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Lie Product Preserving Maps on $M_n(\mathbb{F})$

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Abstract. Bijective maps on matrices over arbitrary fields with sufficiently many elements which preserve Lie product are characterized.

1. Introduction and Statement of the Result

Let $\mathbb{F}$ be a field (i.e. a commutative division ring) and let $M_n(\mathbb{F})$ be the set of all $n \times n$ matrices over $\mathbb{F}$. A map ϕ preserves commutativity if $\phi(A)\phi(B) = \phi(B)\phi(A)$ whenever $AB = BA$, $A, B \in M_n(\mathbb{F})$. If ϕ is bijective and both ϕ and ϕ^{-1} preserve commutativity, then we say that ϕ preserves commutativity in both directions. A map $\phi\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a Lie product $[A, B] = AB - BA$ preserving map if $\phi([A, B]) = \big[\phi(A), \phi(B)\big]$. Commutativity preserving maps are closely related to Lie product preserving maps. Namely, preserving commutativity is the same as preserving zero Lie product. In last decades linear commutativity preserving maps (see for example [3] and references therein) and recently also non-linear commutativity preserving maps were extensively studied. In [16] Šemrl characterized maps on $M_n(\mathbb{C})$, $n \geq 3$, which preserve commutativity in both directions (see also [9]). The same author considered in [15] injective, continuous maps on $M_n(\mathbb{C})$, $n > 3$, that preserve commutativity. Fošner [8] proved, using the real Jordan canonical form, that an analogous result holds true for the set of real matrices $M_n(\mathbb{R})$, $n > 3$. Dolinar et al. classified in [7] maps on $M_n(\mathbb{F})$, where $\mathbb{F}$ is an arbitrary field with sufficiently many elements and $n \geq 3$, which preserve commutativity in both directions.

In [16] Šemrl described the form of bijective, Lie product preserving maps on $M_n(\mathbb{C})$ and Dolinar obtained in [6] a similar result in a more general setting, omitting the assumption of bijectivity (see also [5]). The techniques were mostly developed for the case of complex matrices using Jordan canonical form. The question is whether these or similar techniques work also in the case of a general field $\mathbb{F}$.

Let us provide some examples of Lie product preserving maps on $M_n(\mathbb{F})$. The first example is a similarity transformation $A \mapsto TAT^{-1}$ where $T \in M_n(\mathbb{F})$ is an invertible matrix. Next, every map $A \mapsto A + \psi(A)I$, where $I \in M_n(\mathbb{F})$ is the identity matrix and ψ is a scalar function on $M_n(\mathbb{F})$, is a Lie product preserving map. The same holds true for maps of the following form: $A \mapsto -A^t$ where A^t denotes the transpose of $A \in M_n(\mathbb{F})$. Given a field homomorphism $\sigma\colon \mathbb{F} \to \mathbb{F}$ and a matrix $A \in M_n(\mathbb{F})$, we denote by A^σ the matrix obtained from

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A by applying σ to A entrywise. The map $A \mapsto A^\sigma$ also preserves Lie product. Note that when $\sigma\colon \mathbb{F} \to \mathbb{F}$ is bijective, i.e. a field automorphism, then the map $A \mapsto A^\sigma$ is also bijective. Our result claims that every bijective, Lie product preserving map on $M_n(\mathbb{F})$, where $n \geq 3$ and $\mathbb{F}$ has at least 2^{n-1} elements, or $n = 2$ and char $\mathbb{F} \neq 2$, is a composition of these maps.

Theorem. *Let $\mathbb{F}$ be a field and suppose $\phi\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a bijective map satisfying*

$$\phi([A,B]) = \left[\phi(A), \phi(B)\right] \tag{1}$$

for all $A, B \in M_n(\mathbb{F})$. Then there exist an invertible matrix $T \in M_n(\mathbb{F})$, a field automorphism $\sigma\colon \mathbb{F} \to \mathbb{F}$, and a function $\psi\colon M_n(\mathbb{F}) \to \mathbb{F}$, where $\psi(A) = 0$ for all matrices of trace zero, such that:

(i) *for $n \geq 3$ and $\mathbb{F}$ with at least 2^{n-1} elements, either*

$$\phi(A) = TA^\sigma T^{-1} + \psi(A)I \quad \text{for all} \quad A \in M_n(\mathbb{F}),$$

or

$$\phi(A) = -T(A^\sigma)^t T^{-1} + \psi(A)I \quad \text{for all} \quad A \in M_n(\mathbb{F});$$

(ii) *for $n = 2$ and* char $\mathbb{F} \neq 2$,

$$\phi(A) = TA^\sigma T^{-1} + \psi(A)I \quad \text{for all} \quad A \in M_n(\mathbb{F}).$$

2. Proof

We begin with some easy observations. Let $\mathbb{F}$ be a field and $\phi\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ a map satisfying (1), i.e., $\phi([A,B]) = \left[\phi(A), \phi(B)\right]$ for all $A, B \in M_n(\mathbb{F})$. Note that $\phi(0) = \phi([A,A]) = \left[\phi(A), \phi(A)\right] = 0$, so if $BC = CB$ for $B, C \in M_n(\mathbb{F})$, then $\left[\phi(B), \phi(C)\right] = \phi([B,C]) = \phi(0) = 0$, and hence ϕ preserves commutativity. If $\phi\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is bijective and satisfies (1), then ϕ^{-1} satisfies (1) as well since $\phi([A,B]) = \left[\phi(A), \phi(B)\right]$ implies $[A,B] = \phi^{-1}\left[\phi(A), \phi(B)\right]$ and hence

$$\left[\phi^{-1}(C), \phi^{-1}(D)\right] = \phi^{-1}([C,D])$$

for all $C, D \in M_n(\mathbb{F})$. We may conclude that a Lie product preserving bijective map preserves commutativity in both directions.

Let us mention that the proof of the theorem will follow some techniques and ideas used in the papers mentioned in the introduction, in particular [16, Theorem 2.5] (see also [13]).

(i) We start with the case (i) in the Theorem, when $n \geq 3$ and $\mathbb{F}$ has at least 2^{n-1} elements.

Step 1. Let $A \in M_n(\mathbb{F})$. Then for every idempotent of rank one $P \in M_n(\mathbb{F})$ there exists a rank one matrix $B \in M_n(\mathbb{F})$ such that $PA - AP = PB - BP$.

As usually E_{ij} is the matrix with all entries equal to zero except the (i, j)-entry which is equal to one. Let $C = \left[c_{ij}\right] \in M_n(\mathbb{F})$ be an arbitrary matrix and let

$$D = \begin{bmatrix} 1 & c_{12} & \dots & c_{1n} \\ c_{21} & c_{21}c_{12} & \dots & c_{21}c_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n1}c_{12} & \dots & c_{n1}c_{1n} \end{bmatrix} \in M_n(\mathbb{F}).$$

Observe that D is of rank one and that $E_{11}C - CE_{11} = E_{11}D - DE_{11}$. Let now $P \in M_n(\mathbb{F})$ be any idempotent of rank one. There exists an invertible matrix $T \in M_n(\mathbb{F})$ such that $TE_{11}T^{-1} = P$. As observed, for a matrix $T^{-1}AT$ there exists a rank one matrix $T^{-1}BT$ such that

$$E_{11}T^{-1}AT - T^{-1}ATE_{11} = E_{11}T^{-1}BT - T^{-1}BTE_{11}.$$

Multiplying this equation from the left by T and from the right by T^{-1} we obtain $PA - AP = PB - BP$.

Step 2. For every rank one matrix $A \in M_n(\mathbb{F})$ we obtain that up to similarity, up to the ring automorphism induced by the field automorphism σ, and up to the map $A \mapsto -A^t$, if necessary, $\phi(A) = p_A(A)$, where p_A is a polynomial with coefficients in $\mathbb{F}$ associated to A.

Recall that ϕ preserves commutativity in both directions. The main result in [7] states that there exist a field automorphism $\sigma\colon \mathbb{F} \to \mathbb{F}$ and an invertible matrix S such that one of the following holds:

$\phi(A) = Sp_A(A^\sigma)S^{-1}$ for all $A \in \mathcal{D}_n(\mathbb{F})\cup\mathcal{I}_n^1(\mathbb{F})$;

$\phi(A) = Sp_A(A^\sigma)^tS^{-1}$ for all $A \in \mathcal{D}_n(\mathbb{F})\cup\mathcal{I}_n^1(\mathbb{F})$;

where to each $A \in M_n(\mathbb{F})$ a polynomial p_A with coefficients in $\mathbb{F}$ is associated, $\mathcal{D}_n(\mathbb{F}) \subseteq M_n(\mathbb{F})$ is the subset of all diagonalizable matrices, and $\mathcal{I}_n^1(\mathbb{F})\subseteq M_n(\mathbb{F})$ is the subset of all rank-one matrices.

Observe that the map $\varkappa\colon\ M_n(\mathbb{F}) \to M_n(\mathbb{F})$ defined with $\varkappa(A) = -A^t$ is bijective and that $\varkappa([A,B]) = -(AB-BA)^t = A^tB^t - B^tA^t = [\varkappa(A), \varkappa(B)]$. So in both cases by composing the map ϕ with a ring automorphism induced by the field automorphism σ, we may assume that

$$\phi(A) = p_A(A)$$

for every rank one matrix $A \in M_n(\mathbb{F})$.

Step 3. Assume that $\phi(A) = p_A(A)$ for every rank one matrix $A \in M_n(\mathbb{F})$, where p_A is a polynomial with coefficients in $\mathbb{F}$ associated to A. Then $\phi(A) = A + \lambda_A I$, $\lambda_A \in \mathbb{F}$, for every rank one matrix $A \in M_n(\mathbb{F})$.

First, let us recall that every $A \in M_n(\mathbb{F})$ is similar over $\mathbb{F}$ to the direct sum of companion matrices of appropriate polynomials (called the elementary divisors of A) with coefficients in $\mathbb{F}$ (see [10], page 156). So, if $A \in M_n(\mathbb{F})$ is of rank one, then A is similar to the direct sum

$$\begin{bmatrix} 0 & 0 \\ 1 & \lambda \end{bmatrix} \oplus 0_{n-2}$$

where $0_{n-2} \in M_{n-2}(\mathbb{F})$ is the zero matrix and λ the trace of A. Obviously, if $\lambda = 0$, then A is similar to E_{12}, and if $\lambda \neq 0$, then A is similar to λE_{11}. We may conclude that for a rank one matrix $A \in M_n(\mathbb{F})$ there exists an invertible matrix $T \in M_n(\mathbb{F})$ such that

$$A = TBT^{-1}, \quad \text{where} \quad B = E_{12} \text{ or } B = \lambda E_{11} \text{ with } \lambda \in \mathbb{F} \text{ nonzero.} \tag{2}$$

So, A is a nilpotent matrix of rank one, or A is of the form λP where $\lambda \in \mathbb{F}$ and $P \in M_n(\mathbb{F})$ is an idempotent matrix of rank one.

Suppose first that $P \in M_n(\mathbb{F})$ is any rank one idempotent matrix and let $\lambda \in \mathbb{F}$. Since $\phi(A) = p_A(A)$ for every $A \in M_n(\mathbb{F})$ of rank one and since P is a rank one idempotent matrix, there exist functions $\alpha_P, \mu_P\colon \mathbb{F} \to \mathbb{F}$ such that $\phi(\lambda P) = \alpha_P(\lambda)P + \mu_P(\lambda)I$. We will show that the function α_P is independent of the choice of an idempotent, i.e. $\alpha_P(\lambda) = \alpha(\lambda)$. Let $Q \in M_n(\mathbb{F})$ be an idempotent of rank one and suppose $R \in M_n(\mathbb{F})$ is another idempotent of rank one such that Q are R are orthogonal, i.e. $QR = RQ = 0$. Any commutative subset of the set of all idempotent matrices in $M_n(\mathbb{D})$, where $\mathbb{D}$ is a (not necessarily commutative) division ring and $n \geq 3$, is simultaneously diagonalizable (see [17, Lemma 2.12]). Thus, there exists an invertible matrix $T \in M_n(\mathbb{F})$ such that TQT^{-1} and TRT^{-1} are diagonal idempotents and hence their entries are equal to 0 except one diagonal entry which is equal to 1. Suppose $TQT^{-1} = E_{ii}$ and $TRT^{-1} = E_{jj}$. Then clearly $i \neq j$. Also, $E_{jj}E_{ij} = 0 = E_{ij}E_{ii}$ and $E_{ii}E_{ij} = E_{ij} = E_{ij}E_{jj}$. Let $C = T^{-1}E_{ij}T$. Then $C \in M_n(\mathbb{F})$ is a rank one matrix and since $i \neq j$, C is a nilpotent matrix. Also, $QC = C = CR$ and $RC = 0 = CQ$. By (2), $N^2 = 0$ for every nilpotent matrix N of rank one. So, since $\phi(A) = p_A(A)$ for every rank one matrix $A \in M_n(\mathbb{F})$, we may conclude that $\phi(N) = \lambda_N N + \nu_N I$, $\lambda_N, \nu_N \in \mathbb{F}$, for every nilpotent matrix $N \in M_n(\mathbb{F})$ of rank one. On the one hand we have

$$\left[\phi(\lambda Q), \phi(C)\right] = \phi([\lambda Q, C]) = \phi(\lambda(QC - CQ)) = \phi(\lambda C) = \phi(\lambda(CR - RC)) =$$

$$= \left[\phi(C), \phi(\lambda R)\right] = [\lambda_C C + \nu_C I, \alpha_R(\lambda)R + \mu_R(\lambda)I] = \lambda_C\alpha_R(\lambda)C,$$

while on the other hand we have

$$\left[\phi(\lambda Q), \phi(C)\right] = \left[\alpha_Q(\lambda)Q + \mu_Q(\lambda)I, \lambda_C C + \nu_C I\right] = \alpha_Q(\lambda)\lambda_C C.$$

Since ϕ is an injective map, $\phi(0) = 0$, and $\phi(C) = \lambda_C \alpha_R(1)C$, we may conclude that $\lambda_C \neq 0$. It follows that $\alpha_R(\lambda) = \alpha_Q(\lambda)$.

Suppose now that rank one idempotent matrices $Q, R \in M_n(\mathbb{F})$ are not orthogonal. If $n \geq 4$, then it is easy to see that there exists a rank one idempotent $K \in M_n(\mathbb{F})$ such that

$$KQ = QK = 0 = KR = RK.$$

If $n = 3$, then one can find (see [2, Proof of Theorem 10] and recall that two distinct rank one idempotents are orthogonal if and only if they commute) either a rank one idempotent $K \in M_n(\mathbb{F})$ such that

$$KQ = QK = 0 = KR = RK$$

or two rank one idempotents $K_1, K_2 \in M_n(\mathbb{F})$ such that

$$QK_1 = K_1Q = K_1K_2 = K_2K_1 = K_2R = RK_2 = 0.$$

Thus, $\alpha_Q(\lambda) = \alpha_K(\lambda) = \alpha_R(\lambda)$ in the former case and $\alpha_Q(\lambda) = \alpha_{K_1}(\lambda) = \alpha_{K_2}(\lambda) = \alpha_R(\lambda)$ in the latter case. We conclude that $\alpha_P(\lambda) = \alpha(\lambda)$ is independent of the choice of a rank one idempotent P.

We will now show that $\alpha(\lambda) = \lambda$ for every $\lambda \in \mathbb{F}$. Since E_{11} and $E_{11} + \lambda E_{12}$ are idempotents of rank one, we have

$$\phi([E_{11}, E_{11} + \lambda E_{12}]) = \left[\phi(E_{11}), \phi(E_{11} + \lambda E_{12})\right] =$$

$$= [\alpha(1)E_{11} + \mu_{E_{11}}(1)I, \alpha(1)(E_{11} + \lambda E_{12}) + \mu_{E_{11}+\lambda E_{12}}(1)I] = \alpha(1)^2 \lambda E_{12}.$$

From $\phi([E_{11}, E_{11} + \lambda E_{12}]) = \phi(\lambda E_{12}) = \left[\phi(E_{12}), \phi(\lambda E_{22})\right] = \lambda_{E_{12}} \alpha(\lambda) E_{12}$, it follows $\lambda_{E_{12}} \alpha(\lambda) = \alpha(1)^2 \lambda$. Here $\lambda_{E_{12}} \neq 0$. So,

$$\alpha(\lambda) = \alpha(1)^2 \lambda_{E_{12}}^{-1} \lambda. \tag{3}$$

From

$$\phi(E_{12}) = \phi([E_{11}, E_{11} + E_{12}]) = \alpha(1)^2 E_{12}$$

we obtain

$$\phi(E_{12}) = \phi([E_{11}, E_{12}]) = \left[\phi(E_{11}), \phi(E_{12})\right]$$

$$= \left[\alpha(1)E_{11} + \mu_{E_{11}}(1)I, \alpha(1)^2 E_{12}\right] = \alpha(1)^3 E_{12}.$$

Since ϕ is injective and $\phi(0) = 0$, we have $\phi(E_{12}) \neq 0$ and hence $\alpha(1) \neq 0$. It follows that $\alpha(1) = 1$ since $\alpha(1)^2 = \alpha(1)^3$. From (3) we have $1 = \lambda_{E_{12}}$ and therefore $\alpha(\lambda) = \lambda$. We proved that

$$\phi(\lambda P) = \lambda P + \mu_P(\lambda) I \tag{4}$$

for every rank one idempotent matrix $P \in M_n(\mathbb{F})$.

Let now $A \in M_n(\mathbb{F})$ be any rank one matrix. By (2) it follows that A is similar to E_{12} or to λE_{11} for some nonzero $\lambda \in \mathbb{F}$. In the latter case it immediately follows by (4) that $\phi(A) = A + \lambda_A I$, $\lambda_A \in \mathbb{F}$. In the former case, since $A = TE_{12}T^{-1}$ and $\phi(P) = P + \mu_P(1)I$ for every rank one idempotent matrix $P \in M_n(F)$, we have

$$\phi(A) = \phi\left(\left[TE_{11}T^{-1}, TE_{11}T^{-1} + TE_{12}T^{-1}\right]\right)$$

$$= \left[\phi(TE_{11}T^{-1}), \phi(TE_{11}T^{-1} + TE_{12}T^{-1})\right]$$

$$= \left[TE_{11}T^{-1}, TE_{11}T^{-1} + TE_{12}T^{-1}\right] = TE_{12}T^{-1} = A.$$

To conclude, $\phi(A) = A + \lambda_A I$, $\lambda_A \in \mathbb{F}$, for every rank one matrix $A \in M_n(\mathbb{F})$.

Step 4. Assume that $\phi(A) = A + \lambda_A I$, $\lambda_A \in \mathbb{F}$, for every rank one matrix $A \in M_n(\mathbb{F})$. Then $\phi(A) = A + \lambda_A I$, $\lambda_A \in \mathbb{F}$, for every matrix $A \in M_n(\mathbb{F})$.

Let $A \in M_n(\mathbb{F})$ be any matrix and $P \in M_n(\mathbb{F})$ an idempotent of rank one. By Step 1 there exists a rank one matrix B such that $PA - AP = PB - BP$. It follows

$$PA - AP = [P, B] = [P + \lambda_P I, B + \lambda_B I] = \left[\phi(P), \phi(B)\right] = \phi([P, B]).$$

Since $[P, A] = [P, B]$, we have

$$\begin{aligned} PA - AP &= \phi([P, A]) = \left[\phi(P), \phi(A)\right] \\ &= \left[P + \lambda_P I, \phi(A)\right] = P\phi(A) - \phi(A)P \end{aligned}$$

and therefore

$$P(\phi(A) - A) = (\phi(A) - A)P. \tag{5}$$

Recall that a matrix from $M_n(\mathbb{F})$ which commutes with every rank one idempotent matrix from $M_n(\mathbb{F})$ is in the center of $M_n(\mathbb{F})$. Equation (5) yields that $\phi(A) - A$ is a scalar matrix and hence there exists a scalar $\lambda_A \in \mathbb{F}$ such that $\phi(A) = A + \lambda_A I$ for every matrix $A \in M_n(\mathbb{F})$.

Step 5. Assume that $\phi(A) = A + \lambda_A I$, $\lambda_A \in \mathbb{F}$, for every matrix $A \in M_n(\mathbb{F})$. Then $\phi(A) = A + \psi(A)I$ for every $A \in M_n(\mathbb{F})$, where $\psi\colon M_n(\mathbb{F}) \to \mathbb{F}$ with $\psi(A) = 0$ for every matrix A of trace zero.

Let us define the function $\psi\colon M_n(\mathbb{F}) \to \mathbb{F}$ with $\psi(A) = \lambda_A$, so $\phi(A) = A + \psi(A)I$ for every matrix $A \in M_n(\mathbb{F})$. Let $A \in M_n(\mathbb{F})$ be of trace zero. Then there exist matrices $B, C \in M_n(\mathbb{F})$ such that $A = [B, C]$. This observation was proved by Shoda [14] for the case when $\mathbb{F}$ is a field of characteristic zero and Shoda's result was extended to fields of positive characteristic by Albert and Muckenhoupt [1]. Hence

$$A = [B, C] = [B + \psi(B)I, C + \psi(C)I] = \left[\phi(B), \phi(C)\right] = \phi([B, C]) = \phi(A).$$

Since $\phi(A) = A + \psi(A)I$, we may conclude that $\psi(A) = 0$ for every trace zero matrix A.

To sum up, taking into account our assumptions that $n \geq 3$ and $\mathbb{F}$ is a field with at least 2^{n-1} elements we obtain that a Lie product preserving bijective map $\phi\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is of the following form: $\phi(A) = TA^{\sigma}T^{-1} + \psi(A)I$ for every $A \in M_n(\mathbb{F})$, or $\phi(A) = -T(A^{\sigma})^t T^{-1} + \psi(A)I$ for every $A \in M_n(\mathbb{F})$, where $\psi\colon M_n(\mathbb{F}) \to \mathbb{F}$ with $\psi(A) = 0$ for every matrix A of trace zero.

(ii) We continue the proof of the Theorem with the case (ii) when $n = 2$ and char $\mathbb{F} \neq 2$. We will use some ideas from the proof of Theorem 8.1 in [16]. First, let us prove the following characterization of nilpotent matrices.

Step 6. Let $\mathbb{F}$ be a field with char $\mathbb{F} \neq 2$, let $\mathcal{N}$ be the set of all nilpotent matrices in $M_2(\mathbb{F})$, and let $\mathcal{S} \subseteq M_2(\mathbb{F})$, such that for every $S \in \mathcal{S}$ there exists $A_S \in M_2(\mathbb{F})$ with $[A_S, S] = S$. Then $\mathcal{N} = \mathcal{S}$ and therefore $\phi(\mathcal{N}) = \mathcal{N}$, since ϕ and ϕ^{-1} satisfy (1).

Suppose first $S \in \mathcal{N}$. If $S = 0$, then clearly $S \in \mathcal{S}$. Let S be a nonzero nilpotent matrix. Then S is of rank one and similar to $E_{12} \in M_2(\mathbb{F})$. Since $TE_{11}T^{-1}TE_{12}T^{-1} - TE_{12}T^{-1}TE_{11}T^{-1} = TE_{12}T^{-1}$ holds for every invertible matrix T, we may conclude that $S \in \mathcal{S}$.

Conversely, let $S \in \mathcal{S}$. Then there exists $A_S \in M_2(\mathbb{F})$ such that $[A_S, S] = S$, and therefore S is a trace zero matrix. If S is of rank zero, then $S = 0$ and $S \in \mathcal{N}$. If S is of rank one, then S is similar to the matrix

$$\begin{bmatrix} 0 & 0 \\ 1 & \lambda \end{bmatrix}$$

and λ is equal to the trace of S, hence $\lambda = 0$ and therefore $S \in \mathcal{N}$. Suppose now that $S \in \mathcal{S}$ is of rank two. Then there exists an invertible matrix $T \in M_2(\mathbb{F})$ such that

$$S = T \begin{bmatrix} 0 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix} T^{-1}$$

where $\lambda_1 \neq 0$. Since S is a trace zero matrix, we may conclude that $\lambda_2 = 0$. So,

$$T^{-1}A_S T\begin{bmatrix} 0 & \lambda_1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \lambda_1 \\ 1 & 0 \end{bmatrix} T^{-1}A_S T = \begin{bmatrix} 0 & \lambda_1 \\ 1 & 0 \end{bmatrix}.$$

Let $T^{-1}A_S T = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$. Then

$$\begin{bmatrix} b_2 - \lambda_1 b_3 & b_1\lambda_1 - \lambda_1 b_4 \\ b_4 - b_1 & b_3\lambda_1 - b_2 \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1 \\ 1 & 0 \end{bmatrix}$$

and thus $b_1\lambda_1 - b_4\lambda_1 = \lambda_1$ and $b_4 - b_1 = 1$. So, $b_1 - b_4 = b_4 - b_1$ since $\lambda_1 \neq 0$. We have $-1 = 1$, a contradiction with char $\mathbb{F} \neq 2$. It follows that $S \in \mathcal{S}$ can not be of rank two.

Step 7. For every rank one idempotent $P \in M_2(\mathbb{F})$ we obtain that $\phi(P) = P_1 + \mu_P I$, where P_1 is a rank one idempotent and $\mu_P \in \mathbb{F}$, both depending on P.

Let N be a nilpotent of rank one, so $N = TE_{12}T^{-1}$ for some invertible matrix $T \in M_2(\mathbb{F})$. Let $B \in M_2(\mathbb{F})$, such that $BN - NB = N$, and let $C = \left[\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right] = T^{-1}BT \in M_2(\mathbb{F})$. Then $CE_{12} - E_{12}C = E_{12}$ and therefore $c_1 = c_4 + 1$, $c_3 = 0$. It follows that

$$C = \begin{bmatrix} 1 & c_2 \\ 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \widetilde{P} + c_4 I$$

where $\widetilde{P} \in M_2(\mathbb{F})$ is an idempotent of rank one and $c_4 \in \mathbb{F}$. We may conclude that every matrix $B \in M_2(\mathbb{F})$, such that $BN - NB = N$ for some nilpotent matrix $N \in M_2(\mathbb{F})$ of rank one, may be written as

$$B = \widetilde{P_1} + \alpha I$$

where $\widetilde{P_1} \in M_2(\mathbb{F})$ is an idempotent of rank one and $\alpha \in \mathbb{F}$.

Let now $P \in M_2(\mathbb{F})$ be an idempotent of rank one. Since P is similar to E_{11} and $[E_{11}, E_{12}] = E_{12}$, it follows $PN - NP = N$ for some $N \in \mathcal{N}$. Therefore $\phi(P)\phi(N) - \phi(N)\phi(P) = \phi(N)$. By Step 6, $\phi(N)$ is also a nilpotent and therefore $\phi(P)$ may be written as

$$\phi(P) = P_1 + \mu_P I$$

where $P_1 \in M_2(\mathbb{F})$ is an idempotent of rank one and $\mu_P \in \mathbb{F}$, both depending on P.

Step 8. Let P and Q be rank one idempotents and let $\phi(P) = P_1 + \mu_P I$ and $\phi(Q) = Q_1 + \mu_Q I$ as in Step 7. If P and Q have the same range space or the same null space, then the same is true for rank one idempotents P_1 and Q_1.

Let $P, Q \in M_2(\mathbb{F})$ be rank one idempotents. Suppose first that $\operatorname{Im} P = \operatorname{Im} Q$. As we showed in Step 7, there exists a nilpotent matrix $N \in M_2(\mathbb{F})$ of rank one, such that $PN - NP = N$. Observe that $\operatorname{Im} P = \operatorname{Im} N$. Since $\operatorname{Im} P = \operatorname{Im} Q$, it follows that P and Q are simultaneously similar to E_{11} and $E_{11} + \alpha E_{12}$, hence $QN - NQ = N$ and therefore

$$\phi(P)\phi(N) - \phi(N)\phi(P) = \phi(N) \quad \text{and} \quad \phi(Q)\phi(N) - \phi(N)\phi(Q) = \phi(N).$$

Since $\phi(P) = P_1 + \mu_P I$ and $\phi(Q) = Q_1 + \mu_Q I$, where $P_1, Q_1 \in M_2(\mathbb{F})$ are idempotents of rank one and $\phi(N)$ is a nonzero nilpotent by Step 6, we obtain that $\operatorname{Im} P_1 = \operatorname{Im} \phi(N) = \operatorname{Im} Q_1$.

Suppose now $\operatorname{Ker} P = \operatorname{Ker} Q$. Similarly as in Step 7 we can show that there exists a nilpotent matrix $N \in M_2(\mathbb{F})$ of rank one, such that $NP - PN = N$. Since $\operatorname{Ker} P = \operatorname{Ker} Q$, it follows that P and Q are simultaneously similar to E_{11} and $E_{11} + \alpha E_{21}$, hence $NQ - QN = N$ and therefore $\left[\phi(N), \phi(P)\right] = \phi(N) = \left[\phi(N), \phi(Q)\right]$. Similarly as before we obtain that $\operatorname{Ker} P_1 = \operatorname{Ker} \phi(N) = \operatorname{Ker} Q_1$.

Step 9. For every rank one idempotent $P \in M_2(\mathbb{F})$ and every $\lambda \in \mathbb{F}$ we obtain that $\phi(\lambda P) = \sigma_P(\lambda)P_1 + \omega_{\lambda P} I$, where a rank one idempotent P_1 and a function $\sigma_P\colon \mathbb{F} \to \mathbb{F}$ are both depending on P, and a scalar $\omega_{\lambda P} \in \mathbb{F}$ is depending on λ and P.

Since $[A, E_{11}] = 0$ implies $A = \alpha E_{11} + \beta I$, $\alpha, \beta \in \mathbb{F}$, it follows that for a rank one idempotent P equation $[B, P] = 0$ implies B is in the linear span of $\{P, I\}$. Recall that $\phi(P) = P_1 + \mu_P I$, where P_1 is a rank one idempotent and $\mu_P \in \mathbb{F}$. From $[\lambda P, P] = 0$ it follows that $[\phi(\lambda P), P_1 + \mu_P I] = [\phi(\lambda P), P_1] = \phi(0) = 0$. So

$$\phi(\lambda P) = \sigma_P(\lambda) P_1 + \omega_{\lambda P} I$$

with $\sigma_P \colon \mathbb{F} \to \mathbb{F}$ and $\omega_{\lambda P} \in \mathbb{F}$. Note that $\sigma_P(1) = 1$ and $\sigma_P(0) = 0$.

Step 10. The function $\sigma_P = \sigma$, defined in Step 9, is multiplicative, additive and independent of rank one idempotent P. In addition $\phi(\lambda N) = \sigma(\lambda)\phi(N)$ for every $N \in \mathcal{N}$, $\lambda \in \mathbb{F}$.

Let us first show that σ_P is independent of rank one idempotent P. Suppose $P, Q \in M_2(\mathbb{F})$ are rank one idempotents with the same null space. As we showed in Step 8, there exists $N \in \mathcal{N}$ of rank one such that $[N, P] = N = [N, Q]$. Let $\lambda \in \mathbb{F}$. Then

$$[N, \lambda P] = \lambda N = [N, \lambda Q]$$

and thus by Step 9,

$$\left[\phi(N), \sigma_P(\lambda) P_1\right] = \left[\phi(N), \sigma_Q(\lambda) Q_1\right],$$

where P_1 and Q_1 are rank one idempotents. This yields $\sigma_P(\lambda)\left[\phi(N), P_1\right] = \sigma_Q(\lambda)\left[\phi(N), Q_1\right]$. Since $[N, P] = [N, Q] \neq 0$, we have $\left[\phi(N), P_1\right] = \left[\phi(N), Q_1\right] \neq 0$ and thus $\sigma_P = \sigma_Q$. In a similar way we obtain the same conclusion if P and Q have the same range space.

Suppose P and Q are rank one idempotent matrices which do not have the same null space nor the same range space. Let first $\operatorname{Im} P \neq \operatorname{Ker} Q$. Then there exists a rank one idempotent $R \in M_2(\mathbb{F})$ where $\operatorname{Im} P = \operatorname{Im} R$ and $\operatorname{Ker} Q = \operatorname{Ker} R$. Thus, $\sigma_P = \sigma_R = \sigma_Q$. Suppose now $\operatorname{Im} P = \operatorname{Ker} Q$. Then there exist rank one idempotents $R_1 \neq P$ and R_2, where $\operatorname{Ker} R_1 \neq \operatorname{Im} Q$, such that $\operatorname{Im} P = \operatorname{Im} R_1$, $\operatorname{Ker} R_1 = \operatorname{Ker} R_2$, $\operatorname{Im} R_2 = \operatorname{Im} Q$ and thus $\sigma_P = \sigma_{R_1} = \sigma_{R_2} = \sigma_Q$. We may conclude that $\sigma_p = \sigma$ is independent of P.

Let $N \in M_2(\mathbb{F})$ be a nilpotent matrix of rank one. Then, similarly as we showed in Step 7, there exists a rank one idempotent $P \in M_2(\mathbb{F})$, such that $\operatorname{Im} P = \operatorname{Im} N$ and $PN - NP = N$. Hence $[\lambda P, N] = \lambda N$ for every $\lambda \in \mathbb{F}$. Recall that $\phi(\lambda P) = \sigma(\lambda) P_1 + \omega_{\lambda P} I$, $\lambda \in \mathbb{F}$, and note that $PN - NP = N$ implies $\left[P_1, \phi(N)\right] = \phi(N)$. Thus

$$\phi(\lambda N) = \left[\phi(\lambda P), \phi(N)\right] = \left[\sigma(\lambda) P_1, \phi(N)\right] = \sigma(\lambda)\phi(N).$$

Next we prove that σ is multiplicative. Let N be a nilpotent of rank one and let $\lambda, \mu \in \mathbb{F}$. Then

$$\sigma(\lambda\mu)\phi(N) = \phi(\lambda\mu N) = \sigma(\lambda)\phi(\mu N) = \sigma(\lambda)\sigma(\mu)\phi(N)$$

for every $\lambda, \mu \in \mathbb{F}$, hence σ is multiplicative.

Let us now prove that σ is also additive. There exist an invertible matrix $T \in M_2(\mathbb{F})$ such that $\phi(E_{11}) = TE_{11}T^{-1} + \mu_{E_{11}} I$. We may assume, after composing ϕ with similarity transformation, that $\phi(E_{11}) = E_{11} + \mu_{E_{11}} I$. From $[E_{11}, E_{12}] = E_{12}$ it follows that

$$\phi(E_{12}) = \left[\phi(E_{11}), \phi(E_{12})\right] = \left[E_{11}, \phi(E_{12})\right]. \tag{6}$$

Suppose $\phi(E_{12}) = \left[\begin{smallmatrix} a_1 & \delta \\ a_3 & a_4 \end{smallmatrix}\right]$. From (6) we obtain $a_1 = a_4 = 0$ and $a_3 = -a_3$. Since char $\mathbb{F} \neq 2$, we may conclude that $a_3 = 0$, i.e. $\phi(E_{12}) = \delta E_{12}$. Here $\delta \neq 0$ since $\phi(0) = 0$ and ϕ is injective. Let $\psi \colon M_2(\mathbb{F}) \to M_2(\mathbb{F})$ be the map defined as

$$\psi(A) = \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & \delta^{-1} \end{bmatrix}.$$

Then $\psi(\phi(E_{12})) = \psi(\delta E_{12}) = E_{12}$ and $\psi(\phi(E_{11})) = \psi(E_{11} + \mu_{E_{11}} I) = E_{11} + \mu_{E_{11}} I$, thus we may assume that $\phi(E_{12}) = E_{12}$.

Let $\alpha \in \mathbb{F}$ and let $\phi(E_{11} + \alpha E_{12}) = P_1 + \mu_{E_{11}+\alpha E_{12}} I$. Since $[E_{11} + \alpha E_{12}, E_{12}] = E_{12}$ and $\phi(E_{12}) = E_{12}$, it follows similarly as in Step 7, that $\phi(E_{11} + \alpha E_{12}) = (E_{11} + \beta E_{12}) + \mu_{E_{11}+\alpha E_{12}} I$, for some $\beta \in \mathbb{F}$. So, there exists a function $j\colon \mathbb{F} \to \mathbb{F}$ such that $\phi(E_{11} + \alpha E_{12}) = (E_{11} + j(\alpha)E_{12}) + \mu_{E_{11}+\alpha E_{12}} I$. From $\alpha E_{12} = [E_{11}, E_{11} + \alpha E_{12}]$ we obtain

$$\sigma(\alpha)E_{12} = [E_{11}, E_{11} + j(\alpha)E_{12}] = j(\alpha)E_{12},$$

hence $j = \sigma$. Also, $[E_{11} + \alpha E_{12}, E_{11} + \beta E_{12}] = (\beta - \alpha)E_{12}$ which implies $[E_{11} + \sigma(\alpha)E_{12}, E_{11} + \sigma(\beta)E_{12}] = \sigma(\beta - \alpha)E_{12}$. Thus, $\sigma(\beta) - \sigma(\alpha) = \sigma(\beta - \alpha)$. Since $\sigma(0) = 0$, we obtain that $-\sigma(\alpha) = \sigma(-\alpha)$. So, $\sigma(\beta + \alpha) = \sigma(\beta) - \sigma(-\alpha) = \sigma(\beta) + \sigma(\alpha)$ and therefore σ is also additive.

Step 11. We may assume that $\phi(E_{11} + \alpha E_{12}) - (E_{11} + \alpha E_{12})$ and $\phi(E_{11} + \alpha E_{21}) - (E_{11} + \alpha E_{21})$ are scalar matrices for every $\alpha \in \mathbb{F}$.

Recall that ϕ is bijective, therefore it follows that σ is a field automorphism. Let us compose the map ϕ with the map $\psi\colon M_2(\mathbb{F}) \to M_2(\mathbb{F})$ defined with $\psi(A) = A^{\sigma^{-1}}$, $A \in M_2(\mathbb{F})$. Since $\phi(\lambda P) = \sigma(\lambda)P_1 + \mu_{\lambda P} I$ for a rank one idempotent P, $\phi(\lambda N) = \sigma(\lambda)\phi(N)$ for $N \in \mathcal{N}$, $\phi(E_{11} + \lambda E_{12}) = E_{11} + \sigma(\lambda)E_{12} + \mu_{E_{11}+\lambda E_{12}} I$, and $\phi(E_{12}) = E_{12}$, we may assume that σ is the identity, in particular

$$\phi(E_{11} + \alpha E_{12}) - (E_{11} + \alpha E_{12}),\quad \alpha \in \mathbb{F}, \tag{7}$$

is a scalar matrix.

Similarly as for the function j defined in Step 10, we can see that there exists a function $k\colon \mathbb{F} \to \mathbb{F}$ such that $\phi(E_{11} + \alpha E_{21}) - (E_{11} + k(\alpha)E_{21})$ is a scalar matrix. Assuming that σ is the identity, let us show that k is also the identity. Since $[E_{11} + E_{21}, E_{11} - E_{12}] = E_{11} - E_{12} + E_{21} - E_{22}$ is a nilpotent, it follows that

$$\phi\left([E_{11} + E_{21}, E_{11} - E_{12}]\right) = [E_{11} + k(1)E_{21}, E_{11} - E_{12}] = \begin{bmatrix} k(1) & -1 \\ k(1) & -k(1) \end{bmatrix}$$

is also a nilpotent and hence $k(1) = 0$ or $k(1) = 1$. The former possibility, $k(1) = 0$, can not occur since ϕ is injective. So, $k(1) = 1$. From $E_{21} = [E_{11} + E_{21}, E_{11}]$ it follows

$$\phi(E_{21}) = \left[\phi(E_{11} + E_{21}), \phi(E_{11})\right] = [E_{11} + k(1)E_{21}, E_{11}] = k(1)E_{21} = E_{21}.$$

Also, for any $\alpha \in \mathbb{F}$ we have $\alpha E_{21} = [E_{21}, \alpha E_{11}]$ and $\alpha E_{21} = [E_{11} + \alpha E_{21}, E_{11}]$. By the former equation we obtain

$$\phi(\alpha E_{21}) = \left[\phi(E_{21}), \phi(\alpha E_{11})\right] = [E_{21}, \alpha E_{11}] = \alpha E_{21}$$

and by the latter equation we get

$$\phi(\alpha E_{21}) = \left[\phi(E_{11} + \alpha E_{21}), \phi(E_{11})\right] = [E_{11} + k(\alpha)E_{21}, E_{11}] = k(\alpha)E_{21}.$$

It follows that k is the identity map and therefore

$$\phi(E_{11} + \alpha E_{21}) - (E_{11} + \alpha E_{21}),\quad \alpha \in \mathbb{F}, \tag{8}$$

is a scalar matrix.

Step 12. For every rank one idempotent $P \in M_2(\mathbb{F})$ we obtain that $\phi(P) - P$ is a scalar matrix.

Let $P \in M_2(\mathbb{F})$ be a rank one idempotent. Then

$$P = \begin{bmatrix} \alpha & \beta \\ \gamma & 1-\alpha \end{bmatrix} \quad \text{with} \quad \alpha, \beta, \gamma \in \mathbb{F} \quad \text{and} \quad \alpha(1-\alpha) = \beta\gamma. \tag{9}$$

Let us consider three options: when $\alpha \neq 0$, when $\alpha = 0$ and $P = \begin{bmatrix} 0 & 0 \\ \delta & 1 \end{bmatrix}$, $\delta \in \mathbb{F}$, and when $\alpha = 0$ and $P = \begin{bmatrix} 0 & \lambda \\ 0 & 1 \end{bmatrix}$, $\lambda \in \mathbb{F}$. Note that for $\alpha = 1$, $\beta = 0$, and $\gamma = -\delta$ we have

$$\begin{bmatrix} \alpha & \beta \\ \gamma & 1-\alpha \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \delta & 1 \end{bmatrix} = 0 = \begin{bmatrix} 0 & 0 \\ \delta & 1 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & 1-\alpha \end{bmatrix}.$$

Similarly, for $\alpha = 1$, $\beta = -\lambda$, and $\gamma = 0$ we have

$$\begin{bmatrix} \alpha & \beta \\ \gamma & 1-\alpha \end{bmatrix} \begin{bmatrix} 0 & \lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & 1-\alpha \end{bmatrix}.$$

So, every rank one idempotent is of the form (9) with $\alpha \neq 0$, or it is orthogonal to an idempotent of this form with $\alpha \neq 0$.

First, suppose $P \in M_2(\mathbb{F})$ is of the form (9) with $\alpha \neq 0$. If $\gamma = 0$, then $\alpha(1-\alpha) = 0$ and hence $\alpha = 1$. So, in this case $P = \begin{bmatrix} 1 & \beta \\ 0 & 0 \end{bmatrix}$ and by (7), $\phi(P) - P$ is a scalar matrix. Let now $\gamma \neq 0$. An idempotent of the form (9) with $\alpha \neq 0$ has the same range space as $\begin{bmatrix} 1 & 0 \\ \alpha^{-1}\gamma & 0 \end{bmatrix}$ and the same null space as $\begin{bmatrix} 1 & \alpha^{-1}\beta \\ 0 & 0 \end{bmatrix}$. By (8) and (7) we obtain that

$$\phi\left(\begin{bmatrix} 1 & 0 \\ \alpha^{-1}\gamma & 0 \end{bmatrix}\right) - \begin{bmatrix} 1 & 0 \\ \alpha^{-1}\gamma & 0 \end{bmatrix} \quad \text{and} \quad \phi\left(\begin{bmatrix} 1 & \alpha^{-1}\beta \\ 0 & 0 \end{bmatrix}\right) - \begin{bmatrix} 1 & \alpha^{-1}\beta \\ 0 & 0 \end{bmatrix}$$

are scalar matrices. Since every idempotent is uniquely determined by its range space and its null space, we may conclude by Step 8 that $\phi(P) - P$ is a scalar matrix.

Second, suppose $R \in M_2(\mathbb{F})$ is a rank one idempotent that is orthogonal to a rank one idempotent P of the form (9) with $\alpha \neq 0$, i.e. $RP = PR = 0$. We will show that then $\phi(R) - R$ is a scalar matrix. Let us first prove that for rank one idempotents $M, N \in M_2(\mathbb{F})$, we have $[M, N] = 0$ and $M \neq N$ if and only if $MN = 0 = NM$. If $MN = NM = 0$, then clearly $[M, N] = 0$ and $M \neq N$. Conversely, let $[M, N] = 0$ and $M \neq N$. From $MN = NM$ we may conclude that either $\operatorname{Im} M = \operatorname{Ker} N$ and $\operatorname{Im} N = \operatorname{Ker} M$, and hence $MN = 0 = NM$, or $\operatorname{Im} M = \operatorname{Im} N$ which implies $MN = N$ and $NM = M$, and therefore $M = NM = MN = N$, a contradiction.

Since $RP = PR = 0$, it follows that $[R, P] = 0$ and hence $\left[\phi(R), \phi(P)\right] = 0$. Also, $\operatorname{Im} R = \operatorname{Ker} P$ and $\operatorname{Im} P = \operatorname{Ker} R$. We already know that $\phi(P) - P$ is a scalar matrix and that $\phi(R) - R_1$ is also a scalar matrix for some rank one idempotent matrix $R_1 \in M_2(\mathbb{F})$. Thus, $[R_1, P] = \left[\phi(R), \phi(P)\right] = 0$. Here $R_1 \neq P$ since $\operatorname{Im} R \neq \operatorname{Im} P$ and thus by Step 8, $\operatorname{Im} R_1 \neq \operatorname{Im} P$. It follows that $\operatorname{Im} R_1 = \operatorname{Ker} P$ and $\operatorname{Im} P = \operatorname{Ker} R_1$ which implies $R_1 = R$. We may conclude that $\phi(R) - R$ is indeed a scalar matrix.

Step 13. Assume that $\phi(P) - P$ is a scalar matrix for every rank one idempotent $P \in M_n(\mathbb{F})$. Then $\phi(A) = A + \psi(A)I$ for every $A \in M_n(\mathbb{F})$, where $\psi\colon M_n(\mathbb{F}) \to \mathbb{F}$ with $\psi(A) = 0$ for every matrix A of trace zero.

By Step 9 it follows that $\phi(\lambda P) - \lambda P$ is a scalar matrix for every rank one idempotent $P \in M_2(\mathbb{F})$ and every scalar $\lambda \in \mathbb{F}$. In the same way as in the last paragraph of Step 3 we also obtain that $\phi(N) = N$ for every rank one nilpotent N, so $\phi(A) - A$ is a scalar matrix for every rank one matrix $A \in M_2(\mathbb{F})$. As we showed in Step 4, it follows that $\phi(A) - A$ is a scalar matrix for every $A \in M_2(\mathbb{F})$. Thus, there exists the function $\psi\colon M_2(\mathbb{F}) \to \mathbb{F}$, such that

$$\phi(A) = A + \psi(A)I$$

for every matrix $A \in M_2(\mathbb{F})$ and as we showed in Step 5, $\psi(A) = 0$ for every matrix $A \in M_2(\mathbb{F})$ of trace zero.

Taking into account our assumptions, a bijective Lie product preserving map $\phi\colon M_2(\mathbb{F}) \to M_2(\mathbb{F})$ is of the form:

$$\phi(A) = TA^{\sigma}T^{-1} + \psi(A)I$$

for every $A \in M_2(\mathbb{F})$. Here $\sigma\colon \mathbb{F} \to \mathbb{F}$ is a filed automorphism, and $\psi\colon M_2(\mathbb{F}) \to \mathbb{F}$ maps all trace zero matrices to zero.

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