and suppose that on $\overline{G}$ the following is valid:

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n\,v_n(x)\,, \qquad f(x,t) = \sum_{n=1}^{\infty} f_n(t)\,v_n(x)\,, \quad t \in [0,T]\,. \quad (10.2)$$

Applying the formal scheme of the Fourier method to the problem (1.1)–(3.1), we obtain that a "candidate" for the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} v_n(x) \left[\varphi_n\,e^{-\lambda_n t} + \int_0^t f_n(\tau)\,e^{-\lambda_n (t-\tau)}\,d\tau \right]. \quad (11.2)$$

In order to justify the method, rewrite (11.2) as $u(x,t) = u_1(x,t) + u_2(x,t)$, where, formally,

$$u_1(x,t) = \sum_{n=1}^{\infty} v_n(x)\,\varphi_n\,e^{-\lambda_n t}\,,$$
$$u_2(x,t) = \sum_{n=1}^{\infty} v_n(x) \cdot \int_0^t f_n(\tau)\,e^{-\lambda_n (t-\tau)}\,d\tau\,. \quad (12.2)$$

Proof of Theorem 1.2 , except the "uniqueness part", relies on two lemmas.

Lemma 1.2 . *Suppose that* $q \in C(\overline{G})$, $\varphi \in \overset{\circ}{W}{}_p^{(1)}(G)$ ($1 < p \leq 2$) *, and* φ *satisfies boundary conditions* (5.1) *. Then, the equality*

$$u_1(x,t) \;=\; \sum_{n=1}^{\infty} v_n(x)\,\varphi_n\,e^{-\lambda_n t} \tag{13.2}$$

holds uniformly on $\overline{\Omega}$ *, and the equalities*

$$\begin{aligned} \frac{\partial u_1}{\partial t}(x,t) \;&=\; -\sum_{n=1}^{\infty} \lambda_n\,v_n(x)\,\varphi_n\,e^{-\lambda_n t}\,, \\ \frac{\partial u_1}{\partial x}(x,t) \;=\; \sum_{n=1}^{\infty} v_n'(x)\,\varphi_n\,e^{-\lambda_n t}\,,\quad & \frac{\partial^2 u_1}{\partial x^2}(x,t) \;=\; \sum_{n=1}^{\infty} v_n''(x)\,\varphi_n\,e^{-\lambda_n t} \end{aligned} \tag{14.2}$$

hold uniformly on any rectangle $\overline{\Omega}_\epsilon$ *. The corresponding series are convergent a.u. on* $\overline{\Omega}$ *and* $\overline{\Omega}_\epsilon$ *respectively.*

Lemma 2.2 . *Let us assume:* $q \in C(\overline{G})$; $f \in C(\overline{\Omega})$ *, and* f *satisfies the Hölder condition on* $[0,T]$ *, with an exponent* $\alpha \in (1/2, 1]$ *, uniformly with respect to* $x \in \overline{G}$ *. Then, the equality*

$$u_2(x,t) \;=\; \sum_{n=1}^{\infty} v_n(x) \int_0^t f_n(\tau)\,e^{-\lambda_n (t-\tau)}\,d\tau \tag{15.2}$$

holds uniformly on $\overline{\Omega}$ *, and the equalities*

$$\begin{aligned} \frac{\partial u_2}{\partial t}(x,t) \;&=\; \sum_{n=1}^{\infty} v_n(x)\,\frac{d}{dt}\int_0^t f_n(\tau)\,e^{-\lambda_n (t-\tau)}\,d\tau\,, \\ \frac{\partial u_2}{\partial x}(x,t) \;&=\; \sum_{n=1}^{\infty} v_n'(x) \int_0^t f_n(\tau)\,e^{-\lambda_n (t-\tau)}\,d\tau \end{aligned} \tag{16.2}$$

hold uniformly on any $\overline{\Omega}_\epsilon$ *. The series are convergent a.u. on* $\overline{\Omega}$ *and* $\overline{\Omega}_\epsilon$ *respectively. Moreover, for every point* $(x,t) \in \overline{\Omega}_\epsilon$ *it holds*

$$\frac{\partial^2 u_2}{\partial x^2}(x,t) \;=\; \sum_{n=1}^{\infty} v_n''(x) \int_0^t f_n(\tau)\,e^{-\lambda_n (t-\tau)}\,d\tau\,,$$

the series converges in $L_2(\overline{G})$, $(u_2)''_{x^2} \in C(\overline{G} \times (0,T])$ *, and* u_2 *satisfies the equation* (1.1) *on* Ω *, in the ordinary sense.*

The terminology used above will be more accurately clarified through the proof of Lemma 1.2 .

Having Lemma 1.2 proved, we see that equalities (14.2) hold on Ω , so one can immediately check that the function u_1 belongs to the classes described in 1_u) , satisfies equation (1.1) (with $f = 0$) on Ω in the ordinary sense (by Proposition 2.2), and satisfies boundary conditions (3.1) for any $t \in (0,T]$. Also, Proposition 3.2 (a) obeys the validity of the first decomposition (10.2) , the series being a.u. convergent on $\overline{G}$. So, equality (13.2) and decomposition mentioned give $u_1(x,0) = \varphi(x)$ on $\overline{G}$, wherefrom it follows that u_1 also satisfies boundary conditions (3.1) for $t = 0$. Hence, u_1 is a classical solution to the problem (1.1)–(3.1), with $f = 0$.

On the other hand, Lemma 2.2 shows that u_2 is a classical solution to the problem (1.1)–(3.1), with $\varphi = 0$. Therefore, the function (11.2) will be a classical solution of the general problem (1.1)–(3.1), with the series representation having all the properties stated in Theorem 1.2. The uniqueness of the solution is proved in section 8.2.

In the next two subsections proofs of the lemmas will be given.

4.2. Proof of Lema 1.2

The series (13.2) converges absolutely and uniformly on $\overline{\Omega}$. This is implied by the relation

$$\sum_{n=1}^{\infty} \left| v_n(x)\,\varphi_n\, e^{-\lambda_n t} \right| \;\le\; \sum_{n=1}^{\infty} |\, v_n(x)\,|\,|\,\varphi_n\,|,$$

the majoring series being uniformly convergent on $\overline{\Omega}$ (which follows from Proposition 3.2 (a)). Hence, the first series converges to its sum u_1 uniformly on $\overline{\Omega}$ (which was formulated above as "the equality (13.2) holds uniformly on $\overline{\Omega}$"), and $u_1 \in C(\overline{\Omega})$.

The second part of Lemma 1.2 is based on the following fact: For every $\epsilon > 0$ there exists a constant $K > 0$ such that the estimate

$$e^{-\lambda_n t} \;\le\; \frac{K}{\lambda_n^{3/2}} \tag{17.2}$$

holds for all $\lambda_n > 1$, $t \ge \epsilon$ (see [3], p. 139). Now, by virtue of estimates (4.2), (5.2), (17.2) and the Bessel inequality, the following is valid on each $\overline{\Omega}_\epsilon$:

$$\begin{aligned}
\sum_{n=1}^{\infty} \lambda_n\,|\,v_n(x)\,\varphi_n\,|\,e^{-\lambda_n t} \;&=\; \sum_{0\le\sqrt{\lambda_n}\le 1} (\cdot) \;+\; \sum_{\sqrt{\lambda_n}>1} (\cdot) \\
&\le\; A\,C_0^2\,\|\,\varphi\,\|_{L_1(G)} \;+\; C_0\,K\cdot \sum_{\sqrt{\lambda_n}>1} \frac{|\,\varphi_n\,|}{\lambda_n^{1/2}} \\
\le\; D_3 \;+\; D_4\Bigg(\sum_{k=1}^{\infty} \Bigg(&\sum_{k<\sqrt{\lambda_n}\le k+1} \frac{1}{\lambda_n} \Bigg)\Bigg)^{1/2} \Bigg(\sum_{\sqrt{\lambda_n}>1} |\,\varphi_n\,|^2 \Bigg)^{1/2} \\
&\le\; D_3 \;+\; A^{1/2}\,D_4\,\|\,\varphi\,\|_{L_2(G)}\cdot \Bigg(\sum_{k=1}^{\infty} \frac{1}{k^2} \Bigg)^{1/2}.
\end{aligned} \tag{18.2}$$

This means that the first series (14.2) converges a.u. on $\overline{\Omega}_\epsilon$; especially, the series converges uniformly on the rectangle. It follows, by virtue of equality (13.2), that $(u_1)'_t$ exists on $\overline{\Omega}_\epsilon$ and the first equality (14.2) holds on $\overline{\Omega}_\epsilon$ (which was shortly formulated above as "the first equality (14.2) holds uniformly on $\overline{\Omega}_\epsilon$"). Consequently, it actually holds on Ω. Moreover, $(u_1)'_t \in C(\overline{G}\times(0,T])$.

By differentiating (formally) equality (13.2) with respect to x, we obtain the second series (14.2); it can be estimated from above (on $\overline{\Omega}_\epsilon$) in the following way:

$$\begin{aligned}
\sum_{n=1}^{\infty} |\,v_n'(x)\,\varphi_n\,|\,e^{-\lambda_n t} \;&=\; \sum_{0\le\sqrt{\lambda_n}\le 1} (\cdot) \;+\; \sum_{\sqrt{\lambda_n}>1} (\cdot) \\
\le\; A\,C_0\,C_1\,\|\,\varphi\,\|_{L_1(G)} \;&+\; C_0\,C_1\,K\,\|\varphi\|_{L_1(G)}\cdot \sum_{\sqrt{\lambda_n}>1} \frac{1}{\lambda_n}\,.
\end{aligned}$$

(The estimates (4.2)–(6.2) and (17.2) are used.) Hence, the series converges a.u. on $\overline{\Omega}_\epsilon$, wherefrom it results, by equality (13.2) , that $(u_1)'_x$ exists on $\overline{\Omega}_\epsilon$ and the second equality (14.2) holds on $\overline{\Omega}_\epsilon$ (which was formulated above as "the second equality (14.2) holds uniformly on $\overline{\Omega}_\epsilon$ "). Consequently, this equality is valid on Ω , and $(u_1)'_x \in C(\overline{G} \times (0, T])$.

Finally, for the series $\sum\limits_{n=1}^{\infty} v''_n(x)\, \varphi_n\, e^{-\lambda_n t}$ we obtain, according to estimates (4.2)–(6.2) and (17.2) , that

$$\begin{aligned}\sum_{n=1}^{\infty} |\, v''_n(x)\, \varphi_n\, | e^{-\lambda_n t} \;&=\; \sum_{0\le\sqrt{\lambda_n}\le 1} (\cdot) \;+\; \sum_{\sqrt{\lambda_n}>1} (\cdot) \\ \le\; A\, C_0\, C_2\, \|\, \varphi\, \|_{L_1(G)} \;&+\; C_2 \cdot \sum_{\sqrt{\lambda_n}>1} \lambda_n\, |\, \varphi_n\, |\, e^{-\lambda_n t} \\ &\le\; D_5 \;+\; C_2\, K \cdot \sum_{\sqrt{\lambda_n}>1} \frac{|\, \varphi_n\, |}{\lambda_n^{1/2}}\,,\end{aligned}$$

where $(x, t) \in \overline{\Omega}_\epsilon$. Since the convergence of the majoring series can be proved as in (18.2) , it follows that the third series (14.4) converges a.e. (and, as a consequence, uniformly) on $\overline{\Omega}_\epsilon$. Hence, we can conclude, by the second equality (14.2) , that the derivative $(u_1)''_{x^2}$ exists on the closed rectangle and the third equality (14.2) holds on $\overline{\Omega}_\epsilon$ (which was formulated as "the third equality (14.2) holds uniformly on $\overline{\Omega}_\epsilon$ "). Consequently, this equality is valid on Ω , and $(u_1)''_{x^2} \in C(\overline{G} \times (0, T])$.

Lemma 1.2 is proved.

5.2 . Proof of Lemma 2.2

The lemma was actually proved in [10] . However, in order to keep our paper self–contained, we will expose the major elements of the proof. The central point of the proving procedure is the following one : Proof of existence and continuity of $(u_2)''_{x^2}$ is not based on the direct differentiation of the second series (16.2) (because f is not smooth enough), but on

Proposition 4.2 . *If for each $t \in [\epsilon, T]$ the series*

$$\sum_{n=1}^{\infty} v''_n(x) \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau$$

converges in $L_2(\overline{G})$ to a function $w(\cdot, t) \in C(\overline{G})$, then u_2 has the partial derivative $(u_2)''_{x^2}$ on $\overline{\Omega}_\epsilon$, and $(u_2)''_{x^2}(x, t) = w(x, t)$.

Proof of the proposition relies on Theorem 8.15_2 from [1] . The theorem states : *Suppose that a series $\sum\limits_{n=1}^{\infty} u_n(x)$ ($u_n \in C^{(1)}(\overline{G})$) converges point–wise on $\overline{G}$ to a function $s = s(x)$, and that $\sum\limits_{n=1}^{\infty} u'_n(x)$ converges in $L_2(\overline{G})$ to a function $\sigma \in C(\overline{G})$. Then, s is differentiable on $\overline{G}$, and $s'(x) = \sigma(x)$ on the closed interval.* Now, having the second equality (16.2) proved (see below the second part of the proof of Lemma 2.2), we can apply the theorem to the second series (16.2), and obtain the proposition.

Moreover, we need an appropriate asymptotic formula for the functions

$$F_n(t) \;\stackrel{\text{def}}{=}\; \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau\,, \qquad t \in [\epsilon, T]\,, \tag{19.2}$$

where $\epsilon \in (0,T)$ is an arbitrary number. In paper [10] we established that the following is valid :

$$(\forall t \in [\epsilon, T]) \qquad F_n(t) = \frac{f_n(t)}{\lambda_n} - \frac{f_n(0)}{\lambda_n} e^{-\lambda_n t} + O\Big(\lambda_n^{-(1+\alpha)}\Big), \tag{20.2}$$

where $\Big|O\Big(\lambda_n^{-(1+\alpha)}\Big)\Big| \le \Big(4 + K(\alpha,\epsilon)\Big)\lambda_n^{-(1+\alpha)}$, and the constant $K(\alpha,\epsilon) > 0$ does not depend on t and n (see also [2]).

Let us now turn to the proof of Lemma 2.2 .

The series (15.2) converges a.u. on $\overline{\Omega}$. By estimates (4.2)–(5.2) , this follows from

$$\begin{aligned}\sum_{n=1}^{\infty} |F_n(t)\, v_n(x)| &= \sum_{0\le\sqrt{\lambda_n}\le 1} (\cdot) + \sum_{\sqrt{\lambda_n}>1} (\cdot) \le \\ &\le 2\,A\,C_0^2\,T\,\|f\|_{C(\overline{\Omega})} + 4\,C_0^2\,\|f\|_{C(\overline{\Omega})} \cdot \sum_{\sqrt{\lambda_n}>1} \frac{1}{\lambda_n}\,.\end{aligned}$$

Hence, the series converges to its sum u_2 uniformly on $\overline{\Omega}$, and $u_2 \in C(\overline{\Omega})$.

Consider the existence and continuity of the derivative $(u_2)'_x$. Let $\epsilon \in (0,T)$ be fixed. Differentiating (15.2) with respect to x, by virtue of (19.2) and (20.2) , we obtain the following formal equalities on $\overline{\Omega}_\epsilon$:

$$\begin{aligned}(u_2)'_x(x,t) &= \sum_{0\le\sqrt{\lambda_n}\le 1} F_n(t)\, v'(x) + \sum_{\sqrt{\lambda_n}>1} F_n(t)\, v'_n(x) \\ &= \sum_{0\le\sqrt{\lambda_n}\le 1} F_n(t)\, v'_n(x) + \sum_{\sqrt{\lambda_n}>1} \frac{f_n(t)}{\lambda_n}\, v'_n(x) - \\ &- \sum_{\sqrt{\lambda_n}>1} \frac{f_n(0)}{\lambda_n}\, v'_n(x)\, e^{-\lambda_n t} + \sum_{\sqrt{\lambda_n}>1} O\Big(\lambda_n^{-(1+\alpha)}\Big)\, v'_n(x)\,.\end{aligned} \tag{21.2}$$

Estimates (4.2)–(6.2) give : $\sum\limits_{0\le\sqrt{\lambda_n}\le 1} |F_n(t)\, v'_n(x)| \le 2\,A\,C_0\,C_1\;T\,\|f\|_{C(\overline{\Omega})}$. The other series converge a.u. on $\overline{\Omega}_\epsilon$. Indeed, according to (6.2) , for every $(x,t) \in \overline{\Omega}_\epsilon$ the estimates

$$\left|\frac{f_n(t)}{\lambda_n}\, v'_n(x)\right| \le \frac{C_1\,|f_n(t)|}{\lambda_n^{1/2}}\,, \qquad \Big|O\Big(\lambda_n^{-(1+\alpha)}\Big)\, v'_n(x)\Big| \le \frac{C_1\Big(4 + K(\alpha,\epsilon)\Big)}{\lambda_n^{1/2+\alpha}}$$

are true. Therefore,

$$\sum_{\sqrt{\lambda_n}>1} \left|\frac{f_n(t)}{\lambda_n}\, v'_n(x)\right| \le C_1 \Big(\sum_{n=1}^{\infty} |f_n(t)|^2\Big)^{1/2} \cdot \Big(\sum_{\sqrt{\lambda_n}>1} \frac{1}{\lambda_n}\Big)^{1/2},$$

$$\begin{aligned}\sum_{\sqrt{\lambda_n}>1} \Big|O\Big(\lambda_n^{-(1+\alpha)}\Big)\, v'_n(x)\Big| &\le \Big(4 + K(\alpha,\epsilon)\Big)\, C_1 \cdot \sum_{\sqrt{\lambda_n}>1} \frac{1}{\lambda_n^{1/2+\alpha}} \\ &\le D_6\, A \sum_{k=1}^{\infty} \frac{1}{k^{1+2\alpha}}\,.\end{aligned}$$

Having in mind that the function $\sum\limits_{n=1}^{\infty} |f_n(t)|^2$ is bounded on $[0,T]$, we can conclude that the first and the third series (on the right–hand side of (21.2)) converge a.u. on $\overline{\Omega}_\epsilon$. Then the convergence of the second series follows immediately from the convergence of the first one. Therefore, it is proved that $\sum_{n=1}^{\infty} F_n(t)\, v_n'(x)$ converges a.u. on $\overline{\Omega}_\epsilon$. This and equality (15.2) imply that $(u_2)_x'$ exists on $\overline{\Omega}_\epsilon$, and the second equality (16.2) holds on this closed rectangle. As a consequence, $(u_2)_x' \in C(\overline{G} \times (0,T])$.

Let us now establish the existence and continuity of the partial derivative $(u_2)_t'$ on $\overline{\Omega}_\epsilon$. By (19.2) , $F_n'(t) = f_n(t) - \lambda_n F_n(t)$. Using this and (20.2) , we write the formal equalities on $\overline{\Omega}_\epsilon$,

$$\begin{aligned}(u_2)_t'(x,t) \;&=\; \sum_{n=1}^{\infty} F_n'(t)\, v_n(x) \;=\; \sum_{0 \le \sqrt{\lambda_n} \le 1} (\, f_n(t) - \lambda_n F_n(t)\,)\; v_n(x) \;+ \\ &+ \sum_{\sqrt{\lambda_n} > 1} f_n(0)\, e^{-\lambda_n t}\, v_n(x) \;-\; \sum_{\sqrt{\lambda_n} > 1} O(\lambda_n^{-\alpha}\,)\, v_n(x)\,. \end{aligned} \tag{22.2}$$

The first (finite) sum can be bounded by $2\,(1+T)\,A\, C_0^2\, \|f\|_{C(\overline{\Omega})}$. The first series following this sum converges a.u. on $\overline{\Omega}_\epsilon$. This can be proved by using estimates (4.2) , (5.2) , and (17.2) . For the second series we have

$$\begin{aligned}\sum_{\sqrt{\lambda_n} > 1} \Big| O\Big(\lambda_n^{-\alpha}\Big)\, v_n(x) \Big| \;&\le\; \Big(4 + K(\alpha,\epsilon)\Big)\, C_0 \cdot \sum_{\sqrt{\lambda_n} > 1} \frac{1}{\lambda_n^{\alpha}} \\ &\le\; A\, \Big(4 + K(\alpha,\epsilon)\Big)\, C_0 \cdot \sum_{k=1}^{\infty} \frac{1}{k^{2\alpha}}\;, \end{aligned}$$

where the numerical series converges because of $\alpha \in (1/2, 1]$. Hence, we proved that the series $\sum_{n=1}^{\infty} F_n'(t)\, v_n(x)$ converges a.u. on $\overline{\Omega}_\epsilon$, wherefrom the existence of $(u_2)_t'$ and the first equality (16.2) on $\overline{\Omega}_\epsilon$ follow. Moreover, $(u_2)_t' \in C(\overline{G} \times (0,T])$.

It remains to consider the existence and continuity of the derivative $(u_2)_{x^2}''$. We will start from the series $\sum_{n=1}^{\infty} F_n(t)\, v_n''(x)$. By Proposition 2.2 , we can first write the equalities

$$\begin{aligned} F_n(t)\, v_n''(x) \;&=\; q(x)\, F_n(t)\, v_n(x) \;-\; \lambda_n\, F_n(t)\, v_n(x) \\ &=\; q(x)\, F_n(t)\, v_n(x) \;+\; F_n'(t)\, v_n(x) \;-\; f_n(t)\, v_n(x)\,, \end{aligned}$$

where $(x,t) \in \overline{\Omega}_\epsilon$, and then the formal equality

$$\begin{aligned}\sum_{n=1}^{\infty} F_n(t)\, v_n''(x) \;&=\; q(x) \cdot \sum_{n=1}^{\infty} F_n(t)\, v_n(x) \;+ \\ &+ \sum_{n=1}^{\infty} F_n'(t)\, v_n(x) \;-\; \sum_{n=1}^{\infty} f_n(t)\, v_n(x)\,. \end{aligned} \tag{23.2}$$

Now, for every $t \in [\epsilon, T]$ the three series on the right–hand side in (23.2) converge in $L_2(\overline{G})$ to functions $q \cdot u_2$, $(u_2)_t'$, and f respectively. Since the function

$$w(x,t) \;=\; q(x)\, u_2(x,t) + (u_2)_t'(x,t) - f(x,t)$$

is continuous on $\overline{\Omega}_\epsilon$, we see that the condition imposed in Proposition 4.2 is satisfied by the series (23.2) . That is why the derivative $(u_2)_{x^2}''$ exists on $\overline{\Omega}_\epsilon$, the series $\sum_{n=1}^{\infty} F_n(t)\, v_n''(x)$ converges point–wise on $\overline{\Omega}_\epsilon$ to this derivative, and the equality

$$\frac{\partial^2 u_2}{\partial x^2}(x,t) \;=\; q(x)\, u_2(x,t) \;+\; \frac{\partial u_2}{\partial t}(x,t) \;-\; f(x,t) \tag{24.2}$$

holds on the closed rectangle. The number $\epsilon \in (0,T)$ being arbitrary, it follows that $(u_2)''_{x^2} \in C(\overline{G} \times (0,T])$, and (24.2) shows that u_2 is a solution of the equation (1.1) on Ω (in the ordinary sense).

Proof of Lemma 2.2 is completed.

6.2. A priori estimate

Let u be the classical solution of (1.1)–(3.1) . Then,

$$u(x,t) = u_1(x,t) + u_2(x,t), \tag{25.2}$$

where functions u_i are defined by the series (12.2) . We are going to estimate these functions separately. Let us start with

$$|u_1(x,t)| \le \sum_{n=1}^{\infty} |v_n(x)\,\varphi_n\, e^{-\lambda_n t}| \le \sum_{n=1}^{\infty} |v_n(x)|\,|\varphi_n|. \tag{26.2}$$

Using estimates (4.2) , (8.2) (the second one applied to the function $h \overset{\text{def}}{=} \varphi$), and the inequalities of Hölder and Riesz, we obtain the following chain of equalities and inequalities :

$$\begin{aligned}
&\sum_{n=1}^{\infty} |v_n(x)|\,|\varphi_n| = \sum_{n=1}^{n_0} |v_n(x)|\,|\varphi_n| + \sum_{n=n_0+1}^{\infty} |v_n(x)|\,|\varphi_n| \le \\
&n_0\, C_0^2 \cdot \|\varphi\|_{L_1(G)} + C_0\, D_1 \cdot \sum_{n=n_0+1}^{\infty} \Big[\frac{1}{n} \Big(|a_n(\varphi_{0c} + \varphi_{0s})| + |b_n(\varphi_{0c} + \varphi_{0s})| \Big) \Big] \\
&+ C_0\, D_2 \cdot \sum_{n=n_0+1}^{\infty} \Big[\frac{1}{n^{\beta}} \Big(|a_n(\varphi_{1c} + \varphi_{1s})| + |b_n(\varphi_{1c} + \varphi_{1s})| \Big) \Big] + \sum_{n=n_0+1}^{\infty} \frac{C_0\, D_3}{n^2} \\
&\le E_1 + E_2 \Big(\sum_{n=1}^{\infty} \frac{1}{n^p} \Big)^{1/p} \Big(\sum_{k=0}^{\infty} \Big(|a_k(\cdot)|^r + |b_k(\cdot)|^r \Big) \Big)^{1/r} + \\
&+ E_3 \Big(\sum_{n=1}^{\infty} \frac{1}{n^{\beta p}} \Big)^{1/p} \Big(\sum_{k=0}^{\infty} \Big(|a_k(\cdot\cdot)|^r + |b_k(\cdot\cdot)|^r \Big) \Big)^{1/r} + E_4 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} \le \\
&\le E_1 + E_2 \Big(\sum_{n=1}^{\infty} \frac{1}{n^p} \Big)^{1/p} \cdot \|\varphi_{0c} + \varphi_{0s}\|_{L_p(G)} + \\
&+ E_3 \Big(\sum_{n=1}^{\infty} \frac{1}{n^{\beta p}} \Big)^{1/p} \cdot \|\varphi_{1c} + \varphi_{1s}\|_{L_p(G)} + E_4 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2},
\end{aligned}$$

Here, constants $E_1 - E_4$ have an obvious meaning, and $r > 0$ is a number such that $p^{-1} + r^{-1} = 1$.

Now, by analyzing constants D_i (see (9.2)) and E_j , we may conclude, by (26.2) , that the estimate

$$\max_{(x,t)\in\overline{\Omega}} |u_1(t,x)| \le D_7 \Big(\|\varphi\|_{C(\overline{G})} + \|\varphi'\|_{L_p(G)} \Big), \tag{27.2}$$

holds, where D_7 is a constant not depending on φ , f .

In the case of function u_2 , it holds :

$$\begin{aligned}
|u_2(x,t)| &\le \sum_{n=1}^{\infty} \Big| v_n(x) \cdot \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau \Big| = \sum_{0 \le \sqrt{\lambda_n} \le 1} (\cdot) + \\
&+ \sum_{\sqrt{\lambda_n} > 1} (\cdot) \le 2 A\, C_0^2\, T \|f\|_{C(\overline{\Omega})} + 4 C_0^2 \|f\|_{C(\overline{\Omega})} \cdot \sum_{\sqrt{\lambda_n} > 1} \frac{1}{\lambda_n},
\end{aligned}$$

where $(x,t) \in \overline{G} \times (0,T]$. Having in mind (5.2), we see that the estimate

$$\max_{(x,t)\in\overline{\Omega}} |u_2(x,t)| \;\le\; D_8 \, \|f\|_{C(\overline{\Omega})} \tag{28.2}$$

holds if we put $D_8 \stackrel{\text{def}}{=} 4\,A\,C_0^2\,\big(T + \pi^2/6\big)$.

Finally, from relations (25.2), (27.2)–(28.2) we obtain the estimate (2.2), with $C \stackrel{\text{def}}{=} \max\{D_7\,,\,D_8\}$.

Hence, we can conclude that the assertions of Theorem 1.2, except the one concerning the uniqueness of the solution, are proved.

7.2. On Corollary 1.2

We assume, of course, that functions q, f, φ_1, φ_2 satisfy conditions from Theorem 1.2. Then, the function $u(x,t) \stackrel{\text{def}}{=} w_1(x,t) - w_2(x,t)$ is a classical solution to the problem

$$\begin{gathered} \frac{\partial u}{\partial t}(x,t) \;-\; \frac{\partial^2 u}{\partial x^2}(x,t) \;+\; q(x)\,u(x,t) \;=\; 0\,, \quad (x,t)\in\Omega\,, \\ u(x,0) \;=\; \varphi_1(x) - \varphi_2(x)\,, \quad -1 \le x \le 1\,, \\ u(x,t) \quad \text{satisfies the boundary conditions} \quad (3.1)\,, \quad t \in [0,T]\,. \end{gathered}$$

That is why the corresponding estimate (2.2) for u is valid (see Remark 2.2).

8.2. Uniqueness: Proof of Theorem 2.2

As we have already mentioned, the uniqueness of the classical solution will be proved by a method used in paper [5]. Note that, in our general settings, the use of the method is essentially based on Proposition 3.2 (b).

Suppose that there exist two classical solutions to the problem (1.1)–(3.1) (in the sense of Definition 1.2); let us denote them by w_1, w_2. Then the function $u(x,t) \stackrel{\text{def}}{=} w_1(x,t) - w_2(x,t)$ is a classical solution of the problem

$$\begin{gathered} \frac{\partial u}{\partial t}(x,t) \;-\; \frac{\partial^2 u}{\partial x^2}(x,t) \;+\; q(x)\,u(x,t) \;=\; 0\,, \quad (x,t)\in\Omega\,, \\ u(x,0) \;=\; 0\,, \quad -1 \le x \le 1\,, \\ u(x,t) \quad \text{satisfies the boundary conditions} \quad (3.1)\,, \quad t \in [0,T]\,. \end{gathered} \tag{29.2}$$

Using Definition 1.2 and the above differential equation, we can see that for each $t \in (0,T)$ the function $g_t(x) \stackrel{\text{def}}{=} u(x,t)$ satisfies conditions of Proposition 3.2 (b). Hence, for every $(x,t) \in \overline{G} \times (0,T)$ the equality

$$u(x,t) \;=\; \sum_{n=1}^{\infty} c_n(t)\,v_n(x)\,, \qquad \text{where } c_n(t) \stackrel{\text{def}}{=} \int_{-1}^{1} u(x,t)\,v_n(x)\,dx\,,$$

is valid, the series being a.u. convergent on $\overline{G}$. From u, $u'_t \in C(\overline{G} \times (0,T))$ it follows that $c_n \in C^{(1)}(0,T)$, and

$$c'_n(t) \;=\; \int_{-1}^{1} \frac{\partial u}{\partial t}(x,t)\,v_n(x)\,dx\,.$$

The function c_n is a solution (on $(0,T)$) of the equation $c_n'(t) + \lambda_n c_n(t) = 0$, which follows from the equalities

$$\int_{-1}^{1} \frac{\partial u}{\partial t}(x,t)\, v_n(x)\, dx = \int_{-1}^{1} \Big[\frac{\partial^2 u}{\partial x^2}(x,t) - q(x)\, u(x,t) \Big] v_n(x)\, dx$$

$$= \int_{-1}^{1} u(x,t) \Big[v_n''(x) - q(x)\, v_n(x) \Big] dx = -\lambda_n \cdot \int_{-1}^{1} u(x,t)\, v_n(x)\, dx\,.$$

(The first equality is a consequence of the differential equation (29.2); the second one holds because the functions g_t, v_n belong to the domain of the operator L, and the third equality follows from the differential equation (3.2).) Therefore, we have

$$c_n(t) = B_n\, e^{-\lambda_n t}, \qquad t \in (0,T),$$

where B_n is an arbitrary real constant.

Now, using the initial condition (29.2) and the continuity of u on $\overline{\Omega}$, for every $n \in \mathbb{N}$ we obtain the equalities

$$B_n = \lim_{t \to 0} c_n(t) = \int_{-1}^{1} u(x,0)\, v_n(x)\, dx = 0\,,$$

and conclude that $c_n(t) = 0$ for all $t \in (0,T)$, $n \in \mathbb{N}$. This means that $u(x,t) = 0$ on the set $\overline{G} \times (0,T)$. But again, the function u being continuous on $\overline{\Omega}$, we actually see that $w_1 = w_2$ on $\overline{\Omega}$. Hence, Theorem 2.2 is proved.

Let us now return to Theorem 1.2. By Remark 2.2, the classical solution, "established" by the theorem, must be unique. Now, proof of Theorem 1.2 is completed. ■

9.2. On Remark 1.2

Suppose that the coefficients of the linear forms (3.1) satisfy

$$\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11} \neq 0\,. \tag{30.2}$$

Then, according to Remark 4 in [11], the basic estimate (8.2) remains valid if we require that $h \in W_p^{(1)}(G)$ only. Let us underline the fact that the self–adjoint boundary conditions (b.c.) with the real coefficients were considered in that paper, but all the results proved there hold also in the case of arbitrary self–adjoint b.c. with the complex coefficients.

Note that (30.2) is satisfied if $A_2 \neq 0$, $B_2 \neq 0$ in the case of the separated self–adjoint b.c., and if $k_{12} \neq 0$ in the case of the coupled (real or complex) self–adjoint b.c. (see [14], p. 71). ◇

3. Convergence rate estimates

1.3. Formulation of results

Let $u = u(t,x)$ be the classical solution. As it was shown, this solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} v_n(x) \Big[\varphi_n\, e^{-\lambda_n t} + \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau \Big].$$

For any $\mu > 2$ we can define the partial sum of the order μ :

$$\sigma(x,t;\mu) \overset{\text{def}}{=} \sum_{n=1}^{[\mu]} v_n(x)\Big[\varphi_n e^{-\lambda_n t} + \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau\Big].$$

($[\mu]$ is the entire part of μ .) This section is devoted to the asymptotic behavior of function σ and its derivatives, as $\mu \to +\infty$.

Theorem 1.3 . *Suppose conditions from Theorem* 1.2 *are satisfied. Then, for all* $\epsilon \in (0,T)$, $\delta \in (0, \alpha - 1/2)$ *the following relations hold* :

$$\max_{(x,t)\in\overline{\Omega}} \big| u(x,t) - \sigma(x,t;\mu) \big| = o\Big(\frac{1}{\mu^{1-1/p}}\Big), \qquad \mu \to +\infty\ ; \tag{1.3}$$

$$\max_{(x,t)\in\overline{\Omega}_\epsilon} \big| u(x,t) - \sigma(x,t;\mu) \big| = o\Big(\frac{1}{\mu^{3/2-\delta}}\Big)\ ; \tag{2.3}$$

$$\max_{(x,t)\in\overline{\Omega}_\epsilon} \big| u'_t(x,t) - \sigma'_t(x,t;\mu) \big| = o\Big(\frac{1}{\mu^{1/2}}\Big) + o\Big(\frac{1}{\mu^{2\alpha-1-\delta}}\Big)\ ; \tag{3.3}$$

$$\max_{(x,t)\in\overline{\Omega}_\epsilon} \big| u'_x(x,t) - \sigma'_x(x,t;\mu) \big| = o\Big(\frac{1}{\mu^{1/2-\delta}}\Big)\ . \tag{4.3}$$

Remark 1.3 . The above estimates hold on $\overline{\Omega}$ or $\overline{\Omega}_\epsilon$ uniformly with respect to the *both variables* x , t . If we relax this requirement, then it is possible to prove that, for any fixed $t \in (0,T]$ and $\delta \in (0, 2\alpha - 1)$, the corresponding estimates are valid :

$$\max_{x\in\overline{G}} \big| u(x,0) - \sigma(x,0;\mu) \big| = o\Big(\frac{1}{\mu^{1-1/p}}\Big)\ ;$$
$$\max_{x\in\overline{G}} \big| u(x,t) - \sigma(x,t;\mu) \big| = o\Big(\frac{1}{\mu^{3/2}}\Big)\ ;$$
$$\max_{x\in\overline{G}} \big| u'_t(x,t) - \sigma'_t(x,t;\mu) \big| = o\Big(\frac{1}{\mu^{1/2}}\Big) + o\Big(\frac{1}{\mu^{2\alpha-1-\delta}}\Big)\ ;$$
$$\max_{x\in\overline{G}} \big| u'_x(x,t) - \sigma'_x(x,t;\mu) \big| = o\Big(\frac{1}{\mu^{1/2}}\Big)\ .$$

All the ingredients, necessary for proving the estimates, can be found below, in the reasonings which will be used in the proof of Theorem 1.3 . $\diamond$

2.3 . Proof of estimates (1.3)–(2.3)

Suppose $\mu > 1$. For each $(x,t) \in \overline{\Omega}$ it holds

$$|\, u(x,t) - \sigma(x,t;\mu) \,| \le \sum_{n=[\mu]+1}^{\infty} \big| v_n(x)\, \varphi_n\, e^{-\lambda_n t} \big| + \sum_{n=[\mu]+1}^{\infty} \Big| v_n(x) \cdot \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau \Big|. \tag{5.3}$$

I. We are going to derive two estimates for the first sum. In order to get the estimate (1.3) , we use estimates (4.2) , (8.2) (the second one applied to $h \overset{\text{def}}{=} \varphi$) , and the Hölder inequality . So, for any $\mu > n_0$ and

$(x,t) \in \overline{\Omega}$, we obtain the relations

$$\sum_{n=[\mu]+1}^{\infty} |v_n(x)||\varphi_n| \le$$

$$E_5\Big(\sum_{n=[\mu]+1}^{\infty} \frac{1}{n^p}\Big)^{1/p}\Big(\sum_{n=[\mu]+1}^{\infty}\Big(|a_n(\varphi_{0c}+\varphi_{1c})|^r + |b_n(\varphi_{0c}+\varphi_{1s})|^r\Big)\Big)^{1/r} +$$

$$E_6\Big(\sum_{n=[\mu]+1}^{\infty} \frac{1}{n^{\beta p}}\Big)^{1/p}\Big(\sum_{n=[\mu]+1}^{\infty}\Big(|a_n(\varphi_{1c}+\varphi_{1s})|^r + |b_n(\varphi_{1c}+\varphi_{1s})|^r\Big)\Big)^{1/r} +$$

$$E_7\sum_{n=[\mu]+1}^{\infty}\frac{1}{n^2} \le E_5\Big(\int_{[\mu]+1}^{+\infty}\frac{dt}{t^p}\Big)^{1/p}\cdot \alpha_{1,r}(\mu) + E_6\Big(\int_{[\mu]+1}^{+\infty}\frac{dt}{t^{\beta p}}\Big)^{1/p}\cdot\alpha_{2,r}(\mu) +$$

$$E_7\int_{[\mu]+1}^{+\infty}\frac{dt}{t^2} \le \frac{1}{\mu^{1-1/p}}\cdot\Big(E_8\cdot\alpha_{1,r}(\mu) + \frac{E_9}{\mu^{\beta-1}}\cdot\alpha_{2,r}(\mu) + \frac{E_7}{\mu^{1/p}}\Big),$$

where $r = p/(p-1)$, the constants E_j do not depend on (x,t) and μ, while the functions

$$\alpha_{1,r}(\mu) \overset{\text{def}}{=} \Big(\sum_{n=[\mu]+1}^{\infty}\Big(|a_n(\varphi_{0c}+\varphi_{0s})|^r + |b_n(\varphi_{0c}+\varphi_{0s})|^r\Big)\Big)^{1/r},$$

$$\alpha_{2,r}(\mu) \overset{\text{def}}{=} \Big(\sum_{n=[\mu]+1}^{\infty}\Big(|a_n(\varphi_{1c}+\varphi_{1s})|^r + |b_n(\varphi_{1c}+\varphi_{1s})|^r\Big)\Big)^{1/r}$$

are well defined by the Riesz inequality. Also, $\lim_{\mu\to+\infty}\alpha_{1,r}(\mu) = 0 = \lim_{\mu\to+\infty}\alpha_{2,r}(\mu)$. Therefore, by virtue of the preceding relations, (uniformly on $\overline{\Omega}$) it holds :

$$\sum_{n=[\mu]+1}^{\infty}\Big|v_n(x)\,\varphi_n\,e^{-\lambda_n t}\Big| = o\Big(\frac{1}{\mu^{1-1/p}}\Big). \tag{6.3}$$

In order to get the estimate (2.3) , we will use estimates (4.2) , (7.2) , (17.2) , and the Cauchy–Schwartz inequality. Suppose $\epsilon \in (0,T)$ and $\mu > n_0$ are fixed. Then, for every $(x,t) \in \overline{\Omega}_\epsilon$ we have

$$\sum_{n=[\mu]+1}^{\infty}\Big|v_n(x)\,\varphi_n\Big|\,e^{-\lambda_n t} \le C_0\,K\cdot\sum_{n=[\mu]+1}^{\infty}\frac{|\varphi_n|}{\lambda_n^{3/2}} \le$$

$$\frac{C_0\,K}{\pi^3}\cdot\Big(\sum_{n=[\mu]+1}^{\infty}|\varphi_n|^2\Big)^{1/2}\Big(\sum_{n=[\mu]+1}^{\infty}\frac{1}{n^6}\Big)^{1/2} \le \frac{C_0\,K}{5^{1/2}\,\pi^3}\cdot\frac{\alpha_{3,2}(\mu)}{\mu^{5/2}},$$

where $\alpha_{3,2}(\mu) \overset{\text{def}}{=} \Big(\sum_{n=[\mu]+1}^{\infty}|\varphi_n|^2\Big)^{1/2}$. Hence, (uniformly on $\overline{\Omega}_\epsilon$) it holds :

$$\sum_{n=[\mu]+1}^{\infty}\Big|v_n(x)\,\varphi_n\Big|\,e^{-\lambda_n t} = o\Big(\frac{1}{\mu^{5/2}}\Big). \tag{7.3}$$

II. Let us now estimate the second sum (5.3) . This time we will use the first mean–value theorem for the definite integrals. Suppose $\mu > n_0$. Then, by virtue of estimates (4.2) and (7.2) , for each $(x,t) \in \overline{\Omega}$ and some $\tau^* \in (0,t)$, we obtain :

$$\sum_{n=[\mu]+1}^{\infty} \Big| v_n(x) \cdot \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau \Big| \le 2\, C_0 \cdot \sum_{n=[\mu]+1}^{\infty} \frac{|\, f_n(\tau^*)\,|}{\lambda_n} \le$$

$$\frac{2\, C_0}{\pi^2} \cdot \beta_1(\mu, \tau^*) \cdot \Big(\sum_{n=[\mu]+1}^{\infty} \frac{1}{n^4} \Big)^{1/2} \le \frac{2\, C_0}{3^{1/2}\, \pi^2} \cdot \beta_1(\mu, \tau^*) \cdot \frac{1}{\mu^{3/2}} ,$$

where $\beta_1(\mu, t) \stackrel{\text{def}}{=} \Big(\sum_{n=[\mu]+1}^{\infty} |\, f_n(t)\,|^2 \Big)^{1/2}$. So, we can conclude that, for any $\delta \in (0, 2\,\alpha - 1)$, the estimate

$$\sum_{n=[\mu]+1}^{\infty} \Big| v_n(x) \cdot \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau \Big| = o\Big(\frac{1}{\mu^{3/2-\delta}} \Big) \tag{8.3}$$

holds uniformly on $\overline{\Omega}$.

III. Finally, by (5.3) , (6.3) and (8.3) , we conclude that (1.3) is valid. On the other hand, relations (5.3) , (7.3) , and (8.3) show that (2.3) holds.

3.3 . Proof of estimate (3.3)

Let $\mu > n_0$ and $\epsilon \in (0,T)$ be fixed. By the first equality (14.2) and the first equality (16.2) , we can write

$$|\, u'_t(x,t) - \sigma'_t(x,t;\mu)\,| \le \sum_{n=[\mu]+1}^{\infty} \Big|\, \lambda_n\, v_n(x)\, \varphi_n\, e^{-\lambda_n t} \Big| + \sum_{n=[\mu]+1}^{\infty} \Big|\, v_n(x)\, \frac{d}{dt} \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau \Big|, \qquad (x,t) \in \overline{\Omega}_\epsilon . \tag{9.3}$$

I. In order to estimate the first series above, we will use the Cauchy–Schwartz inequality and estimates (4.2) , (7.2) , (17.2) . Hence, it holds :

$$\sum_{n=[\mu]+1}^{\infty} \lambda_n\, |\, v_n(x)\, \varphi_n\,|\, e^{-\lambda_n t} \le C_0\, K \cdot \sum_{n=[\mu]+1}^{\infty} \frac{|\, \varphi_n\,|}{\lambda_n^{1/2}} \le$$

$$\frac{C_0\, K}{\pi} \cdot \alpha_{3,2}(\mu) \Big(\sum_{n=[\mu]+1}^{\infty} \frac{1}{n^2} \Big)^{1/2} \le \frac{C_0\, K}{\pi} \cdot \alpha_{3,2}(\mu) \cdot \frac{1}{\mu^{1/2}} ,$$

where $\alpha_{3,2}(\mu) \stackrel{\text{def}}{=} \Big(\sum_{n=[\mu]+1}^{\infty} |\, \varphi_n\,|^2 \Big)^{1/2}$. Therefore, the following is valid on $\overline{\Omega}_\epsilon$:

$$\sum_{n=[\mu]+1}^{\infty} \lambda_n\, |\, v_n(x)\, \varphi_n\,|\, e^{-\lambda_n t} = o\Big(\frac{1}{\mu^{1/2}} \Big). \tag{10.3}$$

II. Consider now the second series (9.3) . Starting with equalities (22.2) , we can write the relations

$$
\sum_{n=[\mu]+1}^{\infty} \Big| v_n(x) \frac{d}{dt} \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau \Big| \le
$$

$$
\sum_{n=[\mu]+1}^{\infty} |f_n(0)|\,|v_n(x)|\, e^{-\lambda_n t} + \sum_{n=[\mu]+1}^{\infty} \big| O(\lambda_n^{-\alpha}) \big|\, |v_n(x)| \le
$$

$$
\frac{C_0\, K}{\pi^3} \cdot \beta_1(\mu, 0) \cdot \Big(\sum_{n=[\mu]+1}^{\infty} \frac{1}{n^6} \Big)^{1/2} + \frac{C_0 \big(4 + K(\alpha, \epsilon) \big)}{\pi^{2\alpha}} \cdot \sum_{n=[\mu]+1}^{\infty} \frac{1}{n^{2\alpha}} \le
$$

$$
\frac{C_0\, K}{5^{1/2}\, \pi^3} \cdot \beta_1(\mu, 0) \cdot \frac{1}{\mu^{5/2}} + \frac{C_0 \big(4 + K(\alpha, \epsilon) \big)}{\pi^{2\alpha}} \cdot \frac{1}{\mu^{2\alpha-1-\delta}} \cdot \sum_{n=[\mu]+1}^{\infty} \frac{1}{n^{1+\delta}} ,
$$

where $\delta \in (0, 2\alpha - 1)$ is an arbitrary number. Hence, the estimate

$$
\sum_{n=[\mu]+1}^{\infty} \Big| v_n(x) \frac{d}{dt} \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau \Big| = o\Big(\frac{1}{\mu^{2\alpha-1-\delta}} \Big), \tag{11.3}
$$

holds uniformly on $\overline{\Omega}_\epsilon$.

III. Finally, by virtue of (9.3)–(11.3) , the estimate (3.3) follows.

4.3. Proof of estimate (4.3)

Let $\mu > n_0$ and $\epsilon \in (0, T)$ be fixed. By the second equality (14.2) and the second equality (16.2) , we can write

$$
\begin{aligned}
|u'_x(x,t) - \sigma'_x(x,t;\mu)| \le \sum_{n=[\mu]+1}^{\infty} \big| v'_n(x)\, \varphi_n\, e^{-\lambda_n t} \big| + \\
\sum_{n=[\mu]+1}^{\infty} \Big| v'_n(x) \cdot \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau \Big|, \qquad (x,t) \in \overline{\Omega}_\epsilon .
\end{aligned} \tag{12.3}
$$

I. In order to estimate the first series above, we use (5.2) , (7.2) and (17.2) :

$$
\sum_{n=[\mu]+1}^{\infty} |v'_n(x)\, \varphi_n|\, e^{-\lambda_n t} \le C_1\, K \cdot \sum_{n=[\mu]+1}^{\infty} \frac{|\varphi_n|}{\lambda_n} \le
$$

$$
\frac{C_1\, K}{\pi^2} \cdot \alpha_{3,2}(\mu) \cdot \Big(\sum_{n=[\mu]+1}^{\infty} \frac{1}{n^4} \Big)^{1/2} \le \frac{C_1\, K}{3\,\alpha^2} \cdot \alpha_{3,2}(\mu) \cdot \frac{1}{\mu^{3/2}} ,
$$

wherefrom it follows that

$$
\sum_{n=[\mu]+1}^{\infty} |v'_n(x)\, \varphi_n|\, e^{-\lambda_n t} = o\Big(\frac{1}{\mu^{3/2}} \Big) . \tag{13.3}
$$

II. In the case of the second series (12.3), we can write the relations

$$\sum_{n=[\mu]+1}^{\infty} \Big| v_n'(x) \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau \Big| \le$$

$$C_1 \cdot \Big(\sum_{n=[\mu]+1}^{\infty} \frac{|f_n(t)|}{\sqrt{\lambda_n}} + \sum_{n=[\mu]+1}^{\infty} \frac{|f_n(0)|}{\sqrt{\lambda_n}} \cdot e^{-\lambda_n t} + \sum_{n=[\mu]+1}^{\infty} \frac{\big|4 + K(\alpha,\epsilon)\big|}{\lambda_n^{1/2+\alpha}} \Big) \le$$

$$\frac{C_1}{\pi} \big(\beta_1(\mu,t) + \beta_1(\mu,0)\big) \Big(\sum_{n=[\mu]+1}^{\infty} \frac{1}{n^2} \Big)^{1/2} + \sum_{n=[\mu]+1}^{\infty} \frac{\big|4 + K(\alpha,\epsilon)\big|}{\pi^{1+2\alpha} \cdot n^{1+2\alpha}} \le$$

$$\frac{C_1}{\pi} \big(\beta_1(\mu,t) + \beta_1(\mu,0)\big) \frac{1}{\mu^{1/2-\delta}} \cdot \frac{1}{\mu^{\delta}} + \frac{\big|4 + K(\alpha,\epsilon)\big|}{2\,\alpha\,\pi^{1+2\alpha}} \cdot \frac{1}{\mu^{2\alpha}} .$$

Here, equalities (20.2) and estimates (5.2), (7.2) are used. So, one can conclude that the estimate

$$\sum_{n=[\mu]+1}^{\infty} \Big| v_n'(x) \cdot \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau \Big| = o\Big(\frac{1}{\mu^{1/2-\delta}} \Big), \tag{14.3}$$

holds uniformly on $\overline{\Omega}_\epsilon$.

III. Relations (12.3)–(14.3) show that the estimate (4.3) is valid.

At the end of this proof, let us note the following. Using estimate (7.2) in the previous considerations, we have quietly supposed also that the number μ is such that

$$\gamma + \frac{\rho_n}{n^{\beta-1}} > 0 \qquad \text{for all} \quad n \ge [\mu] + 1,$$

which is possible because the sequence $\{\rho_n\}_{n=n_0}^{\infty}$ is bounded.

Proof of Theorem 1.3 is completed. ■

4. On the problem $[(1.1) - (3.1)]_\infty$

1.4. Existence and uniqueness

Note first that, in accordance with the agreement stated at the end of subsection 2.2, in this section we also work with the interval $G = (-1, 1)$. Hence, we consider the problem of existence of a real–valued function $u = u(x,t)$ defined on the closed rectangle $\overline{\Omega}_\infty = \overline{G} \times [0, +\infty)$, and satisfying: the partial differential equation (1.1) on $G \times (0, +\infty)$, the initial condition (2.1), and the boundary conditions (3.1) for every $t \in [0, +\infty)$.

The following notions will be used. We say that a function $g : \overline{\Omega}_\infty \to \mathbb{R}$ *satisfies the Hölder condition locally on* $[0, +\infty)$, with *an exponent* $\alpha \in (0, 1]$, *uniformly with respect to* $x \in \overline{G}$ if for every $T > 0$ there exists a constant $B_T > 0$ such that

$$(\forall x \in \overline{G})(\forall t,\, t' \in [0, T]) \qquad |g(x,t) - g(x,t')| \le B_T\, |t - t'|^{\alpha} .$$

In this case we write $g \in H_{loc,\alpha}(\overline{G}, [0, +\infty))$. Also, if $T > 0$ and $\epsilon \in (0, T)$, then $\overline{\Omega}_T \overset{\text{def}}{=} \overline{G} \times [0, T]$ and $\overline{\Omega}_{T,\epsilon} \overset{\text{def}}{=} \overline{G} \times [\epsilon, T]$; $\overline{\Omega}_{\infty,\epsilon} \overset{\text{def}}{=} \overline{G} \times [\epsilon, +\infty)$ if $\epsilon \in (0, +\infty)$.

This subsection is devoted to the following assertions.

Theorem 1.4 . *Let us assume*:

1) $q \in C(\overline{G})$.

2) $\varphi \in \overset{\circ}{W}{}_p^{(1)}(G)$ $(1 < p \leq 2)$ *, and* φ *satisfies the boundary conditions* (5.1) .

3) $f \in C(\overline{\Omega}_\infty) \cap H_{loc,\alpha}(\overline{G}, [0, +\infty))$ *, where* $\alpha \in (1/2, 1]$.

Then, the following is valid :

(*a*) *There exists a unique classical solution* $u = u(x, t)$ *of the problem* $[(1.1) - (3.1)]_\infty$. *It belongs to* $C^{(1)}(\overline{G} \times (0, +\infty))$ *, and* $u''_{x^2} \in C(\overline{G} \times (0, +\infty))$.

(*b*) *The solution can be represented as a series converging absolutely on* $\overline{\Omega}_\infty$ *, and uniformly on every* $\overline{\Omega}_T$ *. This series can be differentiated once with respect to* x *or* t *on every closed rectangle* $\overline{\Omega}_{\infty,\epsilon}$ *. The obtained series for the first derivatives of the solution converge absolutely on* $\overline{G} \times (0, T)$ *, and uniformly on any* $\overline{\Omega}_{T,\epsilon}$ *.*

Proof. Let us first note that the conditions from Theorem 1.2 are satisfied on any $\overline{\Omega}_T$, $T > 0$. That is why, by virtue of $\overline{\Omega}_\infty = \bigcup_{T>0} \overline{\Omega}_T$, the real function

$$u(x,t) \overset{\text{def}}{=} \sum_{n=1}^{\infty} v_n(x) \Big[\varphi_n \, e^{-\lambda_n t} + \int_0^t f_n(\tau) \, e^{-\lambda_n (t-\tau)} \, d\tau \Big] \tag{1.4}$$

is well defined on $\overline{\Omega}_\infty$. Also, the restriction $u|_{\overline{\Omega}_T}$ is the classical solution to the problem (1.1)–(1.3) on every $\overline{\Omega}_T$. Based on this fact, one can establish, using propositions (a) , (b) of Theorem 1.2 and the relevant parts of their proofs, that (1.4) is a classical solution of the problem $[(1.1) - (3.1)]_\infty$. The solution is unique, and it obeys all the additional properties stated in Theorem 1.4 . We omit the details. ∎

2.4 . An a priori estimate. Stability

This subsection concerns the self–adjoint operator L such that $\lambda_n > 0$, $n \in \mathbb{N}$. In order to formulate our results, we introduce the function $g(t) \overset{\text{def}}{=} \int_{-1}^{1} \big| f(x,t) \big| \, dx$, $t \in [0, +\infty)$. This function is continuous on $[0, +\infty)$ if $f \in C(\overline{\Omega}_\infty)$.

Theorem 2.4 . *Let the conditions from Theorem* 1.4 *be satisfied. Then, there exists a constant* $\tilde{C} > 0$ *such that for any* $t \in (0, +\infty)$ *the estimate*

$$\|u\|_{C(\overline{G} \times [0,t])} \leq \tilde{C} \left(\|\varphi\|_{C(\overline{G})} + \|\varphi'\|_{L_p(G)} + \max_{0 \leq \tau \leq t} g(\tau) \right) \tag{2.4}$$

holds. The constant does not depend on φ , f .

Proof. As before, we start from the representation $u = u_1 + u_2$, where the functions u_i are defined by (12.2) . Analyzing the proof of estimate (27.2) , we see that the estimate

$$\sup_{(x,t) \in \overline{\Omega}_\infty} | u_1(t,x) | \leq D_7 \left(\| \varphi \|_{C(\overline{G})} + \| \varphi' \|_{L_p(G)} \right) , \tag{3.4}$$

holds, where D_7 is a constant not depending on φ , f (see subsection 6.2).

In the case of function u_2, for any $(x,t) \in \overline{\Omega}_\infty$ the following holds:

$$| u_2(x,t) | \le \sum_{n=1}^{\infty} \left| v_n(x) \cdot \int_0^t f_n(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau \right| = \sum_{0<\sqrt{\lambda_n}\le 1} (\cdot) +$$

$$+ \sum_{\sqrt{\lambda_n}>1} (\cdot) \le \frac{2A\,C_0^2}{\lambda_1} \cdot \max_{0\le\tau\le t} g(\tau) + 2\,C_0^2 \cdot \max_{0\le\tau\le t} g(\tau) \cdot \sum_{\sqrt{\lambda_n}>1} \frac{1}{\lambda_n} .$$

Having in mind (5.2), we see that the estimate

$$\max_{(x,t)\in\overline{G}\times[0,t]} | u_2(x,t) | \le \tilde{D}_8 \cdot \max_{0\le\tau\le t} g(\tau) \tag{4.4}$$

is valid if we put $\tilde{D}_8 = 2A\,C_0^2 \left(1/\lambda_1 + \pi^2/6 \right)$.

By virtue of (3.4)–(4.4), the estimate (2.4) holds, with $\tilde{C} = \max\{ D_7 , \tilde{D}_8 \}$. ■

Corollary 1.4. *Assume, additionally, that the function* $g = g(t)$ *is bounded on* $[0,+\infty)$. *Then*:

(*a*) *The classical solution* u *is a bounded function on* $\overline{\Omega}_\infty$, *and the following estimate holds*:

$$\sup_{(x,t)\in\overline{\Omega}_\infty} | u(x,t) | \le \tilde{C} \left(\|\varphi\|_{C(\overline{G})} + \|\varphi'\|_{L_p(G)} + \sup_{0\le t<+\infty} g(t) \right). \tag{5.4}$$

(*b*) *Let* $w_i = w_i(x,t)$ $(1 \le i \le 2)$ *be the classical solution of the problem* $[(1.1) - (3.1)]_\infty$, *with the initial function* $\varphi_i = \varphi_i(x)$. *There exists a constant* $\tilde{C} > 0$, *not depending on* φ_i *and* f, *such that*

$$\sup_{(x,t)\in\overline{\Omega}_\infty} | w_1(x,t) - w_2(x,t) | \le \tilde{C} \left(\|\varphi_1 - \varphi_2\|_{C(\overline{G})} + \|\varphi_1' - \varphi_2'\|_{L_p(G)} \right). \tag{6.4}$$

Proof. The estimate (5.4) is an immediate consequence of the estimate (2.4). Then, the estimate (6.4) follows from (3.4), by virtue of the fact that $w_1 - w_2$ is a classical solution of the problem $[(1.1) - (3.1)]_\infty$, with $\varphi \overset{\text{def}}{=} \varphi_1 - \varphi_2$ and $f \overset{\text{def}}{=}$ zero–function. ■

3.4. Convergence rate estimates

Analyzing proofs of estimates (1.3)–(2.3), one can see that these estimates are valid in the case considered too. Namely, we have

Theorem 3.4. *Suppose conditions from Theorem* 1.4 *are satisfied, and that the function* f *is bounded on* $\overline{\Omega}_\infty$. *Then, for all* $\epsilon \in (0,+\infty)$, $\delta \in (0, \alpha - 1/2)$ *the following relations hold*:

$$\max_{(x,t)\in\overline{\Omega}_\infty} \left| u(x,t) - \sigma(x,t;\mu) \right| = o\left(\frac{1}{\mu^{1-1/p}} \right), \qquad \mu \to +\infty ;$$

$$\max_{(x,t)\in\overline{\Omega}_{\infty,\epsilon}} \left| u(x,t) - \sigma(x,t;\mu) \right| = o\left(\frac{1}{\mu^{3/2-\delta}} \right).$$

Regarding the estimates for the first derivatives, it is possible to prove that, for any fixed $t \in (0,+\infty)$ and $\delta \in (0, 2\alpha - 1)$, the four estimates from Remark 1.3 are valid. In this case, only conditions from Theorem 1.4 are needed.

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