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On $\mathcal{I}$-Lacunary Statistical Convergence of Weight g of Sequences of Sets

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Abstract. In this paper, following a very recent and new approach of [1], we further generalize recently introduced summability methods in [13] and introduce new notions, namely, $\mathcal{I}$-statistical convergence of weight g and $\mathcal{I}$-lacunary statistical convergence of weight g, where $g : \mathbb{N} \to [0, \infty)$ is a function satisfying $\lim_{n\to\infty} g(n) = \infty$ and $\frac{n}{g(n)} \nrightarrow 0$ as $n \to \infty$, for sequences of sets. We mainly investigate their relationship and also make some observations about these classes. The study leaves a lot of interesting open problems.

1. Introduction

In this section we recall some of the basic concepts related to statistical convergence and lacunary statistical convergence.

The idea of statistical convergence was given by Zygmund [34]in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [31] and Fast [9] and later reintroduced by Schoenberg [30] independently as follows:

If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then $K(m, n)$ denotes the cardinality of the set $K \cap [m, n]$. The upper and lower natural density of the subset K is defined by

$$\overline{d}(K) = \lim_{n\to\infty} \sup \frac{K(1, n)}{n} \text{ and } \underline{d}(K) = \lim_{n\to\infty} \inf \frac{K(1, n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$ then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly $d(K) = \lim_{n\to\infty} \frac{K(1, n)}{n}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to L if for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\}$ has natural density zero.

Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Fridy [10], Kolk [12], Šalát [19], Mursaleen [17], Savaş ([21, 22, 24, 25]). We refer to

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[3–6], where more references can be found. Nuray and Rhoades [18] extended the notion of convergence of set sequences to statistical convergence, and gave some basic theorems. Ulusu and Nuray [32] defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wijsman statistical convergence.

The idea of statistical convergence was further extended to $\mathcal{I}$-convergence in [14] using the notion of ideals of $\mathbb{N}$ with many interesting consequences. More investigations in this direction and more applications of ideals can be found in [7, 8, 15, 27, 28] where many important references can be found.

In another direction, a new type of convergence called lacunary statistical convergence was introduced in [11] as follows: A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. Let $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be lacunary statistically convergent to L (or, S_θ-convergent to L) if for any $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0,$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [11] the relation between lacunary statistical convergence and statistical convergence was established among other things. More results on this convergence can be seen from [16, 20, 23, 25, 26].

Recently in [8, 27] we used ideals to introduce the concepts of $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence which naturally extend the notions of the above mentioned convergence. Recently, Kişi and Savaş [13] defined $\mathcal{I}$- lacunary statistical convergence of sequence of sets.

On the other hand, in [2] a different direction was given to the study of statistical convergence where the notion of statistical convergence of order α, $0 < \alpha < 1$ was introduced by using the notion of natural density of order α (where n is replaced by n^α in the denominator in the definition of natural density).

Very recently, in [1] it has been shown that one can further extend the concept of natural or asymptotic density (as well as natural density of order α) by considering natural density of weight g where $g : \mathbb{N} \to [0, \infty)$ is a function with $\lim_{n\to\infty} g(n) = \infty$ and $\frac{n}{g(n)} \nrightarrow 0$ as $n \to \infty$.

In this paper we combine the approaches of [13] and [1] and introduce new and further general summability methods, namely, $\mathcal{I}$-statistical convergence of weight g and $\mathcal{I}$-lacunary statistical convergence of weight g for sequence of sets.

In this context it should be mentioned that the concept of lacunary statistical convergence of weight g (which happens to be a special case of $\mathcal{I}$-lacunary statistical convergence of weight g) for sequence of sets has also not studied till now. We mainly investigate their relationship and also make some observations about these classes and most importantly the study leaves a lot of interesting open problems.

2. Main Results

In this section we introduce the concept of ideal and some definitions which will be needed in the sequel.

Definition 2.1. A family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an *ideal of* $\mathbb{N}$ if the following conditions hold:
(a) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(b) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$,

Definition 2.2. A non-empty family $\mathcal{F} \subset 2^{\mathbb{N}}$ is said to be an *filter of* $\mathbb{N}$ if the following conditions hold:
(a) $\emptyset \notin F$,
(b) $A, B \in F$ implies $A \cap B \in \mathcal{F}$,
(c) $A \in F$, $A \subset B$ implies $B \in \mathcal{F}$.
If $\mathcal{I}$ is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin I$), then the family of sets $F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in I : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

Definition 2.3. A proper ideal $\mathcal{I}$ is said to be *admissible* if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

Definition 2.4. ([14]) Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$.

(i) The sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of $\mathbb{R}$ is said to be *$\mathcal{I}$-convergent* to $L \in \mathbb{R}$ if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\} \in \mathcal{I}$.

(ii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of $\mathbb{R}$ is said to be *$\mathcal{I}^*$-convergent* to $L \in \mathbb{R}$ if there exists $M \in F(\mathcal{I})$ such that $\{x_n\}_{n\in M}$ converges to L.

We now present the basis of our main discussions. Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. The upper density of weight g was defined in [1] by the formula

$$\overline{d}_g(A) = \limsup_{n\to\infty} \frac{A(1,n)}{g(n)}$$

for $A \subset \mathbb{N}$ where as before $A(1,n)$ denotes the cardinality of the set $A \cap [1,n]$. Then the family

$$\mathcal{I}_g = \{A \subset \mathbb{N} : \overline{d}_g(A) = 0\}$$

forms an ideal. It has been observed in [1] that $\mathbb{N} \in \mathcal{I}_g$ if and only if $\frac{n}{g(n)} \to 0$ as $n \to \infty$ So we additionally assume that $n/g(n) \nrightarrow 0$ as $n \to \infty$ so that $\mathbb{N} \notin \mathcal{I}_g$ and $\mathcal{I}_g$ is a proper admissible ideal of $\mathbb{N}$. The collection of all such weight functions g satisfying the above properties will be denoted by G. As a natural consequence we can introduce the following definition.

Definition 2.5. A sequence (x_n) of real numbers is said to *d_g–statistically convergent* to x if for any given $\varepsilon > 0$, $\overline{d}_g(A(\varepsilon)) = 0$, where $A(\varepsilon)$ is the set defined in Definition 2.4.

Let (X, ρ) be a metric space. For any point $x \in X$ and any nonempty subset A of X, we define the *distance* from x to A by

$$d(x, A) = \inf_{a\in A} \rho(x, A).$$

Definition 2.6. (Baronti & Papini, [2]) Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is *Wijsman convergent* to A if

$$\lim_{k\to\infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

Recently Nuray and Rhoades [18] gave the following two definitions.

Definition 2.7. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is *Wijsman statistical convergent* to A if $\{d(x, A_k)\}$ is statistically convergent to $d(x, A)$, i.e. for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$.

Definition 2.8. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that $\{A_k\}$ is *Wijsman strongly Cesàro summable* to A if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)| = 0.$$

In this case we write $A_k \to A$ $([W\sigma_1])$ or $A_k \overset{[W\sigma_1]}{\to} A$.

Definition 2.9. Let (X, ρ) be a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is *Wijsman $\mathcal{I}-$statistical convergent* to A or $S(I_W)$-convergent to A if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\,(S(\mathcal{I}_W))$.

Definition 2.10. Let (X, ρ) be a metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is *Wijsman $\mathcal{I}-$lacunary statistical convergent* to A to A or $S_\theta(\mathcal{I}_W)$-convergent to A if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\,(S_\theta(\mathcal{I}_W))$.

We now introduce our main definitions.

Definition 2.11. Let (X, ρ) be a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is *Wijsman $\mathcal{I}-$statistical convergent of weight g* to A or $S(\mathcal{I}_W)^g$-convergent to A if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{g(n)}|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A(S(\mathcal{I}_W)^g)$.

The class of all $\mathcal{I}_W$-statistically convergent of weight g sequences will be denoted by simply $S(\mathcal{I}_W)^g$.

Remark 2.12. For $\mathcal{I} = \mathcal{I}_{fin} = \{A \subseteq \mathbf{N} : A \text{ is a finite subset }\}$, $S(I_W)^g$-convergence coincides with Wijsman statistical convergence of weight g which has not studied till now. Further taking $g(n) = n^\alpha$, it reduces to Wijsman asymptotically $\mathcal{I}$- lacunary statistical convergence of weight g, (see, [29]).

Definition 2.13. Let (X, ρ) be a metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is *Wijsman $\mathcal{I}$-lacunary statistically convergent of weight g* to A or $S_\theta(\mathcal{I}_W)^g$-convergent to A if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{r \in \mathbb{N} : \frac{1}{g(h_r)}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\,(S_\theta(\mathcal{I}_W)^g)$.

The class of all Wijsman $\mathcal{I}$-lacunary statistically convergent sequences of weight g will be denoted by $S_\theta(\mathcal{I}_W)^g$.

Remark 2.14. it should be noted that Wijsman lacunary statistical convergence of weight g has not been studied till now. Obviously Wijsman lacunary statistical convergence of weight g is a special case of Wijsman $\mathcal{I}$-lacunary statistical convergence of weight g when we take $\mathcal{I} = \mathcal{I}_{fin}$. So properties of Wijsman lacunary statistical convergence of weight g can be easily obtained from our results with obvious modifications.

Theorem 2.15. *Let $g_1, g_2 \in G$ be such that there exist $M > 0$ and $j_0 \in \mathbb{N}$ such that $\frac{g_1(n)}{g_2(n)} \leq M$ for all $n \geq j_0$. Then $S(\mathcal{I}_W)^{g_1} \subset S(\mathcal{I}_W)^{g_2}$.*

Proof. For any $\varepsilon > 0$,

$$\begin{aligned}\frac{|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|}{g_2(n)} &= \frac{g_1(n)}{g_2(n)} \cdot \frac{|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|}{g_1(n)} \\ &\leq M \cdot \frac{|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|}{g_1(n)}.\end{aligned}$$

for $n \geq j_0$. Hence for any $\delta > 0$,

$$\begin{aligned}&\left\{n \in \mathbb{N} : \frac{|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|}{g_2(n)} \geq \delta\right\} \\ &\subset \left\{n \in \mathbb{N} : \frac{|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|}{g_1(n)} \geq \frac{\delta}{M}\right\} \cup \{1, 2, ..., j_0\}.\end{aligned}$$

If $(A_k) \in \mathrm{S}(\mathcal{I})^{g_1}$ then the set on the right hand side belongs to the ideal $\mathcal{I}$ and so the set on the left hand side also belongs to $\mathcal{I}$. This shows that $S(\mathcal{I})^{g_1} \subset S(\mathcal{I})^{g_2}$. □

Definition 2.16. Let (X, ρ) be a metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be a non-trivial ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is said to be *Wijsman strongly $\mathcal{I}$− lacunary convergent* to A or $N_\theta(\mathcal{I}_W)$-convergent to A if for each $\varepsilon > 0$ and for each $x \in X$,

$$\left\{r \in \mathbb{N} : \frac{1}{g(h_r)} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq \varepsilon\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\,(N_\theta(\mathcal{I}_W)^g)$ and the class of such sequences will be denoted by simply $N_\theta(\mathcal{I}_W)^g$.

Theorem 2.17. *Let (X, ρ) be a metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and A, A_k be non-empty closed subsets of X. Then $A_k \to A(N_\theta(\mathcal{I}_W)^g)$ implies $A_k \to A(S_\theta(\mathcal{I}_W)^g)$.*

Proof. If $\epsilon > 0$ and $A_k \to L(N_\theta(I_W)^g)$, we can write, for each $x \in X$

$$\sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq \sum_{k \in I_r, |d(x, A_k) - d(x, A)| \geq \epsilon} |d(x, A_k) - d(x, A)| \geq \epsilon |\{k \in I_r : \ |d(x, A_k) - d(x, A)| \geq \epsilon\}|$$

and so $$\frac{1}{\epsilon . g(h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq \frac{1}{g(h_r} |\{k \in I_r : \ |d(x, A_k) - d(x, A)| \geq \epsilon\}|.$$

Then for each $x \in X$ and for any $\delta > 0$

$$\{r \in \mathbb{N} : \frac{1}{g(h_r)} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \epsilon\}| \geq \delta\} \subseteq \{r \in \mathbb{N} : \frac{1}{g(h_r)} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq \epsilon.\delta\} \in \mathcal{I}.$$

This proves the result. □

Remark 2.18. In Theorem 2 [33] it was further proved that
(ii) $\{A_k\} \in l_\infty$ and $A_k \to A(S_\theta(\mathcal{I}_W))$ implies $A_k \to A(N_\theta(\mathcal{I}_W)$,
(iii) $S_\theta(\mathcal{I}_W) \cap l_\infty = N_\theta(\mathcal{I}_W) \cap l_\infty$.
It is not clear whether these results hold for any $g \in G$ and we leave it as an open problem.

We will now investigate the relationship between Wijsman $\mathcal{I}$-statistical and Wijsman $\mathcal{I}$-lacunary statistical convergence of weight g.

Theorem 2.19. *Let (X,ρ) be a metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and A, A_k be non-empty closed subsets of X. Then $A_k \to A(S(\mathcal{I}_W)^g)$ implies $A_k \to A(S_\theta(I_W)^g)$ if*

$$\liminf_r \frac{g(h_r)}{g(k_r)} > 1.$$

Proof. Since $\liminf_r \frac{g(h_r)}{g(k_r)} > 1$, so we can find a $H > 1$ such that for sufficiently large r we have

$$\frac{g(h_r)}{g(k_r)} \geq H.$$

Since $x_k \to L(S(\mathcal{I})^g)$, hence for every $\varepsilon > 0$ and sufficiently large r we have

$$\begin{aligned} &\frac{1}{g(k_r)} \left|\{k \leq k_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right| \\ &\geq \frac{1}{g(k_r)} \left|\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right| \\ &\geq H \cdot \frac{1}{g(h_r)} \left|\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right|. \end{aligned}$$

Then for any $\delta > 0$ we get

$$\begin{aligned} &\left\{r \in \mathbb{N} : \frac{1}{g(h_r)} \left|\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right| \geq \delta\right\} \\ &\subseteq \left\{r \in \mathbb{N} : \frac{1}{g(k_r)} \left|\{k \leq k_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right| \geq H\delta\right\} \in \mathcal{I}. \end{aligned}$$

This shows that $x_k \to L(S_\theta(\mathcal{I})^g)$. □

For the next result, as in [29], we assume that the lacunary sequence θ satisfies the condition that for any set $C \in \mathcal{F}(\mathcal{I})$

$$\bigcup \{n : k_{r-1} < n < k_r, r \in C\} \in \mathcal{F}(\mathcal{I}).$$

Theorem 2.20. *Let (X,ρ) be a metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and A, A_k be non-empty closed subsets of X. Then $A_k \to A(S_\theta(\mathcal{I}_W)^g)$ implies $A_k \to A(S(\mathcal{I}_W)^g)$ if $\sup_r \sum_{i=0}^{r-1} \frac{g(h_{i+1})}{g(k_{r-1})} = B(say) < \infty$.*

Proof. Suppose that $A_k \to L(S_\theta(I_W)^g)$ and for $\epsilon, \delta, \delta_1 > 0$ define the sets

$$C = \{r \in \mathbb{N} : \frac{1}{g(h_r)} |\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \epsilon\}| < \delta\}$$

and

$$T = \{n \in \mathbb{N} : \frac{1}{g(n)} |\{k \leq n : |d(x,A_k) - d(x,A)| \geq \epsilon\}| < \delta_1\}.$$

It is obvious from our assumption that $C \in F(I)$, the filter associated with the ideal $\mathcal{I}$. Further observe that

$$A_j = \frac{1}{g(h_j)} |\{k \in I_j : |d(x,A_k) - d(x,A)| \geq \epsilon\}| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\frac{1}{g(n)}|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \epsilon\}| \leq \frac{1}{g(k_{r-1})}|\{k \leq k_r : |d(x, A_k) - d(x, A)| \geq \epsilon\}|$$

$$= \frac{1}{g(k_{r-1})}|\{k \in I_1 : |d(x, A_k) - d(x, A)| \geq \epsilon\}| + \cdots + \frac{1}{g(k_{r-1})}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \epsilon\}|$$

$$= \frac{g(k_1)}{g(k_{r-1})}\frac{1}{g(h_1)}|\{k \in I_1 : |d(x, A_k) - d(x, A)| \geq \epsilon\}| + \frac{g(k_2 - k_1)}{g(k_{r-1})}\frac{1}{g(h_2)}|\{k \in I_2 : |d(x, A_k) - d(x, A)| \geq \epsilon\}| + \cdots +$$

$$+ \frac{g(k_r - k_{r-1})}{g(k_{r-1})}\frac{1}{g(h_r)}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \epsilon\}|$$

$$= \frac{g(k_1)}{g(k_{r-1})}A_1 + \frac{g(k_2 - k_1)}{g(k_{r-1})}A_2 + \cdots + \frac{g(k_r - k_{r-1})}{g(k_{r-1})}A_r$$

$$\leq \sup_{j \in C} A_j . \sup_r \sum_{i=0}^{r-1} \frac{g(k_{i+1} - k_i)}{g(k_{r-1})} < B\delta.$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \subset T$ where $C \in F(\mathcal{I})$ it follows from our assumption on θ that the set T also belongs to $F(\mathcal{I})$ and this completes the proof of the theorem. □

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