Filomat 31:16 (2017), 5263–5269
https://doi.org/10.2298/FIL1716263D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Diophantine Equation $x^2 + 5^a \cdot p^b = y^n$

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Abstract. In this paper, all the solutions of the Diophantine equations $x^2 + 5^a \cdot p^b = y^n$ (for $p = 29, 41$) are given for nonnegative integers $a, b, x, y, n \geq 3$ with x and y coprime.

1. Introduction

Recently, there have been many papers dealing with by the generalized Lebesgue-Nagell equation

$$x^2 + C = y^n \tag{1}$$

where $C > 0$ is a fixed integer and x, y, n are positive integer unknowns with $n \geq 3$. In 1850, V. A. Lebesque [14] proved that this equation has no solution for $C = 1$. Ljunggren [16] solved for $C = 2$ and Nagell [20], [21] solved it for $C = 3, 4$ and 5. J. H. E. Cohn [10] could solve (1) for 77 values of C between 1 and 100. In [19], Mignotte and de Weger dealt with the cases $C = 74$ and 86, which had not been dealt with Cohn. Finally the remaining cases up to 100 were dealt with by Bugeaud, Mignotte and Siksek in [7].

Here we consider the Diophantine equation (1) where $C = q_1^{\alpha_1} \cdot q_2^{\alpha_2} \ldots q_k^{\alpha_k}$ or $C = 2^{\alpha_0} \cdot q_1^{\alpha_1} \cdot q_2^{\alpha_2} \ldots q_k^{\alpha_k}$ are fixed numbers satisfying the following three conditions:

(I) $q_i \equiv 1 \pmod 4$ are primes for all $i = 1, 2 \ldots, k$.

Write $C = d \cdot z^2$ with d is the square-free part of C. Let $h(-d)$ denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Let $rad(n)$ denote the radical of the positive integer n (product of all prime divisors of n).

(II) $rad(h(-d)) \mid 6$ for any decomposition $C = d \cdot z^2$ as above.

(III) $rad(q_i \pm 1) \mid 2 \cdot 3 \cdot 5$ for all $i = 1, \ldots, k$.

In such cases we apply the method used in [4]. If we are able to determine all S-integral points (with S is an explicit set of rational primes) on some associated elliptic curve, then we can completely solve such Diophantine equations. Conditions (I)-(III) above were suggested as a result of section 5 in [4].

In [11], all values of C satisfying conditions (I)-(III) are determined (Lemma 2). Radicals of C take exactly 41 values. Some of the equations $x^2 + C = y^n$ with C listed in Lemma 2 were studied in the literature. These include the cases where $rad(C) \in \{5, 13, 17, 29, 41, 97, 2\cdot5, 2\cdot13, 2\cdot17, 5\cdot13, 5\cdot17, 2\cdot5\cdot13, 2\cdot5\cdot17, 2\cdot29, 2\cdot41\}$.

All solutions of the Diophantine equation (1) where found in [17] and [18] for $rad(C) = 10, 26$; in [11] for $rad(C) = 34, 58, 82$; in [4] for $rad(C) = 65$; in [22] for $rad(C) = 85$; and in [12], [13] for $rad(C) = 130, 170$.

2010 *Mathematics Subject Classification*. 11D61, 11D41.
Keywords. Exponential Diophantine Equations, Primitive divisors of Lehmer sequences.
Received: 22 May 2014; Accepted: 12 April 2017
Communicated by Dragan S. Djordjević
Research supported by the Research Fund of Uludag University KUAP(F)-2015/18
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In [9], the authors gave the complete solutions (n, a, b, x, y) of the Diophantine equation $x^2 + 5^a \cdot 11^b = y^n$ when $gcd(x, y) = 1$, except for the case when $x \cdot a \cdot b$ is odd.

In this paper, we obtain all solutions of the Diophantine equations

$$x^2 + 5^a \cdot p^b = y^n \qquad (p = 29, 41) \tag{2}$$

in integers unknowns x, y, a, b, n under the conditions;

$$x \geq 1,\ y > 1,\ n \geq 3,\ a \geq 0,\ b \geq 0 \qquad x \text{ and } y \text{ are coprime.}$$

We apply the method from [4]. For $n = 3$ and $n = 4$, the problem is reduced to finding all $\{5, p\}$ -integral points on some elliptic curves. For $n \geq 5$, we shall use the primitive divisors of Lucas sequences as in [6] to deduce that only cases $n \in \{5, 7\}$ are possible. In these cases, we again reduce our problem to the computation of all $\{5, p\}$ -integral points on some elliptic curves. The calculations were done using MAGMA, [5]. We now state the two main results of this paper:

Theorem 1.1. *The only solutions of the equation*

$$x^2 + 5^a \cdot 29^b = y^n\ ,\ x, y \geq 1,\ gcd(x, y) = 1,\ n \geq 3,\ a, b \geq 0 \tag{3}$$

are

$$(x, y, a, b) = (2, 9, 2, 1) \qquad when \quad n = 3$$

and

$$(x, y, a, b) = (2, 3, 2, 1) \qquad when \quad n = 6.$$

Theorem 1.2. *The only solutions of the equation*

$$x^2 + 5^a \cdot 41^b = y^n\ ,\ x, y \geq 1,\ gcd(x, y) = 1,\ n \geq 3,\ a, b \geq 0 \tag{4}$$

are

$$(x, y, a, b) = (840, 29, 0, 2) \qquad when \qquad n = 4;$$
$$(x, y, a, b) = (38, 5, 0, 2) \qquad when \qquad n = 5$$

and

$$(x, y, a, b) = (278, 5, 0, 2) \qquad when \qquad n = 7.$$

Note that when $a = 0$, (3) becomes $x^2 + 29^b = y^n$ and $x^2 + 41^b = y^n$, respectively, all solutions of which are already known (see [11]), while when $b = 0$, our equation becomes $x^2 + 5^a = y^n$ and all solutions of which have been found in [2], [3] and [15]. Thus, from now on we shall assume that $a \cdot b > 0$ in (2).

2. Preliminaries

We will determine all the primes $p \equiv 1 \pmod 4$ satisfying the condition (III). First we recall some results:

Lemma 2.1. *([11]) There are exactly eight primes* $p \equiv 1 \pmod 4$ *satisfying the condition (III):* $5, 13, 17, 29, 41, 97, 449, 4801$.

Now we are ready to determine all values of C satisfying (I)-(III).

Lemma 2.2. *([11]) (i) The prime power p^a satisfies the conditions (I)-(III) iff $p \in \{5, 13, 17, 29, 41, 97\}$.*

(ii) The number $C = 2^{a_0} \cdot p^a$ satisfies (I)-(III) iff $p \in \{5, 13, 17, 29, 41\}$.

(iii) The odd number $C = p^a \cdot q^b$ with p, q are different odd primes, satisfies (I)-(III) iff $p \cdot q \in \{5 \cdot 13, 5 \cdot 17, 5 \cdot 29, 5 \cdot 41, 13 \cdot 17, 13 \cdot 29, 13 \cdot 41, 17 \cdot 29, 17 \cdot 41, 17 \cdot 97, 29 \cdot 41\}$.

(iv) The number $C = 2^{a_0} \cdot p^a \cdot q^b$ where p, q are different odd primes satisfies (I)-(III) iff $p \cdot q \in \{5 \cdot 13, 5 \cdot 17, 5 \cdot 41, 13 \cdot 17, 17 \cdot 41\}$.

(v) The odd number $C = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3}$ with p_1, p_2 and p_3 are different odd primes satisfies (I)-(III) iff $p_1 \cdot p_2 \cdot p_3 \in \{5 \cdot 13 \cdot 17, 5 \cdot 13 \cdot 29, 5 \cdot 13 \cdot 41, 5 \cdot 17 \cdot 29, 5 \cdot 17 \cdot 41, 5 \cdot 29 \cdot 41, 13 \cdot 17 \cdot 29, 13 \cdot 17 \cdot 41, 13 \cdot 29 \cdot 41\}$.

(vi) The number $C = 2^{a_0} \cdot p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3}$ where p_1, p_2 and p_3 are different odd primes satisfies (I)-(III) iff $p_1 \cdot p_2 \cdot p_3 \in \{5 \cdot 13 \cdot 29, 5 \cdot 17 \cdot 29, 13 \cdot 17 \cdot 29, 13 \cdot 29 \cdot 41\}$.

(vii) The number C with ≥ 4 different odd prime factors satisfies (I)-(III) iff $C = 5^a \cdot 13^b \cdot 17^c \cdot 41^d$.

Let α, β be two algebraic integers. If $\alpha + \beta$ and $\alpha \cdot \beta$ are nonzero coprime integers and α/β is not a root of unity, then (α, β) is called a Lucas pair. Further, let $k = \alpha + \beta$ and $l = \alpha \cdot \beta$. Then we have

$$\alpha = \tfrac{1}{2}(k + \lambda\sqrt{d})\,,\ \beta = \tfrac{1}{2}(k - \lambda\sqrt{d}) \text{ with } \lambda \in \{\mp 1\},$$

where $d = k^2 - 4l$. We call (k, l) the parameters of the Lucas pair (α, β). Two Lucas pairs (α_1, β_1) and (α_2, β_2) are called equivalent if $\alpha_1/\alpha_2 = \beta_1/\beta_2 = \mp 1$. Given a Lucas pair (α, β), one defines the corresponding sequence of Lucas numbers by

$$L_n(\alpha, \beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad n = 0, 1, 2, \ldots \tag{5}$$

For two equivalent Lucas pairs (α_1, β_1) and (α_2, β_2), we have $L_n(\alpha_1, \beta_1) = \pm L_n(\alpha_2, \beta_2)$ for all $n \geq 0$.
A prime r is called a primitive divisor of $L_n(\alpha, \beta)$, $(n > 1)$ if

$$r \mid L_n(\alpha, \beta) \text{ and } r \nmid d \cdot L_1(\alpha, \beta) \cdots L_{n-1}(\alpha, \beta).$$

Lemma 2.3. *([8]) If r is a primitive divisor of $L_n(\alpha, \beta)$, then*

$$r \equiv e \pmod{n}, \text{ where } e = \left(\tfrac{-4d}{r}\right).$$

Now we give an important result of Bilu, Hanrot and Voutier [6] concerning the existence of primitive divisors of Lucas sequence :

Lemma 2.4. *Let $L_n = L_n(\alpha, \beta)$ be a Lucas sequence. If $n \geq 5$ is a prime, then L_n has a primitive divisor except for finitely many pairs (α, β) which are explicitly determined in Table 1 in [6].*

Proof. Follows by Theorem 1.4 in [6] and Theorem 1 in [1]. □

3. The Case $n = 4$

We now consider the special case of $n = 4$. The situation is rather easy in this case:

Lemma 3.1. *The equation (2) has no solution with $n = 4$ and $a \cdot b > 0$.*

Proof. Let $p \in \{29, 41\}$. Let us rewrite the equation $x^2 + 5^a \cdot p^b = y^4$ in the form $(x/z^2)^2 + A = (y/z)^4$ where A is a 4th power-free positive integer, defined by. $5^a \cdot p^b = A \cdot z^4$ for some integer z. Under these conditions, we can write, $A = 5^\alpha \cdot p^\beta$ with $\alpha, \beta \in \{0, 1, 2, 3\}$ and we obtain the equation

$$V^2 = U^4 - 5^\alpha \cdot p^\beta$$

with $U = y/z$, $V = x/z^2$. We now have to determine all $\{5, p\}$-integral points on these 16 elliptic curves.

Recall that if S is a finite set of prime numbers, then an S-integer is a rational number a/b with coprime integers a and $b > 0$, where the prime factors of b are in S. We can always use MAGMA to determine the $\{5, p\}$-integral points on the above elliptic curves (see [4], p. 176).

Now we give the results of our with MAGMA calculations:

(i) The only $\{5, 29\}$ -integral point on $V^2 = U^4 - 5^{\alpha} \cdot 29^{\beta}$ is $(U, V, \alpha, \beta) = (1, 0, 0, 0)$ with the conditions on x, y and the definition of U, V one can see that there is no solution for this equation.

(ii) The only $\{5, 41\}$-integral point on $V^2 = U^4 - 5^{\alpha} \cdot 41^{\beta}$ is $(U, V, \alpha, \beta) = (1, 0, 0, 0), (29, 840, 0, 2)$. Under the conditions on x, y the definition of U, V which are not convenient for us since they $a = 0$ or $a = b = 0$. This concludes the proof. □

4. The Case $n = 3$

Now we deal with the second separate case of $n = 3$:

Lemma 4.1. *(i) The only solution of the equation* (3) *with* $n = 3$ *and* $ab > 0$ *is* $(x, y, a, b) = (2, 9, 2, 1)$*. In particular, if* $n \geq 3$ *is a multiple of* 3 *and the Diophantine equation* (2) *has an integer valuation* (x, y, a, b)*, then* $n = 6$*. Furthermore when* $n = 6$*, the only solution* (x, y, a, b) *is* $(2, 3, 2, 1)$*.*

(ii) The equation (4) *has no solution with* $n = 3$ *and* $ab > 0$*.*

Proof. Let $p \in \{29, 41\}$. Rewrite the equation $x^2 + 5^a \cdot p^b = y^3$ in the form $(x/z^3)^2 + A = (y/z^2)^3$, where A is a 6th power-free positive integer, defined by $5^a \cdot p^b = Az^6$, with some integer z. Of course, $A = 5^{\alpha} \cdot p^{\beta}$ with $\alpha, \beta \in \{0, 1, 2, 3, 4, 5\}$ and we obtain the equations:

$$V^2 = U^3 - 5^{\alpha} \cdot p^{\beta},$$

with $U = y/z^2$, $V = x/z^3$. We now have to determine the $\{5, p\}$-integral points on these 36 elliptic curves, and to do that, we use again MAGMA.

(i) The only $\{5, 29\}$-integral points on $V^2 = U^3 - 5^{\alpha} \cdot 29^{\beta}$ are $(U, V, \alpha, \beta) \in \{(1, 0, 0, 0), (29, 0, 0, 3), (5, 10, 2, 0), (9, 2, 2, 1), (29, 58, 2, 2), (125, 1390, 2, 2), (145, 1740, 2, 2), (865, 25440, 2, 2), (145, 0, 3, 3)\}$. As the numbers x and y are coprime positive integers, the above solutions lead to only one solution of the original equation, which is $(x, y, a, b) = (2, 9, 2, 1)$.

When $n = 6$, replace n by 3 and y by y^2 to get a solution of equation (3) with $n = 3$ where the value of y being a perfect square. We have only the possibility $(2, 9, 2, 1)$ for (x, y, a, b). Therefore, the only solution of equation (3) with $n = 6$ is $(2, 3, 2, 1)$.

(ii) The only $\{5, 41\}$ -integral points (u, v, α, β) on the curve $V^2 = U^3 - 5^{\alpha} \cdot 41^{\beta}$ are $(1, 0, 0, 0), (41, 0, 0, 3), (41, 246, 1, 2), (5, 10, 2, 0), (41, 164, 2, 2), (5, 0, 3, 0), (205, 0, 3, 3), (125, 950, 4, 2)$ and $(1025, 32800, 4, 2)$ with the conditions on x, y and the definition of U, V one can easily see that none of these leads to a solution of the equation in (1) in the case $n = 3$. This is the required result. □

5. The Case $n \geq 5$ is prime

Lemma 5.1. *Equations* (4) *and* (5) *have no solution with* $n \geq 5$ *prime and* $a.b > 0$*.*

Proof. Suppose that (1) holds with $n \geq 5$, prime. We first rewrite the Diophantine equation $x^2 + 5^a \cdot p^b = y^n$ as $x^2 + d \cdot z^2 = y^n$, where $d \in \{1, 5, p, 5p\}$, $p = 29, 41$, $z = 5^{\alpha} \cdot p^{\beta}$ and the relation between α and β with a and b, respectively, is clear.

If in (4) and (5), $y > 1$ is taken as an even number, we obviously have that x is odd. Since for any odd integer t, we have $t^2 \equiv 1 \pmod 8$ we get that $1 + d \equiv 0 \pmod 8$ by reducing (4) and (5) modulo 8. This leads to $d \equiv 7 \pmod 8$ for $d \in \{1, 5, 29, 145\}$ or $d \in \{1, 5, 41, 205\}$ which gives a contradiction. Hence in what follows we may assume $y > 1$ is odd in (4) and (5) (and hence $x \geq 1$ is even).

We work with the field $K = \mathbb{Q}(\sqrt{-d})$. Since x is even, both factors on the left hand side of the equation $(x + z\sqrt{-d})(x - z\sqrt{-d}) = y^n$ are relatively prime. Hence, the ideal $x + z\sqrt{-d}$ is a q-th power of some element in $\mathbb{Q}_K$, for a prime q. The cardinality of the group of units of $\mathbb{Q}_K$ is 2 or 6, both coprime to q. Furthermore $\{1, (1+\sqrt{-d})/2\}$ is always an integral base for $\mathbb{Q}_K$. Thus, we can finally write the relations

$$x + z\sqrt{-d} = \varphi^q, \qquad \varphi = u + v\sqrt{-d} \tag{6}$$

where $u, v \in \mathbb{Z}$.

Conjugating (7) and subtracting the two relations, we get

$$2\sqrt{-d}\cdot 5^{\alpha}\cdot p^{\beta} = \varphi^q - \overline{\varphi}^q. \tag{7}$$

5.1. The Diophantine equation $x^2 + 5^a \cdot 29^b = y^n$

Since $n \geq 5$, 29 is primitive for L_n by Lemma 3 (n is prime). Thus, $29 \equiv \pm 1 \pmod{n}$ and we conclude that the only possibilities are $n = 7$ and $d = 1$ or $n = 5$ and $d = 2$.

5.1.1. The Case $n = 7$

By means of (8) with $n = 7$ and $d = 1$, we obtain the relation

$$v(7u^6 - 35u^4v^2 + 21u^2v^4 - v^6) = 5^{\alpha}.29^{\beta} \tag{8}$$

Since u and v are coprime, we have the following possibilities:

$$(a)\quad v = \pm 5^{\alpha}\cdot 29^{\beta}, \quad (b)\quad v = \pm 29^{\beta}, \quad (c)\quad v = \pm 5^{\alpha}, \quad (d)\quad v = \pm 1.$$

We need only look at the last two possibilities.

Case 1:. $v = \pm 5^{\alpha}$.

In this case, equation (9) becomes

$$7u^6 - 35u^4v^2 + 21u^2v^4 - v^6 = \pm 29^{\beta}.$$

Dividing both sides by v^6, we obtain

$$7U^3 - 35U^2 + 21U - 1 = D_1 \cdot V^2 \tag{9}$$

where $U = u^2/v^2$, $V = 29^{\beta_1}/v^3$, $\beta_1 = [\beta/2]$, $D_1 = \pm 1, \pm 29$. In this case, as $D_1 = \pm 1$, we have to find the $\{5\}$-integral points on the elliptic curves:

$$7U^3 - 35\gamma U^2 + 21U - \gamma = D_1 \cdot V^2, \quad \gamma = \pm 1. \tag{10}$$

We multiply both sides of (10) by 7^2 to obtain

$$X^3 - 35\gamma \cdot X^2 + 147X - 49\gamma = Y^2,$$

where $(X, Y) = (7\gamma U, 7V)$ are $\{5\}$-integral points on the above elliptic curves.

Using MAGMA, we find $(X, Y) \in \{(1, 8), (58, -293)\}$ (hence $(U, V) \in \{(1/7, 8/7), (58/7, -293/7)\}$ for $\gamma = 1$). These do not lead to any solutions of the equation (4), either.

Consider the case $D_1 = \pm 29$. The unique $\{5\}$-integral point $(2349, -87464)$ on the elliptic curve

$$X^3 - 35\cdot 29X^2 + 21\cdot 7\cdot 29^2X - 7^2\cdot 29^3 = Y^2$$

does not lead us to a solution of (4). With MAGMA, we find the following $\{5\}$-integral points $(-812, 5887), (-377, 6728), (-5, -776), (91, 4648), (1015, 47096), (-340103561/390625, 420852069512/244140625)$ on the elliptic curve

$$X^3 + 35\cdot 29x^2 + 21\cdot 7\cdot 29^2X + 7^2\cdot 29^3 = Y^2.$$

Only the point $(-812, 5887)$ leads to the solution $(x, y, a, b) = (278, 5, 0, 2)$ of our original equation (4), which is not convenient for us since it has $a = 0$.

Case 2:. $v = \pm 1$.

We have to find the integral points on

$$7U^3 - 35U^2 + 21U - 1 = D_1 \cdot V^2 \tag{11}$$

where $D_1 = \pm 1,\ \pm 5,\ \pm 29,\ \pm 145$.

The cases $D_1 = \pm 1,\ \pm 29$ where treated above.

Consider the case $D_1 = \pm 5$. Using MAGMA, we find two solutions $(21, -56)$, $(574, -11557)$ on the curve

$$X^3 - 35 \cdot 5 \cdot X^2 + 21 \cdot 7 \cdot 5^2 X - 7^2 \cdot 5^3 = Y^2$$

and there exists no integral points on the curve

$$X^3 - 35 \cdot 5 \cdot X^2 + 21 \cdot 7 \cdot 5^2 X + 7^2 \cdot 5^3 = Y^2.$$

These do not lead to any solutions of our original equation (4).

Consider the case $D = \pm 145$. Using MAGMA, we only find the solution $(25201, -3586024)$ on the curve

$$X^3 - 35 \cdot 5 \cdot 29X^2 + 21 \cdot 7 \cdot 5^2 \cdot 29^2 X - 7^2 \cdot 5^3 \cdot 29^3 = Y^2,$$

and we find another solution $(696, 10933)$ on the curve

$$X^3 - 35 \cdot 5 \cdot 29X^2 + 21 \cdot 7 \cdot 5^2 \cdot 29^2 X + 7^2 \cdot 5^3 \cdot 29^3 = Y^2.$$

These also do not lead to any solutions of (4).

5.1.2. *Case* $n = 5$

Using (8) with $n = 5,\ d = 2$, we obtain the relation

$$v(5u^4 - 20u^2v^2 + 4v^4) = 5^{\alpha} \cdot 29^{\beta}. \tag{12}$$

As in the case $n = 7$, we only need to check the values $v = \pm 5^{\alpha},\ v = \pm 1$.

In the first case, the Diophantine equation (12) is $5u^4 - 20u^2v^2 + 4v^4 = \pm 29^{\beta}$. Dividing both sides by v^4, we obtain

$$5U^4 - 20U^2 + 4 = D_1 V^2, \tag{13}$$

where $U = u/v,\quad V = 29^{\beta_1}/v^2,\quad \beta_1 = [\beta/2]$ and $D_1 = \pm 1,\ \pm 29$. Using MAGMA, we find three $\{5\}$-integral points $(0, 2)$, $(2, 2)$, $(-2, 2)$ on the curve (13) with $D_1 = \pm 1$, and no other points in the remaining cases. These points do not lead to solution of our original equation (1).

In the second case, the Diophantine equation (12) is $5U^4 - 20U^2 + 4 = \pm 5^{\alpha} \cdot 29^{\beta}$. we need to find the integral points on the curves $5U^4 - 20U^2 + 4 = D_1 V^2$, for $D_1 = \pm 1,\ \pm 5,\ \pm 29,\ \pm 145$. MAGMA finds three solutions $(0, 2)$, $(2, 2)$, $(-2, 2)$. None of points leads to any solutions of equation (2).

5.2. *The Diophantine equation* $x^2 + 5^a.41^b = y^n$

Since $n \geq 5$, by using Lemma 3, 41 is primitive for L_n. Thus, $41 \equiv \pm 1 \pmod{n}$ and we now see that the only possibilities are $n = 5$ and $d = 1$ or $n = 5$ and $d = 2$.

Using (8) with $n = 5$, $d = 2$, we obtain

$$v(5u^4 - 20u^2v^2 + 4v^4) = 5^{\alpha} 41^{\beta}. \tag{14}$$

Therefore we only need to check $v = \pm 5^{\alpha},\ v = \pm 1$.

In the first case the Diophantine equation is $v(5u^4 - 20u^2v^2 + 4v^4) = \pm 41^{\beta}$. Dividing both sides by v^4, we obtain

$$5U^4 - 20U^2 + 4 = D_1 V^2,$$

where $U = u/v,\ V = 41^{\beta_1}/v^2,\quad \beta_1 = [\beta/2]$ and $D_1 = \pm 1,\ \pm 41$. Using MAGMA, we find three {5}-integral points $(0,2),(2,2),(2,-2)$ on (14) with $D_1 = 1$, and none in the reamining cases. These points do not lead to any solutions of equation (4).

In the second case the Diophantine equation is $v(5u^4 - 20u^2v^2 + 4v^4) = 5^{\alpha} \cdot 41^{\beta}$. We need to find integral points on the curves $v(5U^4 - 20U^2 + 4) = D_1V^2$, for $D_1 = \pm 1,\ \pm 5,\ \pm 41,\ \pm 205$. MAGMA finds three solutions $(0,2),(2,2),(2,-2)$. These points do not lead either to any solutions of our original equation (4).

Using (8) with $n = 5,\ d = 1$, we obtain the relation

$$v(5u^4 - 20u^2v^2 + 4v^4) = 5^{\alpha}41^{\beta}.$$

In case $v = \pm 5^{\alpha}$, we obtain $5u^4 - 10u^2v^2 + v^4 = \pm 41^{\beta}$. MAGMA then finds the {5}-integral points on

$$5U^4 - 10U^2 + 1 = D_1V^2 \text{ for } D_1 = \pm 1,\ \pm 41,$$

which are $(0,1)$ if $D_1 = 1, (1,-2), (-1,-2)$ if $D_1 = -1$, and finally $(2,1),(-2,1)$ if $D_1 = 41$. The point $(2,1)$ gives a new solution $(x,y,a,b) = (38,5,0,2)$ of the equation (4) which is not conventient for us since it has $a = 0$.

In case $v = \pm 1$, we obtain $5u^4 - 10u^2v^2 + 4v^4 = 5^{\alpha}41^{\beta}$ MAGMA finds the integral points on

$$5U^4 - 10U^2 + 1 = D_1Y^2 \text{ for } D_1 = \pm 1,\ \pm 5,\ \pm 41,\ 205.$$

These points are $(2,41),(-2,41)$ for $D_1 = 41$. The point $(2,1)$ gives the solution $(x,y) = (38,5)$ of (4) again. This solution is not convenient for us since it has $a = 0$. This completes the proof of lemma. □

Acknowledgements 5.2. *The author specially thanks to Dr. Gokhan Soydan for his orientation and encouragements.*

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